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CLIFFORD L^2-COHOMOLOGY ON THE COMPLETE KÄHLER MANIFOLDS

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0. Introduction

In the study of a manifold M, the exterior algebra Λ^*M plays an important role. In fact, the de Rham cohomology theory gives many informations of a manifold. Another important object in the study of a manifold is its Clifford algebra $Cl(M)$, generated by the tangent space. It carries an intrinsic first order elliptic operator D, the Dirac operator. There is a canonical vector (but not algebra) bundle isomorphism $\Lambda^*(M) \to Cl(M)$. In $\Lambda^*(M)$, the Dirac operator D is $D \cong d+\delta$, where d is the exterior differential and δ is the adjoint operator of d. Therefore many results of the Clifford theory yield the results of the de Rham theory([8]). Moreover the calculus of the pair $Cl(M)$, D carries over formally to bundles of modules over $Cl(M)$. On Kähler manifolds, we obtain operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ such that $\mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0$, $\mathcal{D}+\bar{\mathcal{D}} = \frac{1}{2}D$ and $\bar{\mathcal{D}}$ is the formal adjoint of $\mathcal{D}$. Using these operators, M. L. Michelsohn([10]) studied the Clifford and spinor cohomology theory and proved some vanishing theorems on compact Kähler manifold. In this paper, we study the Clifford L^2-cohomology theory, the decomposition theorem for the L^2-Clifford algebra $L^2(\mathbb{C}l^{p,q}(M))$ and prove some vanishing theorems on complete Kähler manifold.

1. Preliminaries

Let M be a $2n$-dimensional Kähler manifold with almost complex

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structure J and with connection ∇. Let $Cl(M)$ be the Clifford bundle generated by the tangent bundle TM. Now we define a derivation $\mathcal{J}_0 : Cl(M) \to Cl(M)$ induced by J as follows:

$$\mathcal{J}_0(v_1 \cdots v_k) = \sum_{j=1}^{k} v_1 \cdots Jv_j \cdots v_k \tag{1.1}$$

for $v_1, \cdots, v_k \in TM$, where " $\cdot$ " is the Clifford multiplication. If it is clear from the context which multiplication is meant,we omit the Clifford multiplication " $\cdot$ ". To study $\mathcal{J}_0$ effectively we consider the complexification $\mathbb{C}l(M) = Cl(M) \otimes_{\mathbb{R}} \mathbb{C}$. This algebra has a natural basis given as follows: Let $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ be an orthonormal basis of T_xM. Let $T_x^{1,0}$(resp. $T_x^{0,1}$) be the i eigenspace (resp. $-i$ eigenspace) of J in $T_xM \otimes \mathbb{C}$. Put

$$\xi_k = \frac{1}{2}\{e_k - iJe_k\}, \quad \bar{\xi}_k = \frac{1}{2}\{e_k + iJe_k\}.$$

Then $\xi_1, \cdots, \xi_n$(resp. $\bar{\xi}_1, \cdots, \bar{\xi}_n$) is the basis of $T_x^{1,0}$(resp. $T_x^{0,1}$). And $\{\xi_k, \bar{\xi}_k\}$ has the following properties;

(1.2)
$$\xi_k\bar{\xi}_\ell + \bar{\xi}_k\xi_\ell = \xi_k\bar{\xi}_\ell + \bar{\xi}_\ell\xi_k = -\delta_{k\ell}, \quad \xi_k\xi_\ell = -\xi_\ell\xi_k, \quad \bar{\xi}_k\bar{\xi}_\ell = -\bar{\xi}_\ell\bar{\xi}_k.$$

Denote $\xi_K\bar{\xi}_I = \xi_{k_1} \cdots \xi_{k_r}\bar{\xi}_{i_1} \cdots \bar{\xi}_{i_s}$, where K and I range over all strictly ascending multiindices from $\{1, \cdots, n\}$. For convenience we set $\mathcal{J} = \frac{1}{i}\mathcal{J}_0$. Then by the derivation property, we have

$$\mathcal{J}(\xi_K\bar{\xi}_I) = (|K| - |I|)\xi_K\bar{\xi}_I, \tag{1.3}$$

where $|K|, |I|$ denote the lengths of K and I. This gives a decomposition

$$\mathbb{C}l(M) = \bigoplus_{p=-n}^{n} \mathbb{C}l^p(M),$$

where $\mathbb{C}l^p(M) = \{\phi \in \mathbb{C}l(M) \mid \mathcal{J}\phi = p\phi\}$.

