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# THE UNIFORM ASYMPTOTIC STABILITY AND THE UNIFORM ULTIMATE BOUNDEDNESS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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## 1. Introduction

The purpose of this paper is to present the uniform asymptotic stability theorem and the uniform boundedness and uniform ultimate boundedness theorem for functional differential equations.

We consider a system of functional differential equations with finite delay. Also we consider a system of functional differential equations with unbounded delay. For $x \in \mathbf{R}^n$, $|x|$ denotes a usual norm in $\mathbf{R}^n$, and $W_i$ denotes a continuous function from $\mathbf{R}_+$ into $\mathbf{R}_+$, such that $W_i(0) = 0$ and $W_i$ is strictly increasing on $\mathbf{R}_+ = [0, \infty)$. Our main goal is to generalize a theorem in [5] and to present sufficient conditions to ensure that solutions of a system of functional differential equations with unbounded delay are uniformly bounded and uniformly ultimately bounded. Liapunov methods are used throughout.

## 2. Uniform Asymptotic Stability for Functional Differential Equations with Finite Delay

In this section we consider the system

$$x'(t) = F(t, x_t) \tag{1}$$

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where $x_t$ is the translation of $x$ on $[t-h,t]$ back to $[-h,0]$, where $h>0$ is a fixed constant, and $x'$ denotes the right-hand derivative. The following notation will be used.

For $h>0$, $C$ denotes the space of continuous functions mapping $[-h,0]$ into $\mathbf{R}^n$, and for $\phi\in C$, $||\phi|| = \sup_{-h\le s\le 0}|\phi(s)|$ and $|||\phi||| = [\int_{-h}^{0}|\phi(s)|^2\,ds]^{\frac{1}{2}}$. Also, $C_H$ denotes the set of $\phi\in C$ with $||\phi||<H$. If $x$ is a continuous function of $u$ defined for $-h\le u<A$, with $A>0$, and if $t$ is a fixed number satisfying $0\le t<A$, then $x_t$ denotes the translation of the restriction of $x$ to $[t-h,t]$ so that $x_t$ is an element of $C$ defined by $x_t(\theta)=x(t+\theta)$ for $-h\le\theta\le 0$. We denote by $x(t)\equiv x(t_0,\phi)$ a solution of (1) with the initial condition $x_{t_0}(t_0,\phi)=\phi\in C$ and we denote by $x(t,t_0,\phi)$ the value of $x(t_0,\phi)$ at $t$.

It is supposed that $F:\mathbf{R}_+\times C_H\to\mathbf{R}^n$ is continuous and takes bounded sets into bounded sets ; where $0<H\le\infty$. It is well known ([1],[4]) that for each $t_0\in\mathbf{R}_+$ and each $\phi\in C_H$ there is at least one solution $x(t_0,\phi)$ defined on an interval $[t_0,t_0+\alpha)$ with $\alpha>0$ and, if there is an $H_1<H$ with $|x(t,t_0,\phi)|\le H_1$, then $\alpha$ may be considered to be $\infty$.

A Liapunov functional is a continuous function $V(t,\phi):\mathbf{R}_+\times C_H\to\mathbf{R}_+$ whose derivative along a solution of (1) satisfies some specific relation. The derivative of a Liapunov functional $V(t,\phi)$ along a solution $x(t)$ of (1) may be defined by

$$V'_{(1)}(t,\phi)=\limsup_{\delta\to 0^+}\{V(t+\delta,x_{t+\delta}(t,\phi))-V(t,\phi)\}/\delta.$$

For a function $a:R\to R$, we define $a_+$ and $a_-$ by $a_+=\max\{a,0\}$ and $a_-=\max\{-a,0\}$, respectively.

DEFINITION 2.1. Let $F(t,0)=0$, for all $t\ge 0$.

(a) The zero solution of (1) is said to be *stable* if for each $\epsilon>0$ and $t_0\ge 0$ there is a $\delta>0$ such that $[\varphi\in C_\delta, t\ge t_0]$ imply $|x(t,t_0,\phi)|<\epsilon$.

(b) The zero solution of (1) is *uniformly stable* (U.S.) if it is stable and if $\delta$ is independent of $t_0$.

(c) The zero solution of (1) is *asymptotically stable* (A.S.) if it is stable and if for each $t_0\ge 0$ there is a $\sigma>0$ such that $\phi\in C_\sigma$ implies that $x(t,t_0,\phi)\to 0$ as $t\to\infty$.

(d) The zero solution of (1) is *uniformly asymptotically stable* (U.A.S.) if it is U.S. and if there is an $\delta_0 > 0$ and for each $\epsilon > 0$ there exists $T > 0$ such that $[t_0 \in \mathbf{R}_+, \phi \in C_{\delta_0}, t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \epsilon$.

