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AN EXISTENCE RESULT OF POSITIVE SOLUTIONS FOR SINGULAR SUPERLINEAR BOUNDARY VALUE PROBLEMS AND ITS APPLICATIONS

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1. Introduction

In this paper, we are concerned with the existence of positive solutions for the boundary value problems of the form;

$$(1) \qquad \begin{aligned} &u''(t) + q(t)g(u(t)) = 0, \ 0 < t < 1 \\ &u(0) = 0 = u(1), \end{aligned}$$

where q is singular at 0 and/or 1.

By a positive solution of (1), we understand a solution $u \in C[0,1] \cap C^2(0,1)$ (or $C^2(0,1]$ or $C^2[0,1)$, depending of the singularity) which is positive on $(0,1)$ and satisfies both the equation on $(0,1)$ and the boundary condition in (1).

Introduce the notation

$$g_0 = \lim_{u \to 0^+} \frac{g(u)}{u}, \quad g_\infty = \lim_{u \to \infty} \frac{g(u)}{u},$$

then $g_0 = 0$ and $g_\infty = \infty$ correspond to the superlinear problem, and in this case, we show that (1) has a positive solution. This result is motivated by Erbe and Wang [2] who considered regular problems.

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As applications, we first consider Dirichlet boundary value problems of the generalized Emden-Fowler equations

$$(2) \qquad \begin{aligned} u''(t) + q(t)u(t)^p = 0, \\ u(0) = 0 = u(1), \end{aligned}$$

where $p \in \mathbf{R}$. When q is not continuous at the end points of $(0,1)$ or $p < 0$, the problem is singular.

Taliaferro [6] has studied the existence and uniqueness of positive solutions of (2) for $p < 0$ with the shooting method. Zhang [7] has considered the problem for $0 < p < 1$. Precisely, under the assumptions $q \in C(0,1)$, $q > 0$ on $(0,1)$, he proved that a necessary and sufficient condition for the existence of positive solutions is

$$\int_0^1 s(1-s)q(s)ds < \infty.$$

using the method of upper and lower solutions.

For the remaining case $p > 1$, so called superlinear problem, related results have not been founded yet and in this paper, we prove that a sufficient condition for the existence of positive solutions is the same as given in Zhang. We also give a uniqueness result of this problem.

As the second application, we consider the existence of positive radial solutions for the semilinear elliptic problems, in particular, the existence of decaying positive radial solutions with zero boundary condition. So we consider

$$(3) \qquad \begin{aligned} &\Delta u + |x|^{-\lambda} f(|x|)g(u(x)) = 0, \ \text{ in } \Omega, \\ &u = 0, \qquad \text{if } |x| = r_o, \ r_o > 0, \\ &u \to 0 \qquad \text{as } |x| \to \infty, \end{aligned}$$

where $\Omega = \{x \in \mathbf{R}^n : |x| > r_o\}$ and $n \geq 3$.

Noussair and Swanson [4] obtained positive solutions for the problem of joint dependence nonlinear term, say $f(x,u)$ which is assumed to have polynomial growth in u as $u \to \infty$ with bounded continuous coefficients. Bandle and Marcus [1] considered the problems when $f \equiv 1$. Noussair, Swanson and Jianfu [5] studied the existence of positive solutions when the nonlinearity involves a critical growth.

In this paper, under suitable conditions on f and g, we obtain an existence result of positive radial solutions for (3) when $\lambda < 2(n-1)$.

2. Main results

Let us consider the problem (1). Our main existence result is

THEOREM 1. *Assume the following conditions:*

(a_1) $q \in C((0,1),(0,\infty))$ *satisfies* $\int_0^1 s(1-s)q(s)ds < \infty$.

(b_1) $g \in C([0,\infty),[0,\infty))$ *satisfies* $g_0 = 0$ *and* $g_\infty = \infty$.

Then (1) *has at least one positive solution.*

The following lemma is well known and crucial in our arguments, see Guo and Lakshmikantham [3] for proof and further discussion of the fixed point index.