We now introduce two intrinsically defined linear maps $\mathcal{L}, \bar{\mathcal{L}} : \mathbb{C}l(M) \to \mathbb{C}l(M)$ as follows; For any $\varphi \in \mathbb{C}l(M)$, set

$$\mathcal{L}(\varphi) = -\sum_{k=1}^{n} \xi_k\varphi\bar{\xi}_k, \quad \bar{\mathcal{L}}(\varphi) = -\sum_{k=1}^{n} \bar{\xi}_k\varphi\xi_k. \tag{1.4}$$

These operators are independent of the Hermitian basis chosen to define them. We consider the operator $\mathcal{H} = [\mathcal{L}, \bar{\mathcal{L}}]$. Then they satisfy the following relations;

$$[\mathcal{L}, \bar{\mathcal{L}}] = \mathcal{H}, \quad [\mathcal{H}, \mathcal{L}] = 2\mathcal{L}, \quad [\mathcal{H}, \bar{\mathcal{L}}] = -2\bar{\mathcal{L}}. \tag{1.5}$$

Hence they define a representation of $s\ell(2, \mathbb{C})$, the Lie algebra of $SL(2, \mathbb{C})$, on $\mathbb{C}l(M)$. Since each of the operators $\mathcal{L}, \bar{\mathcal{L}}$ and $\mathcal{H}$ commutes with $\mathcal{J}$, we can define the subspaces

$$\mathbb{C}l^{p,q}(M) = \{\varphi \in \mathbb{C}l(M) \mid \mathcal{H}\varphi = q\varphi,\ \mathcal{J} = p\varphi\}$$

and obtain a decomposition([10])

$$\mathbb{C}l(M) = \bigoplus_{p,q} \mathbb{C}l^{p,q}(M). \tag{1.6}$$

Proposition 1.1([10]). *For each $\xi \in T^{1,0}(M)$, one has that $\xi \cdot \mathbb{C}l^{p,q} \subseteq \mathbb{C}l^{p+1,q+1}$ and $\bar{\xi} \cdot \mathbb{C}l^{p,q} \subseteq \mathbb{C}l^{p-1,q-1}$. Furthermore, if $\xi \neq 0$, the sequences*

$$\cdots \xrightarrow{\lambda_\xi} \mathbb{C}l^{p-1,q-1} \xrightarrow{\lambda_\xi} \mathbb{C}l^{p,q} \xrightarrow{\lambda_\xi} \mathbb{C}l^{p+1,q+1} \longrightarrow \cdots$$
$$\cdots \xleftarrow{\lambda_{\bar{\xi}}} \mathbb{C}l^{p-1,q-1} \xleftarrow{\lambda_{\bar{\xi}}} \mathbb{C}l^{p,q} \xleftarrow{\lambda_{\bar{\xi}}} \mathbb{C}l^{p+1,q+1} \longleftarrow \cdots,$$

where λ_ξ denotes left Clifford multiplication by ξ, are exact.

Moreover, these subspaces $\mathbb{C}l^{p,q}$ have the following properties: If $q - s \neq p + r$, then $\mathbb{C}l^{p,q} \cdot \mathbb{C}l^{r,s} = \{0\}$. Otherwise, $\mathbb{C}l^{p,q} \cdot \mathbb{C}l^{r,q-p-r} \subseteq \mathbb{C}l^{p+r,q-r}$. In particular, $\mathbb{C}l^{p,q} \cdot \mathbb{C}l^{p,q} \subseteq \mathbb{C}l^{p,q}$, $\mathbb{C}l^{k,k} \cdot \mathbb{C}l^{\ell,-\ell} \subseteq \mathbb{C}l^{k+\ell,k-\ell}$ and $\mathbb{C}l^{k,\ell} \cdot \mathbb{C}l^{-k,\ell} \subseteq \mathbb{C}l^{0,\ell+k}$.