DEFINITION 2.2. A measurable function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ is said to satsify the condition $(I)$ if for any $\epsilon > 0$ there exists a $\theta = \theta(\epsilon) > 0$ such that

$$(I) \qquad \int_t^{t+\epsilon} \eta(s)ds > \theta$$

for all $t \geq 0$.

REMARK 2.1. It is easy to check that the condition $(I)$ implies that for any $\theta > 0$ there exists an $\epsilon = \epsilon(\theta) > 0$ such that $\int_t^{t+\epsilon} \eta(s)ds > \theta$ for all $t \geq 0$.

DEFINITION 2.3. Let $U : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ be continuous and locally Lipschiztian in $x \in \mathbf{R}^n$. Then the derivative of $U(t, x(t))$ along a solution $x(t)$ of (1) is defined as

$$U'_{(1)}(t, x(t)) = \limsup_{\delta \to 0^+} \{U(t + \delta, x(t) + \delta F(t, x_t)) - U(t, x(t))\}/\delta.$$

REMARK 2.2. (i) It is easy to check that

$$\limsup_{\delta \to 0^+} \frac{1}{\delta}\{U(t + \delta, x(t + \delta)) - U(t, x(t))\} = U'_{(1)}(t, x(t))$$

for any solution $x(t)$ of (1).

(ii) If $U(t, x(t))$ has continuous partial derivatives of the first order, we have

$$U'_{(1)}(t, x(t)) = grad\ U \cdot F + \partial U/\partial t.$$

In presenting the uniform asymptotic stability theorem for functional differential equations with finite delay, the following theorem is basic.

THEOREM A. [5] *Let $H > 0$ and $V : \mathbf{R}_+ \times C_H \to \mathbf{R}_+$ be continuous and locally Lipschitz in $\phi$, and let $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ be a measurable function such that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)\,ds = \infty$ uniformly with respect to $t \in \mathbf{R}_+$.*

*Suppose there exist wedges $W_1, W_2$ and $W_3$ such that, for all $t \geq 0$ and $\phi$ in $C_H$,*

$$\text{(i)}\quad W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(||\phi||),$$

$$\text{(ii)}\quad V'_{(1)}(t,x_t) \leq -\eta(t)W_3(|||x_t|||), \text{ and}$$

$$\text{(iii)}\quad \int_0^t F(s,x_s)ds \text{ is uniformly continuous}$$

*for any bounded solution $x(t)$ of (1) on $\mathbf{R}_+$. Then the zero solution of (1) is uniformly asymptotically stable.*

THEOREM 2.1. *Let $H > 0$ and $V : \mathbf{R}_+ \times C_H \to \mathbf{R}_+$ be continuous and locally Lipschitzian in $\phi$, and let $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ be a measurable function that satisfies the condition (I). Suppose that $U : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}_+$ is continuous and locally Lipschitzian in $x \in \mathbf{R}^n$ and that there exist wedges $W_1, W_2, W_3, W_4$ and $W_5$ such that, for any $\phi$ in $C_H$ and any solution $x(t)$ of (1),*

$$\text{(i)}\quad W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(||\phi||),$$

$$\text{(ii)}\quad V'_{(1)}(t,x_t) \leq -\eta(t)W_3(|||x_t|||), \text{ and}$$

$$\text{(iii)}\quad W_4(|x(t)|) \leq U(t,x(t)) \leq W_5(|x(t)|).$$

*Furthermore, suppose that either*

$$\int_0^t \{U'_{(1)}(s,x(s))\}_+\,ds \qquad \text{or} \qquad \int_0^t \{U'_{(1)}(s,x(s))\}_-\,ds$$

*is uniformly continuous for any bounded solution $x(t)$ of (1) on $\mathbf{R}_+$. Then the zero solution of (1) is uniformly asymptotically stable.*

*Proof.* First we will show that the zero solution of (1) is uniformly stable. Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ with $0 < W_2(\delta) < W_1(\epsilon)$. Let $\phi \in C_\delta$ and $t_0 \geq 0$. Then for any $t \geq t_0$

$$\begin{aligned} W_1(|x(t)|) &\leq V(t,x_t) \leq V(t_0,\phi) \\ &< W_2(\delta) < W_1(\epsilon), \end{aligned}$$

where $x(t) = x(t, t_0, \phi)$ is a solution of (1). That is, $|x(t)| < \epsilon$ for all $t \geq t_0$. This implies that the zero solution of (1) is uniformly stable.

