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SYMMETRIC CURRENTS OF A TWO-LAYER FLUID WITH FREE SURFACE OVER AN ELLIPTIC OBSTRUCTION

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1. Introduction

This paper concerns the symmetric wave solutions between two immiscible, inviscid, and incompressible fluids of different but constant densities in the presence of small elliptic obstruction of compact support at the rigid bottom when the effect of gravity is considered (Fig. 1). We assume that the upper boundary is a free surface and the two dimensional obstruction is moving along the lower rigid boundary at a constant speed. By choosing a coordinate system moving with the object, the fluid motion becomes steady. Two critical speeds are obtained, near either one of which an FKdV for steady flow can be derived and has been studied extensively in [1] and [2]. Forbes [3], Belward and Forbes [4], Sha and Vanden-Broeck [5], and Moni and King [6] studied steady flow of a two layer fluid over a bump or a step bounded by a free

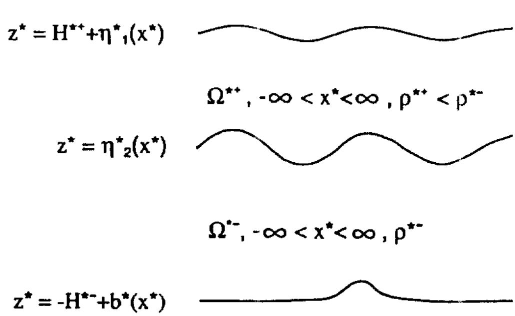

Fig. 1. Fluid Domain

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of rigid boundary numerically. An asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen [7] on the basis of FKdV theory, and with surface tension by Choi *et.al.* [8]. The case of free upper boundary was studied with surface tension by Choi *et.al.* [9] asymptotically on the basis of EKdV theory. In the case considered here, when the wave speed is near the smaller critical speed for internal wave, the nonlinear term in the FKdV may vanish and the derivation of FKdV fails. To overcome this difficulty, a refined asymptotic method is used to derive the Steady Modified K-dV equation with forcing term (SFMKdV) in the following form:

$$(A\eta_2^2 + B)\eta_{2x} + C\eta_{2xxx} + Db_x = 0,$$

where A to D are constants depending on several parameters and $b(x)$ is a function with compact support due to the obstruction on the rigid lower boundary. We investigate solutions of the SFMKdV, which represent possible interfacial wave forms.

In section 2, we formulate the problem and develop the asymptotic scheme to derive the SFMKdV. In section 3, existence theorems are proved and numerical solutions of soliton-like solutions and symmetric wave solutions are presented for different values of parameters. The parameters are determined along the density ratios of the two fluids, depth ratio of the two, and the perturbation of horizontal velocity at far upstream.

2. Formulation and Successive Approximate Equations

We consider steady internal gravity waves between two immiscible, inviscid and incompressible fluids of constant but different densities bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction of compact support. The domains of the upper fluid with a constant density ρ^{*+} and the lower fluid with a constant density ρ^{*-} are denoted by Ω^{*+} and Ω^{*-} respectively (Fig. 1). Assume that the small obstruction is moving with a constant speed C. In reference to a coordinate system moving with the obstruction, the flow is steady and moving with the speed C far upstream. The governing equations and boundary conditions are given by the following Euler equations:

In $\Omega^{*\pm}$,

$$
\begin{aligned}
u^{*\pm}_{x^*} + v^{*\pm}_{y^*} &= 0, \\
u^{*\pm}u^{*\pm}_{x^*} + v^{*\pm}u^{*\pm}_{y^*} &= -p^{*\pm}_{x^*}/\rho^{*\pm}, \\
u^{*\pm}v^{*\pm}_{x^*} + v^{*\pm}v^{*\pm}_{y^*} &= -p^{*\pm}_{y^*}/\rho^{*\pm} - g,
\end{aligned}
$$

at the free surface, $y^* = h^{*+} + \eta_1^*$,

$$
\begin{aligned}
u^{*+}\eta^*_{1x^*} - v^{*+} &= 0, \\
p^{*+} &= 0\,;
\end{aligned}
$$