2. Clifford coholomogy group

We recall some facts from [3]: Consider Hilbert spaces H_i $(0 \leq i \leq N)$, $H_{N+1} := 0$ and closed operators $D_i : H_i \to H_{i+1}$, with D_i^*, the adjoint operator. Let $domD_i$ be the domain of D_i and $ranD_i$ the range of D_i. We then assume that

$$ranD_i \subset domD_{i+1} \quad \text{and} \quad D_{i+1} \circ D_i = 0.$$

Thus we obtain a complex

$$(2.1)\qquad 0 \longrightarrow domD_0 \xrightarrow{D_0} domD_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} domD_N \longrightarrow 0$$

in the sense of homological algebra with additional functional analytic structure, which is called a *Hilbert complex*. We will abbreviate the complex (2.1) as $(domD, D)$.

LEMMA 2.1([3])(THE WEAK HODGE DECOMPOSITION). *Let* $(domD, D)$ *be a Hilbert complex. Then for each* i, *we have an orthogonal decomposition*

$$(2.2)\qquad H_i = \hat{\mathcal{H}}_i \oplus \overline{imD_{i-1}} \oplus \overline{imD_i^*}$$

where $\hat{\mathcal{H}}_i := KerD_i \cap KerD_{i-1}^*$.

Put $\Delta_i := D_iD_i^* + D_i^*D_i$. Then we have

LEMMA 2.2([3]). $\hat{\mathcal{H}}_i = Ker\Delta_i$.

Now, let $E_i \to M$ $(0 \le i \le N)$ be hermitian vector bundles over a Riemannian manifold M and $d_i := \Gamma_{cpt}(E_i) \to \Gamma_{cpt}(E_{i+1})$ differential operators such that $d_i \circ d_{i-1} = 0$. Denote the formal adjoint d_i^t by d_i^t. Then d_i has a closed extension $d_{i,\max}$ in the Hilbert space $H_i := L^2(E_i)$ given by

$$d_{i,\max} := (d_{i,\min}^t)^*,$$

where $d_{i,\min}$ is the minimal extension or the closure of d_i. Then we have

LEMMA 2.3([3]). *If* $(\Gamma_{cpt}(E_i), d_i)$ *is an elliptic complex, then*

$$\cdots \longrightarrow domd_{i-1,\max} \xrightarrow{d_{i-1,\max}} domd_{i,\max} \xrightarrow{d_{i,\max}} domd_{i+1,\max} \longrightarrow \cdots$$

is a Hilbert complex.

Suppose now that M is a complete Kähler manifold. We introduce two differential operators $\mathcal{D}, \bar{\mathcal{D}} : \Gamma\mathbb{C}l(M) \to \Gamma\mathbb{C}l(M)$ by the formulas

$$(2.3)\qquad \mathcal{D} = \sum_j \xi_j \nabla_{\bar{\xi}_j}, \quad \bar{\mathcal{D}} = \sum_j \bar{\xi}_j \nabla_{\xi_j},$$

where ∇ is the cannonical connection. Since ∇ preserves the subbundles $\Gamma\mathbb{C}l^{p,q}(M)$, we have

$$\mathcal{D}(\Gamma\mathbb{C}l^{p,q}) \subset \Gamma\mathbb{C}l^{p+1,q+1}, \quad \bar{\mathcal{D}}(\Gamma\mathbb{C}l^{p,q}) \subset \Gamma\mathbb{C}l^{p-1,q-1}$$

for all p and q. Then we have the following well known fact:

THEOREM 2.4([10]). *The operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoints of one another on $\Gamma_{cpt}\mathbb{C}l(M)$, the set of all sections with the compact support. And they satisfy*

$$\mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0.$$

Furthermore, the complex

$$\cdots \xrightarrow{\mathcal{D}} \Gamma\mathbb{C}^{p-1,q-1} \xrightarrow{\mathcal{D}} \Gamma\mathbb{C}^{p,q} \xrightarrow{\mathcal{D}} \Gamma\mathbb{C}^{p+1,q+1} \xrightarrow{\mathcal{D}} \cdots$$

is elliptic.