Let $0 < H' < H$ and take $\delta_0 = \delta_0(H')$ of uniform stability. For any $\epsilon > 0$, we will show that there is a $T = T(\epsilon) > 0$ such that any solution $x(t, t_0, \phi)$ of (1) with $||\phi|| < \delta_0$ satisfies $|x(t, t_0, \phi)| < \epsilon$ for any $t \geq t_0 + T$. Let $\delta = \delta(\epsilon)$ be the constant for uniform stability. Suppose that a solution $x(t) = x(t, t_0, \phi)$, $||\phi|| < \delta_0$, satisfies $||x_t(t_0, \phi)|| \geq \delta$ for any $t \geq t_0$. Then we have $t^* \in [t, t+h]$ for each $t \geq t_0$ such that $|x(t^*)| \geq \delta$. Also we can choose a constant $\theta = \theta(\epsilon) > 0$ with $W_4(\delta) > W_5(\theta)$ and $0 < \theta < \delta$.

Now we claim that there is an $L = L(\epsilon) > 0$ such that there is at least $t' \in [t, t+L]$ with $|x(t')| \leq \theta$ for any $t \geq t_0$. By assumption on $\eta$ we note that there is an $L = L(\epsilon) > 0$ with $L > 2h$ such that

$$\int_{t+h}^{t+L} \eta(s)\, ds > W_2(\delta_0)/W_3(\sqrt{h}\theta)$$

for all $t \geq t_0$. If $|x(t)| > \theta$ were true for all $t \in [t_*, t_* + L]$ with some $t_* \geq t_0$, then we would have

$$\begin{aligned} 0 \leq V(t_* + L) &\leq V(t_0, \phi) - \int_{t_0}^{t_*+L} \eta(s) W_3(|||x_t(s)|||)\, ds \\ &\leq W_2(\delta_0) - W_3(\sqrt{h}\theta) \int_{t_*+h}^{t_*+L} \eta(s)\, ds \\ &< W_2(\delta_0) - W_3(\sqrt{h}\theta) \frac{W_2(\delta_0)}{W_3(\sqrt{h}\theta)} = 0, \end{aligned}$$

a contradiction.

Now, we shall assume that $\displaystyle\int_0^t \{U'_{(1)}(s, x(s)\}_+\, ds$ is uniformly continuous on $R_+$. Then we may choose a sequence

$$t_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_i < \beta_i < \cdots$$

such that, for $i = 1, 2, 3, \cdots$,

$$|x(\alpha_i)| = \theta, |x(\beta_i)| \geq \delta, \theta \leq |x(t)| \text{ for any } \ t \in [\alpha_i, \beta_i],$$

and $\alpha_i, \beta_i \in I_i = [t_0 + (2i-1)L, t_0 + 2iL]$. Thus we have

$$W_4(|x(\beta_i)|) \leq U(\beta_i, x(\beta_i)) = \int_{\alpha_i}^{\beta_i} U'_{(1)}(s, x(s))\, ds + U(\alpha_i, x(\alpha_i))$$
$$\leq \int_{\alpha_i}^{\beta_i} \{U'_{(1)}(s, x(s))\}_+\, ds + W_5(|x(\alpha_i)|)$$

and

$$0 < W_4(\delta) - W_5(\theta) \leq \int_{\alpha_i}^{\beta_i} \{U'_{(1)}(s, x(s))\}_+\, ds.$$

By assumption there exists a $\rho > 0$ with $\beta_i - \alpha_i \geq \rho$ for $i = 1, 2, 3, \cdots$

Case 1) $\rho < h$

We note that

$$|||x_t||| = \left[\int_{-h}^{0} |x_t(s)|^2\, ds\right]^{\frac{1}{2}} = \left[\int_{t-h}^{t} |x(s)|^2\, ds\right]^{\frac{1}{2}} \geq (\sqrt{h-\rho})\theta$$

for all $t \in [\beta_i, \beta_i + \rho] = J_i$ with $i = 1, 2, 3, \cdots$. Then we have

$$0 \leq \lim_{t\to\infty} V(t, x_t) \leq V(t_0, \phi) - W_3\{\sqrt{h-\rho}\theta\} \int_J \eta(s)\, ds = -\infty,$$

where $J = \cup_{i=1}^{\infty} J_i$, a contradiction. Let $N$ be the smallest positive integer such that

$$W_2(\delta_0) - W_3(\sqrt{h-\rho}\theta) \sum_{i=1}^{N} \int_{\beta_i}^{\beta_i+\rho} \eta(s)\, ds < 0.$$

Then $N$ only depends on $\epsilon$ and we can take $T = (2N+1)L$ such that, at some $\tau \in [t_0, t_0 + T]$ $\quad ||x_\tau(t_0, \phi)|| < \delta$, which implies $|x(t)| < \epsilon$ for all $t \geq t_0 + T$.