at the interface, $y^* = \eta_2^*$,

$$
\begin{aligned}
p^{*+} - p^{*-} &= 0, \\
u^{*\pm}\eta^*_{2x^*} - v^{*\pm} &= 0,
\end{aligned}
$$

at the rigid bottom, $y^* = -h^{*-} + b^*(x^*)$,

$$
v^{*-} - b^*_{x^*}u^{*-} = 0,
$$

where $u^{*\pm}$ and $v^{*\pm}$ are horizontal and vertical velocities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant. We define the following nondimensional variables:

$$
\begin{aligned}
&\epsilon = H/L << 1, \quad \eta_1 = \epsilon^{-1}\eta_1^*/h^{*-}, \quad \eta_2 = \epsilon^{-1}\eta_2^*/h^{*-}, \\
&p^{\pm} = p^{*\pm}/(gh^{*-}\rho^{*-}), \quad (x, y) = (\epsilon x^*, y^*)/h^{*-}, \\
&(u^{\pm}, v^{\pm}) = (gh^{*-})^{-1/2}(u^{*\pm}, \epsilon^{-1}v^{*\pm}), \\
&\rho^{+} = \rho^{*+}/\rho^{*-} < 1, \quad \rho^{-} = \rho^{*-}/\rho^{*-} = 1, \quad U = C/(gh^{*-})^{1/2}, \\
&h = h^{*+}/h^{*-}, b(x) = b^*(x^*)(h^{*-}\epsilon^3)^{-1},
\end{aligned}
$$

where L is the horizontal scale, H is the vertical scale, $b(x) = b^*(x)(h^{*-}\epsilon^3)^{-1}$, h^{*+} and h^{*-} are the equilibrium depths of the upper and lower fluids at $x^* = -\infty$ respectively, and $y^* = -h^{*-} + b^*(x)$ is the equation of the obstruction. In terms of the nondimensional quantities, the above equations become in $\Omega^{\pm}$,

(1) $$u_x^{\pm} + v_y^{\pm} = 0,$$

(2) $$u^{\pm}u_x^{\pm} + v^{\pm}u_y^{\pm} = -p_x^{\pm}/\rho^{\pm},$$

(3) $$\epsilon^2 u^{\pm}v_x^{\pm} + \epsilon^2 v^{\pm}v_y^{\pm} = -p_y^{\pm}/\rho^{\pm} - 1,$$

at $y = h + \epsilon\eta_1$,

(4) $$p^+ = 0,$$

(5) $$\epsilon u^+\eta_{1x} - v^+ = 0;$$

at $y = \epsilon\eta_2$,

(6) $$\epsilon u^-\eta_{2x} - v^- = 0,$$

(7) $$\epsilon u^+\eta_{2x} - v^+ = 0,$$

(8) $$p^+ - p^- = 0;$$

at $y = -1 + \epsilon^3 b(x)$,

(9) $$v^- = \epsilon^3 u b_x,$$

where $b(x)$ has a compact support.

In the following, we use a unified asymptotic method to derive the equations for $\eta_1(x)$ and $\eta_2(x)$. We assume that $u^{\pm}, v^{\pm}$, and $p^{\pm}$ are functions of x, y near the equilibrium state $u^{\pm} = u_0$, $v^{\pm} = 0$, $p^+ = -\rho^+ y + \rho^+ h$ and $p^- = -\rho^- y + \rho^+ h$, where u_0 is a constant, and possess asymptotic expansions:

(10)
$$(u^{\pm}, v^{\pm}, p^{\pm}) = (u_0, 0, -\rho^{\pm}y + \rho^+ h) + \epsilon(u_1^{\pm}, v_1^{\pm}, p_1^{\pm}) + \epsilon^2(u_2^{\pm}, v_2^{\pm}, p_2^{\pm}) + \epsilon^3(u_3^{\pm}, v_3^{\pm}, p_3^{\pm}) + O(\epsilon^4).$$