Now we set

$$\Delta := \mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}. \tag{2.4}$$

Then Δ is a formally self-adjoint elliptic operator. To understand Δ we introduce two "real" operators on $\mathbb{C}l(M)$:
(2.6)

$$D = \sum_j \{e_j \nabla_{e_j} + (Je_j)\nabla_{Je_j}\}, \quad D^c = \sum_j \{e_j \nabla_{Je_j} - (Je_j)\nabla_{e_j}\}.$$

The first operator is called the *Dirac operator*. Then we can easily see that

$$\mathcal{D} = \frac{1}{4}(D + iD^c), \quad \bar{\mathcal{D}} = \frac{1}{4}(D - iD^c). \tag{2.7}$$

Since $\mathcal{D}^2 = 0$, we have that $D^2 = (D^c)^2$ and $DD^c + D^cD = 0$. It follows that

$$\Delta = \frac{1}{4}D^2. \tag{2.7}$$

Since D is essentially self-adjoint, we have

$$KerD = KerD^2 = Ker\Delta. \tag{2.8}$$

Now, we consider the usual inner product

$$\ll \varphi_1, \varphi_2 \gg = \int_M < \varphi_1, \varphi_2 > \tag{2.9}$$

for any $\varphi_1, \varphi_2 \in \Gamma_{cpt}\mathbb{C}l(M)$. Let $L^2(\mathbb{C}l^{p,q}(M))$ be the completion of $\Gamma_{cpt}\mathbb{C}l^{p,q}$ with respect to $\ll\, ,\, \gg$. We recall that the operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoint to one another with respect to $\ll\, ,\, \gg$. Then $\mathcal{D}$ and $\bar{\mathcal{D}}$ have closed extensions in $L^2(\mathbb{C}l^{p,q}(M))$ defined by

$$\mathcal{D}_{\max} := (\bar{\mathcal{D}}_{\min})^*, \quad \bar{\mathcal{D}}_{\max} := (\mathcal{D}_{\min})^*, \tag{2.10}$$

where $\bar{\mathcal{D}}_{\min}$(resp. $\mathcal{D}_{\min}$) is a minimal extension of $\bar{\mathcal{D}}$(resp. $\mathcal{D}$) and $(\ )^*$ is the adjoint operator of $(\ )$ with respect to $\ll\, ,\, \gg$. Since Δ and D are essentially self-adjoints, we have $\mathcal{D}_{\max} = \mathcal{D}_{\min}$ and $\bar{\mathcal{D}}_{\max} = \bar{\mathcal{D}}_{\min}$([3]). And hence we denote the closed extensions as the same symbols. Consequently, from Lemma 2.3 and Theorem 2.4, we obtain the Hilbert complexes

(2.11)

$$\cdots \xrightarrow{\mathcal{D}} L^2(\mathbb{C}l^{p-1,q-1}(M)) \xrightarrow{\mathcal{D}} L^2(\mathbb{C}l^{p,q}(M)) \xrightarrow{\mathcal{D}} L^2(\mathbb{C}l^{p+1,q+1}(M)) \xrightarrow{\mathcal{D}} \cdots,$$

$$\cdots \xleftarrow{\bar{\mathcal{D}}} L^2(\mathbb{C}l^{p-1,q-1}(M)) \xleftarrow{\bar{\mathcal{D}}} L^2(\mathbb{C}l^{p,q}(M)) \xleftarrow{\bar{\mathcal{D}}} L^2(\mathbb{C}l^{p+1,q+1}(M)) \xleftarrow{\bar{\mathcal{D}}} \cdots.$$

Now, we put

$$L^2\mathcal{H}^{p,q} := Ker\mathcal{D}/\overline{Im\mathcal{D}} \cap L^2(\mathbb{C}l^{p,q}(M)), \tag{2.12}$$

$$L^2\hat{\mathcal{H}}^{p,q} := Ker\mathcal{D} \cap Ker\bar{\mathcal{D}} \cap L^2(\mathbb{C}l^{p,q}(M)), \tag{2.13}$$

$$L^2H^{p,q} := Ker\Delta \cap L^2(\mathbb{C}l^{p,q}(M)). \tag{2.14}$$

Here $L^2\mathcal{H}^{p,q}$ and $L^2H^{p,q}$ are called the *Clifford L^2-cohomology group* and *L^2-harmonic space*, respectively. Then we have

COROLLARY 2.5. *Let M be a complete Kähler manifold. Then we have*

$$L^2(\mathbb{C}l^{p,q}(M)) = L^2\hat{\mathcal{H}}^{p,q} \oplus \overline{Im\mathcal{D}} \oplus \overline{Im\bar{\mathcal{D}}},$$

and

$$L^2\mathcal{H}^{p,q} \cong L^2\hat{\mathcal{H}}^{p,q} \cong L^2H^{p,q}.$$

Proof. The first follows from Lemma 2.3 and Lemma 2.4. The second is obvious from [3, Lemma 3.2].