Case 2) $\rho \geq h$

Now we note that

$$|||x_t||| = \left[\int_{-h}^{0} |x_t(s)|^2\, ds\right]^{\frac{1}{2}} = \left[\int_{t-h}^{t} |x(s)|^2\, ds\right]^{\frac{1}{2}} \geq \sqrt{\frac{h}{2}}\theta$$

for all $t \in [\beta_i, \beta_i + \frac{h}{2}] = J_i$ with $i = 1, 2, 3, \cdots$. Then we have

$$0 \le \lim_{t\to\infty} V(t, x_t) \le V(t_0, \phi) - W_3(\sqrt{\frac{h}{2}}\theta) \int_J \eta(s)\, ds = -\infty,$$

where $J = \cup_{i=1}^{\infty} J_i$, a contradiction. Let $N'$ be the smallest positive integer such that

$$W_2(\delta_0) - W_3(\sqrt{\frac{h}{2}}\theta) \sum_{i=1}^{N} \int_{\beta_i}^{\beta_i + \frac{h}{2}} \eta(s)\, ds < 0.$$

Then $N'$ only depends on $\epsilon$ and we can take $T = (2N' + 1)L$ such that, at some $\tau \in [t_0, t_0 + T]$ $\|x_\tau(t_0, \phi)\| < \delta$, which $|x(t)| < \epsilon$ for all $t \ge t_0 + T$. Hence the proof is complete.

REMARK 2.3. The condition that either

$$\int_0^t \{U'_{(1)}(s, x(s))\}_+\, ds \qquad \text{or} \qquad \int_0^t \{U'_{(1)}(s, x(s))\}_-\, ds$$

is uniformly continuous for any bounded solution $x(t)$ of (1) on $\mathbf{R}_+$ is satisfied if

$$-p(t) \le U'_{(1)}(t, x(t)) \qquad \text{or} \qquad U'_{(1)}(t, x(t)) \le q(t),$$

where $p,\ q : \mathbf{R}_+ \to \mathbf{R}_+$ are measurable functions such that $\int_0^t p(s)\, ds$ and $\int_0^t q(s)\, ds$ are uniformly continuous on $\mathbf{R}_+$.

Consider the scalar differential equation (1) and consider $U(t, x(t)) = |x(t)|$. Then the condition that $\int_0^t F(s, x_s) ds \equiv \int_0^t x'(s) ds$ is uniformly continuous for any bounded solution $x(t)$ of (1) on $\mathbf{R}_+$ is satisfied if the condition (A) : $\int_0^t |x'(s)| ds$ is uniformly continuous for any bounded solution $x(t)$ of (1) on $\mathbf{R}_+$ holds. Assume that the condition (A) holds. Then we see that

$$\int_0^t U'_{(1)}(s, x(s))_+ ds \qquad \text{and} \qquad \int_0^t U'_{(1)}(s, x(s))_- ds$$

are uniformly continuous for any bounded solution $x(t)$ of (1) on $\mathbf{R}_+$, since $U'_{(1)}(t, x(t))_+ = (|x(t)|')_+ \le |x'(t)|$ and $U'_{(1)}(t, x(t))_- = (|x(t)|')_- \le |x'(t)|$. Hence Theorem 2.1 improves Theorem A.

REMARK 2.4. Consider the scalar equation

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} \lambda(s)x(s)ds \tag{1.1}$$

where $a, b, \lambda : \mathbf{R}_+ \to \mathbf{R}$ are continuous and $a(t) \geq 0$ for all $t \geq 0$.

From Theorem A we see that one of the sufficient conditions ensuring the uniform asymptotic stability of the zero solution of (1.1) is that the functions

$$\int_0^t a(s)ds \qquad \text{and} \qquad \int_0^t |b(s)|(\int_{s-h}^{s} |\lambda(u)|du)ds$$

are uniformly continuous on $\mathbf{R}_+$, since one of the sufficient conditions must satisfy the condition (iii) in Theorem A.