LEMMA 1. *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets in E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$

or

(ii) $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Proof of Theorem 1. First, it is well known that the problem (1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)q(s)g(u(s))ds,$$

where $G(t,s)$ is the Green's function corresponding to the linear homogeneous problem explicitly written by

$$G(t,s) = \begin{cases} s(1-t) & \text{for } 0 \le s \le t \\ t(1-s) & \text{for } t \le s \le 1. \end{cases}$$

Thus (1) is equivalent to the fixed point equation

$$u = Tu$$

in $E = C([0,1])$, where $T : E \to E$ is given by

$$Tu(t) = \int_0^1 G(t,s)q(s)g(u(s))ds.$$

By the condition (a_1), T is completely continuous on the cone of non-negative functions in E. We define a cone K in E by

$$K = \{u \in E | u(t) \geq 0,\ t \in [0,1],\ \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \frac{1}{4} \|u\|_\infty\},$$

then it is not hard to check $T(K) \subset K$.

Second, by (a_1), we may choose $\eta > 0$ so that $\eta \int_0^1 s(1-s)q(s)ds \leq 1$. Since $g_0 = 0$, there exists $R_1 > 0$ such that $g(u) \leq \eta u$, for $0 < u \leq R_1$. Let $\Omega_1 = \{u \in E : \|u\|_\infty < R_1\}$, then Ω_1 is bounded open in E and $0 \in \Omega_1$. Moreover, let $u \in K \cap \partial\Omega_1$, then $u \in K$, $\|u\|_\infty = R_1$, and thus

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)q(s)g(u(s))ds \\ &\leq \int_0^1 s(1-s)q(s)g(u(s))ds \\ &\leq \eta \int_0^1 s(1-s)q(s)u(s)ds \\ &\leq \eta \int_0^1 s(1-s)q(s)\|u\|_\infty ds \leq \|u\|_\infty. \end{aligned}$$

Therefore

$$\|Tu\|_\infty \leq \|u\|_\infty, \quad \text{for all } u \in K \cap \partial\Omega_1.$$

Third, choose $\mu > 0$ such that $\frac{\mu}{4} \int_{1/4}^{3/4} G(\frac{1}{2}, s)ds > 1$. Since $g_\infty = \infty$, there exists $R > 0$ such that $q_o g(u) \geq \mu u$, for all $u \geq R$, where $q_o = \min_{t \in [1/4, 3/4]} q(t)$. Let $R_2 = \max\{2R_1, 4R\}$ and $\Omega_2 = \{u \in E : \|u\|_\infty < R_2\}$, then Ω_2 is bounded open in E and $\bar{\Omega}_1 \subset \Omega_2$. We show $\|Tu\|_\infty \geq \|u\|_\infty$, for all $u \in K \cap \Omega_2$, so let $u \in K$ and $\|u\|_\infty = R_2$, then $\min_{t \in [1/4, 3/4]} u(t) \geq 1/4 \|u\|_\infty \geq R$. Thus $q(t)g(u(t)) \geq \mu u(t)$, for all $t \in [1/4, 3/4]$ and

$$\begin{aligned} Tu(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2}, s)q(s)g(u(s))ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)q(s)g(u(s))ds \geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)u(s)ds \\ &\geq \frac{\mu}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)\|u\|_\infty ds > \|u\|_\infty. \end{aligned}$$

Therefore $\|Tu\|_\infty \geq \|u\|_\infty$, for $u \in K \cap \partial\Omega_2$, and by Lemma 1, T has a fixed point u in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $R_1 \leq \|u\|_\infty \leq R_2$. Futhermore, since $G(t,s)q(s) > 0$ for all $s \in (0,1)$, it follows that $u > 0$ on $(0,1)$, and this completes the proof.

Let $p > 1$ and let $g(u) = u^p$. Then g obviously satisfies (b_1) in Theorem 1 and we obtain an existence result of positive solutions for (2) as follows;

COROLLARY 1. *Let $p > 1$ and assume (a_1) in Theorem 1. Then (2) has at least one positive solution.*

We have a similar result as Theorem 1 when q is singular at 0.

THEOREM 2. *Assume (b_1) in Theorem 1 and*

(a_2) $q \in C((0,1],(0,\infty))$ *satisfies* $\int_0^1 sq(s)ds < \infty$.

Then (1) has at least one positive solution.