By inserting (10) into (1) to (4) and (7) to (9) and arranging the resulting equations according to the powers of ϵ, it follows that $(u_0, 0, -\rho^{\pm}$

$y + \rho^+ h)$ are the solutions of the zeroth order system of equations and the equations of the order ϵ are as follows:

$$u^\pm_{1x} + v^\pm_{1y} = 0, \tag{11}$$

$$u_0 u^\pm_{1x} = -p^\pm_{1x}/\rho^\pm, \tag{12}$$

$$p^\pm_{1y} = 0; \tag{13}$$

at $y = h$,

$$p^+_1 + \eta_1 p^+_{0y} = 0, \tag{14}$$

at $y = 0$,

$$p^+_1 - p^-_1 + \eta_2(p^+_{0y} - p^-_{0y}) = 0, \tag{15}$$

$$u_0 \eta_{2x} - v^+_1 = 0; \tag{16}$$

at $y = -1$

$$v^-_1 = 0. \tag{17}$$

Hereafter for the sake of convenience we shall use ρ to denote ρ^+ and set ρ^- equal to 1. From (13), $p^\pm_1$ are functions of x only. $p^+_1 = \rho\eta_1$ by (14) and $p^-_1 = \rho\eta_1 + \eta_2(1-\rho)$ by (15). We can find $v^\pm_1$ by using (11), (12), (15), and (17) so that

$$\begin{aligned} v^+_1 &= y(\eta_{1x}/u_0) + u_0\eta_{2x}, \\ v^-_1 &= (y+1)(\rho\eta_{1x} + (1-\rho)\eta_{2x})/u_0. \end{aligned} \tag{18}$$

$u^\pm_1$ are also derived from (11)

$$\begin{aligned} u^+_1 &= -\eta_1/u_0, \\ u^-_1 &= (-\rho\eta_1 - (1-\rho)\eta_2)/u_0, \end{aligned} \tag{19}$$

where we assume $\eta_1(x=-\infty) = \eta_2(x=-\infty) = 0$, $u^\pm_1(x=-\infty) = 0$.

Similarly, we can find $p^\pm_2, v^\pm_2, u^\pm_2, p^\pm_3, v^\pm_3, u^\pm_3$ in terms of η_1 and η_2 without using the kinematic conditions (5) and (6). From (5) and(6), and the asymptotic expansion of u^- and v^-, we have at $y = h$,

(20)
$$\begin{aligned}
&u_0\eta_{1x} - v_1^+ + \epsilon(u_1^+\eta_{1x} - \eta_1 v_{1y}^+ - v_2^+)\\
&+ \epsilon^2(u_2^+\eta_{1x} + \eta_1\eta_{1x}u_{1y}^+ - v_{1yy}^+\eta_1^2 - \eta_1 v_{2y}^+ - v_3^+) + O(\epsilon^3) = 0,
\end{aligned}$$

and at $y = 0$,

(21)
$$\begin{aligned}
&u_0\eta_{2x} - v_1^- + \epsilon(u_1^-\eta_{2x} - \eta_2 v_{1y}^- - v_2^-)\\
&+ \epsilon^2(u_2^-\eta_{2x} + \eta_2\eta_{2x}u_{1y}^- - v_{1yy}^-\eta_2^2 - \eta_2 v_{2y}^- - v_3^-) + O(\epsilon^3) = 0.
\end{aligned}$$

Then we make use of these equations to find the equations of the free surface $\eta_1(x)$ and the interface $\eta_2(x)$. By substituting $u_0, u_1^\pm, v_1^\pm, u_2^\pm, v_2^\pm, v_3^\pm$ into (20) and (21) and eliminating η_1, we obtain

$$\begin{aligned}
(22)\qquad &(u_0 - \rho c_1/u_0 - (1-\rho)/u_0)\eta_{2x} + \epsilon(E\eta_2\eta_{2x})\\
&+ \epsilon^2(F_1\eta_2^2\eta_{2x} + F_2\eta_{2x} + F_3\eta_{2xxx} + F_4 b_x)\\
&+ O(\epsilon^3) = 0,
\end{aligned}$$