REMARK ([10]). We study the relationship between Dolbeault cohomology and Clifford cohomology. First, we prepare the some facts: Let $\Lambda^{r,s}(M)$ be the standard Dolbeault decomposition of $\Lambda^*(M) \otimes \mathbb{C}$. Then there are operators

$$\partial : \Gamma\Lambda^{r,s} \longrightarrow \Gamma\Lambda^{r+1,s}, \quad \bar{\partial} : \Gamma\Lambda^{r,s} \longrightarrow \Gamma\Lambda^{r,s+1}$$

given by the formulas;

$$\partial = \sum_j \bar{\xi}_j \wedge \nabla_{\xi_j}, \quad \bar{\partial} = \sum_j \xi_j \wedge \nabla_{\bar{\xi}_j}, \tag{2.15}$$

where ∇ is the Kähler connection and $\{\xi_j, \bar{\xi}_j\}$ is as before. The formal adjoints of ∂ and $\bar{\partial}$ are given respectively by

$$\partial^* = -\sum_j i(\xi_j)\nabla_{\bar{\xi}_j}, \quad \bar{\partial}^* = -\sum_j i(\bar{\xi}_j)\nabla_{\xi_j}, \tag{2.16}$$

where $i(\cdot)$ denotes the interior product. It is well known that under the isomorphism $\mathbb{C}l(M) \cong \Lambda^*(M)\otimes\mathbb{C}$, we have $\mathcal{D} \cong \bar{\partial}+\partial^*$ and $\bar{\mathcal{D}} \cong \partial+\bar{\partial}^*$. Note that the (p,q)-decomposition of $\mathbb{C}l(M)$ constructed above does not directly correspond to the Dolbeault decomposition. In fact,

$$\mathbb{C}l^{p,*}(M) \cong \bigoplus_{s-r=p} \Lambda^{r,s}(M), \tag{2.17}$$

where $\mathbb{C}l^{p,*}(M) = \oplus_q \mathbb{C}l^{p,q}(M)$. Moreover,

$$H^{s-r,n-r-s}(M) \cong H^{r,s}_{Dol}(M) \tag{2.18}$$

where $H^{r,s}_{Dol}(M) = H \cap \Lambda^{r,s}(M)$, H is the harmonic space. The relations (2.17) and (2.18) hold for the space of L^2 sections.

3. Vanishing theorems

In this section, we shall prove some vanishing theorems under various curvature conditions. Let M be a Kähler manifold and consider a hermitian vector bundle $S \rightarrow M$ of left modules over $\mathbb{C}l(M)$ with a hermitian metric $< , >$ such that:

(1) Module multiplication by unit tangent vectors is unitary, i.e.,

$$< \xi \cdot \phi, \psi > + < \phi, \bar{\xi} \cdot \psi >= 0, \tag{3.1}$$

for any $\phi, \psi \in \Gamma(S)$ and $\xi \in \Gamma(TM) \otimes \mathbb{C}$

(2) With respect to the canonical hermitian connection, covariant differentiation is a derivation of module multiplication. That is, for $\phi \in \Gamma(\mathbb{C}l(M))$ and $s \in \Gamma(S)$, we have

$$\nabla(\phi \cdot s) = (\nabla \phi) \cdot s + \phi \cdot (\nabla s). \tag{3.2}$$

Now, we recall some basic results from [10]. For each j, we set $\omega_j = -\xi_j \bar{\xi}_j$, $\bar{\omega}_j = -\bar{\xi}_j \xi_j$. To each (possibly empty) subset $I = \{i_1, \cdots, i_p\} \subseteq \{1, \cdots, n\}$ with complementary subset $\{j_1, \cdots, j_{n-p}\}$ we set $\omega_I = \omega_{i_1} \cdots \omega_{i_p} \bar{\omega}_{j_1} \cdots \bar{\omega}_{j_{n-p}}$ and we denote $|I| = p$. Then we have

$$1 = \prod_{j=1}^{n} (\omega_j + \bar{\omega}_j) = \sum_{r=1}^{n} \pi_r, \tag{3.3}$$