Proof. The proof generally follows that of Theorem 1, and it is enough to check the second part in the proof of Theorem 1.
By (a_2), we may choose $\eta > 0$ so that $\eta \int_0^1 sq(s)ds \leq 1$. Since $g_0 = 0$, there exists $R_1 > 0$ such that $g(u) \leq \eta u$, for $0 < u \leq R_1$. Let $\Omega_1 = \{u \in E : \|u\|_\infty < R_1\}$, and let $u \in K \cap \partial\Omega_1$, then

$$
\begin{aligned}
Tu(t) &= (1-t)\int_0^t sq(s)g(u(s))ds + t\int_t^1 (1-s)q(s)g(u(s))ds \\
&\leq (1-t)\int_0^1 sq(s)g(u(s))ds \\
&\leq \eta \int_0^1 sq(s)u(s)ds \\
&\leq \eta \int_0^1 sq(s)\|u\|_\infty ds \leq \|u\|_\infty.
\end{aligned}
$$

Therefore

$$\|Tu\|_\infty \leq \|u\|_\infty, \quad \text{for all } u \in K \cap \partial\Omega_1,$$

and the proof is done.

Since the problem having the singularity at 0 and the problem having the singularity at 1 are equivalently transformed, we get the following corollary;

Corollary 2. *Assume* (b_1) *in Theorem 1 and*

(a_3) $q \in C([0,1),(0,\infty))$ *satisfies* $\int_0^1 (1-s)q(s)ds < \infty$.

Then (1) *has at least one positive solution.*

3. Uniqueness

Let u be a positive solution of (1) and let $L_u = \max_{t\in[0,1]} g(u(t))$. Then $q(t)g(u(t)) \le L_n q(t)$ and $\int_0^1 |u''(t)|dt \le L_n \int_0^1 q(t)dt < \infty$, provided by $q \in L^1[0,1]$. Thus both $u'(0^+)$ and $u'(1^-)$ exist and consequently, all positive solutions of (1) are of $C^1[0,1] \cap C^2(0,1)$. Based on this fact, we obtain the existence of a unique positive solution for (1) as follows;

Theorem 3. *Assume*

(a_4) $q \in C((0,1),(0,\infty))$ *satisfies* $\int_0^1 q(s)ds < \infty$.

(b_1) $g \in C([0,\infty),[0,\infty))$ *satisfies* $g_0 = 0$ *and* $g_\infty = \infty$.

(b_2) g *is increasing and* $\frac{g(u)}{u}$ *is strictly monotone.*

Then (1) *has a unique positive solution.*

Proof. Suppose that (1) has two distinct positive solutions u_1 and u_2. We first show that there exist two ordered positive solutions of (1). Let $\phi(t) = \min\{u_1(t), u_2(t)\}$, $t \in [0,1]$. Using the fact that g is increasing, $G(t,s), q(t) \ge 0$, we get

$$\begin{aligned} T\phi(t) &= \int_0^1 G(t,s)q(s)g(\phi(s))ds \\ &\le \min\{\int_0^1 G(t,s)q(s)g(u_1(s))ds,\ \int_0^1 G(t,s)q(s)g(u_2(s))ds,\} \\ &= \min\{u_1(t),\ u_2(t)\} = \phi(t) \end{aligned}$$

Thus $(T^n\phi)_{n=0}^{\infty}$ decreases to a positive solution u_3 of (1) and

$$u_3(t) \le \min\{u_1(t),\ u_2(t)\}, \text{ for all } t \in [0,1].$$

Therefore we have two ordered positive solutions u_3 and one of u_1 or u_2 whether u_1 and u_2 are ordered or not. So for convinience, assume that u and v are two ordered positive solutions of (1) with $u(t) \le v(t)$, for all $t \in [0,1]$. Let $z(t) = u(t)v'(t) - u'(t)v(t)$. Since $u, v \in C^1[0,1]$, $z(0) = 0 = z(1)$ and

$$\begin{aligned} z'(t) &= u(t)v''(t) - u''(t)v(t) \\ &= q(t)u(t)v(t)\Big(\frac{g(u(t))}{u(t)} - \frac{g(v(t))}{v(t)}\Big) \quad \text{on } (0,1). \end{aligned}$$

Since $\frac{f(u)}{u}$ is strictly monotone, either

$$z'(t) > 0 \quad \text{or} \quad z'(t) < 0 \quad \text{on } (0,1).$$

This contradicts to $z(0) = 0 = z(1)$, and the proof is complete.

Finally, we obtain the uniqueness of positive solutions for the generalized Emden-Fowler equations.