where if we let $c_1 = (2u_0^2 - (1-\rho))/(\rho + u_0^2 - h)$, $D_1 = u_0/(\rho + u_0^2 - h)$, $\lambda = u_2^\pm(-\infty)$, and $R = \rho c_1 + 1 - \rho$, then

$$\begin{aligned}
E =\ & -(R^2 + 2Ru_0^2)u_0^{-3} - \rho D_1((hc_1^2 - R^2)u_0^{-4}\\
&+ (2c_1^2 - 2R - 2c_1)u_0^{-2}),\\
F_1 =\ & -\rho D_1 u_0^{-1}((3c_1^3 - 3c_1^2 + R^2/2)u_0^{-3} + (3hc_1^3/2 - 3R^3/2)u_0^{-5}\\
&+ 3D_1(\rho u_0^{-1} + \rho R u_0^{-3})((3R/2 + c_1 - c_1^2)u_0^{-1}\\
&+ (R^2/2 - hc_1^2/2)u_0^{-3}) - 3R^2u_0^{-3}/2 - 3R^3u_0^{-5}/2),\\
F_2 =\ & \lambda((-\rho D_1 u_0^{-1})(2 + Ru_0^{-2} - c_1 - hc_1u_0^{-2}) + (1 + Ru_0^{-2})),\\
F_3 =\ & (-\rho D_1 u_0^{-1})(-c_1(\rho h^2/2 + \rho/3)u_0^{-1} - (u_0^2\rho h + (1-\rho)/3)u_0^{-1}\\
&+ (c_1(\rho h^3/3)/u_0\rho) + u_0h^2/2)\\
&- c_1(\rho h^2/2 + \rho/3)u_0^{-1} - (u_0^2\rho h + (1-\rho)/3)u_0^{-1},\\
F_4 =\ & \rho D_1 - u_0.
\end{aligned}$$

3. Steady Modified KdV Equation with Forcing (SFMKdV)

From the zeroth order term of (22), we obtain

$$u_0 - (\rho c_1/u_0) - (1-\rho)/u_0 = 0,$$

and by the expression for c_1 in (22), it follows that

$$u_0^4 - (1+h)u_0^2 + h(1-\rho) = 0, \tag{23}$$

and

$$u_0^2 = (1 + h \pm ((1-h)^2 + 4\rho h)^{1/2})/2.$$

We denote the two values of u_0^2 by u_{01}^2 and u_{02}^2 respectively corresponding to the plus and minus signs. Without loss of generality we assume u_{01} and u_{02} are both positive and call them critical speeds, near each of which a nonlinear theory for the motion of the interface has to be developed.

Next we consider the coefficients of $\eta_2\eta_{2x}$ in the first order terms of the equation (22). If E in (22) is not zero, an FKdV can be derived if we assume $b(x) = b^*(x^*)(h^{*-}\epsilon^2)^{-1}$ and $x = \epsilon^{1/2}x^*/h^{*-}$ in nondimensional varibles and similar results as in [1] can be obtained. However, E may vanish. First, let us simplify the expression of E,

$$\begin{aligned}
E &= -((\rho c_1 + 1 - \rho)^2/u_0^3) - 2((\rho c_1 + 1 - \rho)/u_0) \\
&\quad - \rho D_1[-2((\rho c_1 + 1 - \rho)/u_0) - ((\rho c_1 - \rho + 1)^2/u_0^3) \\
&\quad + 2(c_1^2/u_0) + h(c_1^2/u_0^3) - 2(c_1/u_0)]/u_0 \\
&= 3(u_0\rho)^{-1}(u_0^2 + \rho - h)(\rho(u_0^2 h - u_0^4 - u_0^2 + 1) - u_0^4 + 2u_0^2 - 1)\,, \\
&= 3u_0(1-u_0^2)(\rho h(u_0^2 + \rho - h))^{-1}(u_0^4 + (1-2h)u_0^2 + h^2 - 1)\,.
\end{aligned}$$