where $\pi_r = \sum_{|I|=r} \omega_I$. Moreover, we have an orthogonal decomposition of the bundle

$$S = \bigoplus_{r=0}^{n} S^r, \quad S^r = \pi_r \cdot S. \tag{3.4}$$

Then the complex

$$0 \longrightarrow \Gamma_{cpt}(S^0) \xrightarrow{\mathcal{D}} \Gamma_{cpt}(S^1) \xrightarrow{\mathcal{D}} \cdots \xrightarrow{\mathcal{D}} \Gamma_{cpt}(S^n) \longrightarrow 0 \tag{3.5}$$

is elliptic. By Lemma 2.3, its completion becomes a Hilbert complex. Similarly with Corollary 2.5, we have

$$L^2\mathcal{H}^r(M,S) \cong L^2\hat{\mathcal{H}}^r(M,S) \cong L^2H^r(M,S). \tag{3.6}$$

Now, we define invariant operators on $\Gamma(S)$ by

$$\begin{aligned} \nabla^*\nabla &= -\sum_j \nabla_{\xi_j, \bar{\xi}_j}, & \bar{\nabla}^*\bar{\nabla} &= -\sum_j \nabla_{\bar{\xi}_j, \xi_j}, \\ \mathcal{R} &= \sum_{j,k} \xi_j \bar{\xi}_k R_{\bar{\xi}_j, \xi_k}, & \bar{\mathcal{R}} &= \sum_{j,k} \bar{\xi}_j \xi_k R_{\xi_j, \bar{\xi}_k}, \end{aligned} \tag{3.7}$$

where $R_{V,W} = \nabla_{V,W} - \nabla_{W,V}$ is the curvature tensor and where $\nabla_{V,W} = \nabla_V \nabla_W - \nabla_{\nabla_V W}$ is the invariant second covariant derivative. Then we have the following result([10]):

PROPOSITION 3.1. *For any two sections $s_1,\ s_2 \in \Gamma(S)$, at least one of which has compact support, the following holds:*

$$\int_M < \nabla^*\nabla s_1, s_2 >= \int_M < \nabla s_1, \nabla s_2 >,$$

where $< \nabla s_1, \nabla s_2 >=< \nabla_{\bar{\xi}_i} s_1, \nabla_{\bar{\xi}_i} s_2 >$. Hence $\nabla^*\nabla$ is a formally self adjoint, nonnegative operator. Similary, this holds for $\bar{\nabla}^*\bar{\nabla}$. Moreover, the zero order operators $\mathcal{R}$ and $\bar{\mathcal{R}}$ are self-adjoint.

Moreover, by the straight calculation, we obtain the Bochner-Weitzenböck type formula([10]);

$$\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} = \nabla^*\nabla + \mathcal{R} = \bar{\nabla}^*\bar{\nabla} + \bar{\mathcal{R}}. \tag{3.8}$$

From this formula, we obtain the first important consequence

THEOREM 3.2. *For any $s \in dom\mathcal{D}\bar{\mathcal{D}} \cap dom\bar{\mathcal{D}}\mathcal{D}$, we have*

$$\|\mathcal{D}s\|^2 + \|\bar{\mathcal{D}}s\|^2 = \|\nabla s\|^2 + \ll \mathcal{R}s, s \gg = \|\bar{\nabla} s\|^2 + \ll \bar{\mathcal{R}}s, s \gg, \tag{3.9}$$

where $\|\nabla s\|^2 = \ll \nabla_{\bar{\xi}_j} s, \nabla_{\bar{\xi}_j} s \gg$ and $\|\bar{\nabla} s\|^2 = \ll \nabla_{\xi_j} s, \nabla_{\xi_j} s \gg$.

Proof. First we consider a function ω_ℓ such that $0 \leq \omega_\ell(x) \leq 1$ for any $x \in M$, $supp\ \omega_\ell \subset B(x_0, 2\ell)$, $\omega_\ell(x) = 1$ for any $x \in B(x_0, \ell)$, $\lim_{\ell\to\infty} \omega_\ell = 1$ and $|d\omega_\ell| \leq C/\ell$ almost everywhere on M, where C is a positive constant independent of $\ell \in \mathbb{R}_+$, $x_0 \in M$ and $B(x_0, r)$ is the Riemannian open ball with radius r and center x_0.