To apply Theorem 2.1 to the equation (1.1) we consider $U(t, x(t)) = |x(t)|$. Then we have

$$\begin{aligned} U'_{(1.1)}(t, x(t)) &= |x(t)|' \\ &\leq -a(t)|x(t)| + |b(t)| \int_{t-h}^{t} |\lambda(s)||x(s)|ds \\ &\leq |b(t)| \int_{t-h}^{t} |\lambda(s)||x(s)|ds \end{aligned}$$

and

$$U'_{(1.1)}(t, x(t))_+ \leq |b(t)| \int_{t-h}^{t} |\lambda(s)||x(s)|ds.$$

If $x(t)$ is a bounded solution of (1.1) and $|x(t)| \leq M$ for some $M > 0$, then we have

$$U'_{(1.1)}(t, x(t))_+ \leq M|b(t)| \int_{t-h}^{t} |\lambda(s)|ds.$$

Now we note that the condition that $\int_0^t |b(s)|(\int_{s-h}^{s} |\lambda(u)|du)ds$ is uniformly continuous on $\mathbf{R}_+$ implies that $\int_0^t U'_{(1.1)}(s, x(s))_+$ is uniformly continuous on $\mathbf{R}_+$. Therefore, the strong condition that $\int_0^t a(s)ds$ is uniformly continuous on $\mathbf{R}_+$ is redundant under Theorem 2.1. For additional sufficient conditions ensuring the uniform asymptotic stability of the zero solution of (1.1) see [5, Theorem 5.1] and [7, Example 3.4].

## 3. Uniform Boundedness and Uniform Ultimate Boundedness for Functional Differential Equations with Unbounded Delay

It is well known that for a functional differential equation with unbounded delay which is sufficiently well posed in a certain phase space, existence of a periodic solution follows directly if solutions are uniformly bounded and uniformly ultimately bounded with respect to this phase space. Thus the properties of uniform boundedness and uniform ultimate boundedness frequently emerge as central. The purpose of this section is to develop conditions ensuring such boundedness properties for solutions of unbounded delay functional differential equations.

In this section we consider a system of the functional differential equations with unbounded delay

$$x'(t) = F(t, x(s); \alpha \leq s \leq t), \qquad t > 0 \tag{2}$$

where $-\infty \leq \alpha \leq 0$ (note that $\alpha$ could be $-\infty$), $F$ is a function determined by $t$ and the value of $x(s)$ for $\alpha \leq s \leq t$ and taking values in $\mathbf{R}^n$.

For a $t_0 \geq 0$ and a bounded continuous function $\phi : [\alpha, t_0] \to \mathbf{R}^n$ (denote $[\alpha, t_0]$ by $(-\infty, t_0]$ if $\alpha = -\infty$), the solution of (2), denoted by $x(t, t_0, \phi)$, is a continuous function satisfying (2) on an interval $[t_0, t_0 + \beta)$ for some $\beta > 0$ with $x(t, t_0, \phi) = \phi(t)$ for all $\alpha \leq t \leq t_0$. We assume that $F$ satisfies appropriate conditions to guarantee the existence and uniqueness of solutions (cf. Burton[1] or Driver[4]).

Let $C(t)$ denote the function space with $\phi \in C(t)$ if $\phi : [\alpha, t] \to \mathbf{R}^n$ is continuous and bounded. For $\phi \in C(t)$ we define its norm as follows:

$$\|\phi\| = \|\phi\|^{[\alpha, t]} = \sup_{\alpha \leq \theta \leq t} |\phi(\theta)|$$

Further, for any $H > 0$ and $t_0 \geq 0$, let $C_H(t_0)$ be the subset of $C(t_0)$ such that $\phi \in C(t_0)$ and $\|\phi\| = \|\phi\|^{[\alpha, t_0]} < H$.

A Liapunov functional is a continuous function $V(t, x(\cdot)) : \mathbf{R}_+ \times C(t) \to \mathbf{R}_+$ whose derivative along a solution $x(t)$ of (2) satisfies some specific relation. The derivative of a Liapunov functional $V(t, x(\cdot))$ along a solution $x(t)$ of (2) may be defined in several equivalent ways. If $V$ is differentiable, the natural derivative is obtained using the chain rule. Then $V'_{(2)}(t, x(\cdot))$ denotes the derivative of functional $V$ with respect to (2) defined by

$$V'_{(2)}(t, x(\cdot)) = \limsup_{\delta \to 0^+} \{V(t + \delta, x(\cdot)) - V(t, x(\cdot))\}/\delta.$$

DEFINITION 3.1. The solutions of (2) are *uniformly bounded* (U.B.) if for each $B_1 > 0$ there is a $B_2 > 0$ such that $[t_0 \in \mathbf{R}_+, \phi \in C_{B_1}(t_0), t \geq t_0]$ imply that $|x(t, t_0, \phi)| < B_2$.

DEFINITION 3.2. The solutions of (2) are *uniformly ultimately bounded* (U.U.B.) with bound $B$ if for each $B_3 > 0$ there is a $T = T(B_3) > 0$ such that $[t_0 \in \mathbf{R}_+, \phi \in C_{B_3}(t_0), t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < B$.