COROLLARY 3. *Let $p > 1$ and assume (a_4) in Theorem 2. Then (2) has a unique positive solution.*

4. Existence of positive radial solutions

Let us consider the semilinear elliptic problems of the form;

$$\begin{aligned} &\Delta u + |x|^{-\lambda} f(|x|)g(u(x)) = 0, \text{ in } \Omega, \\ &u = 0, \qquad \text{if } |x| = r_o, \\ &u \to 0 \qquad \text{as } |x| \to \infty, \end{aligned} \tag{3}$$

where $\Omega = \{x \in \mathbf{R}^n : |x| > r_o\}$ and $n \geq 3$. For any real number λ satisfying $\lambda < 2(n-1)$, we prove the existence of positive radial solutions for (3) if g and f satisfies the following conditions;

(b_1) $g \in C([0,\infty),[0,\infty))$ satisfies $g_0 = 0$ and $g_\infty = \infty$.

(c_1) $f \in C([r_o,\infty),(0,\infty))$ satisfies $\int_{r_o}^{\infty} x^{1-\lambda} f(x)dx < \infty$.

We are concerned with radial solutions, thus for the radial variable $r = |x|$, we write (3) as

$$\begin{aligned} &u''(r) + \frac{n-1}{r}u'(r) + f(r)g(u(r)) = 0, \\ &u(r_o) = 0, \\ &u(r) \to 0 \qquad \text{as } r \to \infty. \end{aligned} \tag{3'}$$

Setting $s = r^{2-n}$, $v(s) = u(r(s))$, and then $t = (r_o^{2-n} - s)/r_o^{2-n}$, $z(t) = v(s)$, we rewrite (3′) as

$$z''(t) + q(t)g(z(t)) = 0,$$
$$z(0) = 0 = z(1),$$

where $q(t) = \frac{r_o^{2-\lambda}}{(n-2)^2}(1-t)^{\frac{-2(n-1)+\lambda}{n-2}} f(r_o(1-t)^{\frac{-1}{n-2}})$. Thus by (c_1), $q \in C([0,1),(0,\infty))$ is singular at 1 satisfying $\int_0^1 (1-s)q(s)ds < \infty$, and for the problem in this section, it suffices to consider the existence of positive solutions of the problem (3) with the conditions on q described above. Therefore by Corollary 2, we obtain an existence result for problem (3).

COROLLARY 4. *Let $\lambda < 2(n-1)$, and assume (b_1) and (c_1). Then (3) has at least one positive radial solution for all $0 < r_o < \infty$.*

It is easy to check that if $\int_{r_o}^{\infty} x^{n-1-\lambda} f(x)dx < \infty$, then $\int_0^1 q(s)ds < \infty$. Thus we obtain a uniqueness result for (3) as follows;

COROLLARY 5. *Let $\lambda < 2(n-1)$, and assume (b_1) and (b_2). Moreover, assume*

(c_2) $f \in C([r_o,\infty),(0,\infty))$ *satisfies* $\int_{r_o}^{\infty} x^{n-1-\lambda} f(x)dx < \infty$.

Then (3) has a unique positive radial solution for all $0 < r_o < \infty$.

EXAMPLE. If $g(u) = u^p$, $p > 1$ or $g(u) = e^u - u$, then (3) has at least one positive radial solution or a unique positive radial solution provided $\int_{r_o}^{\infty} x^{1-\lambda} f(x)dx < \infty$ or $\int_{r_o}^{\infty} x^{n-1-\lambda} f(x)dx < \infty$, respectively.

References

1. Bandle C. and Marcus M., *The positive radial solutions of a class of semilinear elliptic equations*, J. reine angew. Math. **401** (1989), 25-59.
2. Erbe L. H. and Wang H., *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120(3)** (1994), 743-748.
3. Guo D. and Lakshmikantham V., *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL, 1988.
4. Noussair E. S. and Swanson C. A., *A positive solutions of semilinear elliptic problems in unbounded domains*, J. Diff. Eqns. **57** (1985), 349-372.
5. Noussair E. S., Swanson C. A. and Jianfu Y., *Positive finite energy critical solutions of semilinear elliptic problems*, Can. J. Math. **44** (1992), 1014-1029.

6. Taliaferro S. D., *A nonlinear singular boundary value problem*, Nonlinear Analysis, T.M.A. **3** (1979), 897-904.
7. Zhang Y., *Positive solutions of singular sublinear Emden-Fowler boundary value problems*, J. Math. Anal. Appl. **185** (1994), 215-222.

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