For any $s \in L^2(S)$, we calculate $\ll \mathcal{D}\bar{\mathcal{D}}s + \bar{\mathcal{D}}\mathcal{D}s, \omega_\ell^2 s \gg$ on $B(2\ell)$. We choose $\{\xi_j, \bar{\xi}_j\}$ such that $(\nabla\xi_j)_x = (\nabla\bar{\xi}_j)_x = 0$. By the definition of $\bar{\mathcal{D}}$ and (3.2), we get

$$\ll \mathcal{D}\bar{\mathcal{D}}s, \omega_\ell^2 s \gg = 2 \ll \omega_\ell \bar{\mathcal{D}}s, \bar{\xi}_j(\nabla_{\xi_j}\omega_\ell)s \gg + \|\omega_\ell\bar{\mathcal{D}}s\|^2.$$

Using (3.1) and $\xi\bar{\xi} + \bar{\xi}\xi = -\|\xi\|^2$, we obtain

$$\|\xi \cdot s\|^2 + \|\bar{\xi} \cdot s\|^2 = \|\xi\|^2\|s\|^2.$$

Hence we get $\|\xi \cdot s\| \leq \|\xi\|\|s\|$ for any $\xi \in TM \otimes \mathbb{C}$. Therefore, by this inequality and Schwarz inequality. we have

$$\begin{aligned} | \ll \bar{\mathcal{D}}s, \bar{\xi}_j(\nabla_{\xi_j}\omega_\ell)s \gg | &\leq \|\bar{\mathcal{D}}s\|\|\bar{\xi}_j(\nabla_{\xi_j}\omega_\ell)s\| \leq \|\bar{\mathcal{D}}s\|\|(\nabla_{\xi_j}\omega_\ell)s\| \\ &\leq \|\bar{\mathcal{D}}s\|\|\nabla_{\xi_j}\omega_\ell\|\|s\| \leq \frac{C}{\ell}\|\bar{\mathcal{D}}s\|\|s\|. \end{aligned}$$

Since $\|s\|$ and $\|\bar{\mathcal{D}}s\|$ are finite, letting $\ell \to \infty$, we have $\ll \bar{\mathcal{D}}s, \bar{\xi}_j(\nabla_{\xi_j}\omega_\ell)s \gg \to 0$. This implies that $\ll \mathcal{D}\bar{\mathcal{D}}s, s \gg = \|\bar{\mathcal{D}}s\|^2$. Similarly, we get $\ll \bar{\mathcal{D}}\mathcal{D}s, s \gg = \|\mathcal{D}s\|^2$. On the other hand, by Proposition 3.1 and (3.2), we have

$$\ll \nabla^*\nabla s, \omega_\ell^2 s \gg = 2 \ll \omega_\ell \nabla s, \nabla\omega_\ell \cdot s \gg + \|\omega_\ell \nabla s\|^2.$$

By similar method, we have $| \ll \omega_\ell \nabla s, \nabla\omega_\ell \cdot s \gg | \to 0$ as $\ell \to \infty$. Hence we have $\ll \nabla^*\nabla s, s \gg = \|\nabla s\|^2$. Hence we complete the proof of the first equation of (3.9). For the second part, the proof is similar. □

From Theorem 3.2, we have

$$2(\|\mathcal{D}s\|^2 + \|\bar{\mathcal{D}}s\|^2) = \|\nabla s\|^2 + \|\bar{\nabla}s\|^2 + \ll (\mathcal{R} + \bar{\mathcal{R}})s, s \gg .$$

Hence for any $s \in Ker\mathcal{D} \cap Ker\bar{\mathcal{D}}$, if $R = \mathcal{R} + \bar{\mathcal{R}}$ is non-negative, then we have $\|\nabla s\| = \|\bar{\nabla}s\| = 0$. This implies that s is a parallel section. In addition, if R is positive at some point, then $s = 0$. Hence we have