THEOREM 3.1. *Let $M > 0$, and let $V(t, \phi(\cdot)) : \mathbf{R}_+ \times C(t) \to [0, \infty)$ be continuous and locally Lipschitz in $\phi$. Suppose that there exist wedges $W_1, W_2, W_3, W_4, W_5$ and continuous functionals $D(t, \phi(\cdot)), E(t, \phi(\cdot)) : \mathbf{R}_+ \times C(t) \to [0, \infty)$ such that*

*(i)* $W_1(|\phi(t)|) \leq V(t, \phi(\cdot)) \leq W_2(D(t, \phi(\cdot)))$,

*(ii)* $V'_{(2)}(t, x(\cdot)) \leq -W_3(E(t, x(\cdot))) + M$,

*(iii)* $D(t, \phi(\cdot)) \leq W_4(\|\phi\|)$, *and*

*(iv)* $D(t, x(\cdot)) \leq W_5(E(t, x(\cdot)))$

*for any $t \in \mathbf{R}_+$ and any solution $x(t)$ of (2). Then solutions of (2) are U.B. and U.U.B.*

*Proof.* Let $B_1 > 0$, $\phi \in C(t_0)$ with $\|\phi\| < B_1$ be given and let $x(t) = x(t, t_0, \phi)$ be a solution of (2). Integrate

$$V'_{(2)}(t, x(\cdot)) \leq -W_3(E(t, x(\cdot))) + M$$

from $t_1$ to $t_2$ with $t_1 \leq t_2$. Then we have

$$\begin{aligned} 0 &\leq \int_{t_1}^{t_2} W_3(E(s, x(\cdot)))\, ds \\ &\leq V(t_1, x(\cdot)) - V(t_2, x(\cdot)) + M(t_2 - t_1). \end{aligned}$$

Now, consider $v(s) = V(s, x(\cdot))$ on any interval $[t_0, L]$ for any $L \geq t_0 + M$. Since $v(s)$ is continuous, it has a maximum at some $\bar{t} \in [t_0, L]$. Suppose $\bar{t} \leq t_0 + M$; then

$$v(t) \leq v(\bar{t}) \leq v(t_0) + M(\bar{t} - t_0) \leq W_2 \circ W_4(B_1) + M^2$$

for any $t \in [t_0, L]$, and thus

$$|x(t)| \leq W_1^{-1}\{W_2 \circ W_4(B_1) + M^2\} \equiv \beta_1$$

for any $t \in [t_0, L]$. If $\bar{t} \in [t_0 + M, L]$, then $v'_{(2)}(\bar{t}) \geq 0$ and hence $E(\bar{t}, x(\cdot)) \leq W_3^{-1}(M)$. Thus we have

$$\begin{aligned} W_1(|x(t)|) &\leq v(t) \leq v(\bar{t}) \leq W_2(D(\bar{t}, x(\cdot))) \\ &\leq W_2 \circ W_5(E(\bar{t}, x(\cdot))) \leq W_2 \circ W_5 \circ W_3^{-1}(M) \end{aligned}$$

and

$$|x(t)| \leq W_1^{-1} \circ W_2 \circ W_5 \circ W_3^{-1}(M) \equiv \beta_2$$

for any $t \in [t_0, L]$. Therefore,

$$|x(t)| \leq \max\{\beta_1, \beta_2\} \equiv \beta$$

for any $t \in [t_0, L]$. Since $L$ is arbitrary, $|x(t)| \leq \beta \equiv \beta(B_1)$ for any $t \geq t_0$. Thus solutions of (2) are uniformly bounded.

For the U.U.B., let $\eta > 0$ be given and find $\theta = \theta(\eta) > 0$ with $\theta \geq \eta$ such that $[t_0 \geq 0, \|\phi\| < \eta]$ imply that $|x(t, t_0, \phi)| < \theta$ by U.B. Let $U = W_3^{-1}(2M)$. Then, if

$$E(t, x(\cdot)) \geq W_3^{-1}(2M),$$

we have

$$V'_{(2)}(t, x(\cdot)) \leq -W_3(E(t, x(\cdot))) + M \leq -2M + M < 0.$$

Now we note that

$$0 \leq V(t, x(\cdot)) \leq W_2(D(t, x(\cdot))) \leq W_2(W_4(\|x\|^{[\alpha, t]})) \leq (W_2 \circ W_4)(\theta)$$

for any $t \geq t_0$. That is, $V(t, x(\cdot))$ is nonnegative and bounded on $\mathbf{R}_+ \times C(t)$. Now, we can choose a sufficiently large integer $N = N(\eta)$ such that for any interval $[t, t + NM]$ with $t \geq t_0$, there is some $\bar{t} \in (t, t + MN)$ with $E(\bar{t}, x(\cdot)) < U$. Consider the intervals

$$I_1 = [t_0, t_0+MN], \quad I_2 = [t_0+MN, t_0+2NM], \ldots I_i = [t_0+(i-1)NM,$$

$t_0 + iNM], \ldots$ and select $t_i \in I_i$ such that $v(t_i)$ is the maximum on $I_i$. In case $t_i = t_0 + (i-1)NM$ with $E(t_i, x(\cdot)) > U$ then by choice of $N$, there is a first $\overline{t_i} \in [t_0 + (i-1)NM, t_0 + iNM]$ such that

$$E(\overline{t_i}, x(\cdot)) = U.$$

Now, instead of the above choice for $I_i$, in this case we pick

$$I_i = [\overline{t_i}, t_0 + iNM]$$

and let

$$v(t_i) = \max_{s \in I_i} v(s).$$

Therefore, in any case we have

$$E(t_i, x(\cdot)) \leq U, \quad i = 1, 2, 3, \ldots$$

Now, consider the intervals

$$L_2 = [t_2 - M, t_2], \quad L_3 = [t_3 - M, t_3], \ldots L_i = [t_i - M, t_i], \ldots$$

For each $i = 2, 3, 4, \ldots$ we have two cases.
Case 1) $v(t_i) + 1 \geq v(s)$ for all $s \in L_i$.
Case 2) $v(t_i) + 1 < v(s_i)$ for some $s_i \in L_i$.

Note that in case 2, $s_i \notin I_i$ since $v(t_i)$ is the maximum on $I_i$. If there is no gap between $I_{i-1}$ and $I_i$, then $s_i \in I_{i-1}$. If there is a gap and $s_i \in [t_0 + (i-1)NM, \overline{t_i}]$, then we have $E(t, x(\cdot)) \geq U$ and thus $v'_{(2)}(t) \leq 0$ on $[t_0 + (i-1)NM, \overline{t_i}]$. Hence

$$v(t_{i-1}) \geq v(t_0 + (i-1)NM) \geq v(s_i) > v(t_i) + 1.$$

In either case we have

$$v(t_i) + 1 < v(t_{i-1})$$

since $v(t_{i-1})$ is the maximum on $I_{i-1}$. By the boundedness of $v(t)$, there is an integer $N^* > 0$ such that case 2 holds on no more than $N^* = N^*(\eta)$ consecutive intervals $L_i$. Thus, on some $L_j$ with $j \leq N^* = N^*(\eta)$ we have

$$v(t_j) + 1 \geq v(s) \quad \text{for all} \quad s \in L_j = [t_j - M, t_j].$$

Now we note that

$$v(t_j) \leq W_2(D(t_j, x(\cdot))) \leq W_2 \circ W_5(U)$$

and

$$v(t) \leq v(t_j) + 1 \leq W_2 \circ W_5(U) + 1 \quad \text{for all} \quad t \in L_j.$$

Thus we try to show that

$$v(t) \leq (W_2 \circ W_5)(U) + 1 \equiv D \quad \text{for any} \quad t \geq t_j.$$

To see this, let $t_p$ be the first $t > t_j$ with $v(t_p) = D$. Then

$$v'_{(2)}(t_p) \geq 0$$

and

$$v(t_p) \leq W_2(D(t_p, x(\cdot))) \leq W_2 \circ W_5(E(t_p, x(\cdot))) < W_2 \circ W_5(U),$$

which is a contradiction. Hence, for $t \geq t_0 + N^*NM$, we have

$$W_1(|x(t)|) \leq v(t) \leq D$$

and

$$|x(t)| \leq W_1^{-1}(D).$$

Thus solutions of (2) are uniformly ultimately bounded. Hence we complete the proof.

EXAMPLE 3.1. Consider the following scalar differential equation

$$x'(t) = -a(t)x(t) + \alpha \int_0^t c(t-s)x(s)\,ds + f(t), \quad t \geq 0, \tag{2.1}$$

where $a, c, f : [0,\infty) \to (-\infty, \infty)$ are continuous, $a(t) \geq 0$ for $t \geq 0$, $\alpha > 0$ is a constant. Suppose
(i) $\int_0^\infty |c(v)|\,dv \leq c^* < \infty$ for some $c^* > 0$ ; $\int_t^\infty |c(v)|\,dv \leq K|c(t)|$ for some $K > 0$, and $\int_t^\infty |c(v)|\,dv \in L^1[0,\infty)$,
(ii) there exist some constants $\beta > 0$ and $\gamma > 0$ such that $a(t) - \beta c^* \geq \gamma$, $\alpha - \beta + \beta K(a(t) - \beta c^*) \leq 0$, $|f(t)| \leq M < \infty$ for some $M > 0$. Then the solutions of (2.1) are U.B. and U.U.B.