THEOREM 3.3. *Let M be a complete Kähler manifold and let S be any hermitian vector bundle of modules over $\mathbb{C}l(M)$. If R is non-negative and positive at some point of M, then the Clifford L^2-cohomology group is trivial. This is,*

$$L^2\mathcal{H}^r(M, S) = \{0\}, \quad \textit{for any } r = 0, 1, \cdots, n.$$

Moreover, on $TM \subset \mathbb{C}l(M)$, we have ([8])

$$\mathcal{R} + \bar{\mathcal{R}} = \frac{1}{2}Ric.$$

Thus, from (3.6) and Theorem 3.3, we have

COROLLARY 3.4. *On the complete Kähler manifold, if the Ricci curvature is non-negative and positive at some point, then every L^2-harmonic 1-form is necessary zero.*

Now, we shall consider some special cases of Theorem 3.3. To begin, we suppose that M is a Kähler spin manifold, i.e., we assume that there exists a principal Spin-bundle, $P_{Spin}(M) \to M$, with a $Spin_{2n}$-equivalent map $\tau : P_{Spin}(M) \longrightarrow P_{SO}(M)$, to the bundle of real oriented orthonormal frame on M. The *bundle of spinors*, S, is then defined to be vector bundle associated to the unitary representation Δ of $Spin_{2n}$ given by the unique irreducible complex representation of Cl_{2n}, i.e., $S = P_{Spin} \times_\Delta \mathbb{C}^{2^n}$. This bundle is naturally a bundle of modules over $\mathbb{C}l(M)$ and carries a canonical connection induced from the lift of the riemannian connection on $P_{SO}(M)$. Since M is Kähler, this bundle S is naturally holomorphic and its connection is hermitian. On this bundle S, the curvature tensor R^S is given by

$$R^S_{V,W} = \frac{1}{4} \sum_{\alpha,\beta=1}^{2n} < R_{V,W} X_\alpha, X_\beta > X_\alpha X_\beta, \tag{3.10}$$

where $X_1, \cdots, X_{2n}$ is any real orthonormal basis of the tangent space. Choosing a basis $e_1, \cdots, Je_n$, we can write R^S as

$$R^S_{V,W} = 2 \sum_{j,k=1}^{n} < R_{V,W}\xi_j, \bar{\xi}_k > \bar{\xi}_j \xi_k + \sum_{j=1}^{n} < R_{V,W}\xi_j, \bar{\xi}_j > .$$

Hence we have

$$\begin{aligned} \mathcal{R}^S &= \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k R^S_{\bar{\xi}_j, \xi_k} \\ &= \sum_{i,j,k=1}^{n} < R_{\xi_i, \bar{\xi}_i} \bar{\xi}_j, \xi_k > \xi_j \bar{\xi}_k \\ &= -\frac{1}{2} \sum_{j,k=1}^{n} Ric(\bar{\xi}_j, \xi_k) \xi_j \bar{\xi}_k, \end{aligned} \tag{3.11}$$

where *Ric* is Ricci tensor on M([10]). Since *Ric* is hermitian symmetric, we may choose our basis so that $Ric(\bar{\xi}_j, \xi_k) = 1/2\lambda_j \delta_{jk}$, where $\lambda_j = Ric(e_j, e_j) = Ric(Je_j, Je_j)$, for $j = 1, \cdots, n$, are the eigenvalues. Then we have

$$\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} = \nabla^* \nabla + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \omega_j = \bar{\nabla}^* \bar{\nabla} + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \bar{\omega}_j. \tag{3.12}$$

We note that $\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla} = \frac{1}{2}\tilde{\nabla}^*\tilde{\nabla}$ where

$$\tilde{\nabla}^*\tilde{\nabla} = -\sum_j (\nabla_{e_j,e_j} + \nabla_{Je_j,Je_j}) \tag{3.13}$$

is a self-adjoint, elliptic operator whose kernel is the space of parallel sections. We note that the scalar curvature κ of M is given by

$$\kappa = trace_R(Ric) = 2\sum_j \lambda_j. \tag{3.14}$$

Hence we get

THEOREM 3.5([10]). *On the spinor bundle S, we have*

$$4(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}) = \tilde{\nabla}^*\tilde{\nabla} + \frac{1}{4}\kappa,$$

where κ is the scalar curvature of M.

Summing up Theorem 3.3 and Theorem 3.5, we have

THEOREM 3.6. *Let M be a complete Kähler spin manifold. If $\kappa \geq 0$ for all $x \in M$ and $\kappa > 0$ for some point $x_0 \in M$, then there are no non-trivial L^2-harmonic spinors.*

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