*Proof.* We consider the following Liapunov functional

$$\begin{aligned} V(t,\phi(\cdot)) &= |\phi(t)| + \beta \int_0^t \int_t^\infty |c(u-s)|\,du|\phi(s)|\,ds \\ &= D(t,\phi(\cdot)) \\ &= E(t,\phi(\cdot)) \end{aligned}$$

Then we have

$$\begin{aligned} D(t,\phi(\cdot)) &= E(t,\phi(\cdot)) \\ &\leq ||\phi|| + \beta||\phi|| \int_0^t \int_t^\infty |c(u-s)|\,du\,ds \\ &\leq ||\phi|| + \beta L||\phi|| \end{aligned}$$

for some $L$ with $0 < L < \infty$ and

$$\begin{aligned} V'_{(2.1)}(t,x(\cdot)) \leq & -a(t)|x(t)| + \alpha \int_0^t |c(t-s)|\,|x(s)|\,ds + |f(t)| \\ & + \beta \int_t^\infty |c(u-t)|\,du\,|x(t)| - \beta \int_0^t |c(t-s)|\,|x(s)|\,ds \end{aligned}$$

$$\begin{aligned}
&\leq (-a(t)+\beta c^*)|x(t)| + (\alpha-\beta)\int_0^t |c(t-s)|\,|x(s)|\,ds + |f(t)| \\
&= (-a(t)+\beta c^*)\{E(t,x(\cdot)) - \beta\int_0^t\int_t^\infty |c(u-s)|\,du\,|x(s)|\,ds\} \\
&\quad + (\alpha-\beta)\int_0^t |c(t-s)|\,|x(s)|\,ds + |f(t)| \\
&\leq (-a(t)+\beta c^*)E(t,x(\cdot)) + \beta K(a(t)-\beta c^*)\int_0^t |c(t-s)|\,|x(s)|\,ds \\
&\quad + (\alpha-\beta)\int_0^t |c(t-s)|\,|x(s)|\,ds + |f(t)| \\
&\leq (-a(t)+\beta c^*)E(t,x(\cdot)) + |f(t)| \\
&\leq -\gamma E(t,x(\cdot)) + M.
\end{aligned}$$

That is, all conditions in Theorem 3.1 are satisfied. Therefore, the solutions of (2.1) are U.B. and U.U.B.

REMARK 3.1. We could specify $a(t)$, $c(t)$, $f(t)$ and $\alpha$ to satisfy the assumptions in the previous example. For instance, if $a(t) \geq 2$ and $f(t)$ is a bounded continuous for on $\mathbf{R}_+$ with $c(t) = e^{-2t}$ and $\alpha = 1$, then

(i) $\int_0^\infty |c(v)\,dv = \int_0^\infty e^{-2v}\,dv = \frac{1}{2}$, $\int_t^\infty |c(v)|\,dv = \frac{1}{2}e^{-2t} \leq \frac{1}{2}|c(t)|$ and thus $c^* = \frac{1}{2}$, $K = \frac{1}{2}$, more over, $\displaystyle\int_0^\infty |c(v)|\,dv\,ds = \int_0^\infty \frac{1}{2}e^{-2s}\,ds = \frac{1}{4} < \infty$,

(ii) there exists $\beta = 3$ such that

$$a(t) - \beta c^* = a(t) - \frac{3}{2} \geq \frac{1}{2}$$

and

$$\alpha - \beta + \beta K(a(t) - \beta c^*) \leq |-3 + \frac{3}{2}(\frac{1}{2})| = -2 + \frac{3}{4} < 0.$$

Therefore, the solutions of (2.1) are U.B. and U.U.B.

REMARK 3.2. Zhang has obtained some result about the uniform boundedness and the uniform ultimate boundedness for a functional differential equations with unbounded delay [10, Theorem 1] and applied his result for the equation (2.1) to study such boundedness properties of solutions of (2.1). In his application the function $f(t)$ in (2.1) has a strong restriction as follows :

$$\int_0^\infty |f(s)|(\exp \int_0^s \eta(u)du)ds \leq M < \infty$$

for some measurable function $\eta(t) \geq 0$ on $[0, \infty)$.

Also our new independent result [Theorem 3.1] can be applied to the equation (2.1) to study such boundedness properties of solutions of (2.1). In our application the function $f(t)$ in (2.1) may have a flexible restriction as follows:

$$|f(t)| \leq M < \infty \qquad \text{for all} \quad t \in [0, \infty).$$

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