Then $E = 0$ implies $u_0^4 + (1-2h)u_0^2 + h^2 - 1 = 0$. Let $u_0 = u_{01}$ or u_{02}. Then

$$u_{01}^4 + (1-2h)u_{01}^2 + h^2 - 1 = 1 + h\rho + (2-h)((1-h)^2 + 4\rho h)^{1/2}, \tag{24}$$

$$u_{02}^4 + (1-2h)u_{02}^2 + h^2 - 1 = 1 + h\rho - (2-h)((1-h)^2 + 4\rho h)^{1/2}. \tag{25}$$

Equation (24) tells us that E does not vanish if we take u_{01} as a critical speed. Suppose both sides of (24) vanish. Then real u_{01}^2 implies $h < 5/4$ and the right hand side of (24) is greater than zero. This is a contradiction. Thus the only possible case for $E = 0$ is that the critical speed u_0^2 is equal to u_{02}^2, and it is easy to show that $E = 0$ if $u_0^2 = u_{02}^2$, and

$$1 + h\rho = (2 - h)((1 - h)^2 + 4\rho h)^{1/2}. \tag{26}$$

With the conditions (21) and (25), from (22), we obtain a Steady FMKdV,

$$F_1 \eta_2^2 \eta_{2x} + F_2 \eta_{2x} + F_3 \eta_{2xxx} + F_4 b_x = 0, \tag{27}$$

where

$$\begin{aligned} F_1 &= 3u_0(4\rho + 3h - u_0^2)\,, \\ F_2 &= \lambda(2(1 + h)u_0^2 - 4h(1 - \rho))u_0^{-2}\,, \\ F_3 &= u_0^{-1}(h(1 + h) - u_0^2(h^2 + 1 + 3\rho h)\,, \\ F_4 &= u_0(h - u_0^2)\,. \end{aligned}$$

The coefficients F_1 to F_4 here are the simplified forms of F_1 to F_4 in the previous section by using (23). The sign of $F_3 F_1$ determines the existence of solutions of (27). In the following sections, we assume $F_3 F_1 > 0$ and the case for $F_3 F_1 < 0$ is considered in subsequent study [10].

3.1. Symmetric soliton-like waves

We assume $U = u_0 + \lambda\epsilon^2 + O(\epsilon^3)$ and consider (27) for $F_1/F_3 > 0$ and $F_2/F_3 < 0$. (27) can be rewritten as

$$\eta_{2xxx} = -A_1 \eta_2^2 \eta_{2x} + A_2 \eta_{2x} + A_3 b_x, \tag{28}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3 > 0$, $A_3 = -F_4/F_3$. When $b_x \equiv 0$, (28) has soliton solutions whose value is 0 at $x = \pm\infty$ for $A_2 \geq 0$:

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\operatorname{sech}((A_2)^{1/2}x), \tag{29}$$

For $A_2 \leq 0$, there is no soliton solution. The solutions in (29) are obtained as in the classical case by taking the limit of elliptic functions in the periodic solutions of (28) for $b_x = 0$ when the wave length tends to infinity. Next we consider (28) when $b_x \neq 0$ but of compact support.

We look for a solution $\eta_2(x)$ such that $A_2 > 0$ and

$$\lim_{|x|\to\infty} (d/dx)^j \eta_2(x) = 0 \qquad j = 0, 1, 2.$$

Integrating (28) from $-\infty$ to x, it follows that

$$A_2\eta_2 - \eta_{2xx} = A_1\eta_2^3 - A_3 b(x), \quad -\infty < x < \infty. \tag{30}$$

(30) can be converted to the following integral equation:

$$\eta_2(x) = \int_{-\infty}^{\infty} K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3 b(\xi))d\xi\,,$$

where $K(x,\xi) = \exp(-\sqrt{A_2}|x-\xi|)/(2\sqrt{A_2})$ is a Green function of $A_2K(x,\xi) - K_{xx}(x,\xi) = \delta(x,\xi), -\infty < x < \infty$.
Define

$$\begin{aligned}
T(\eta_2) &= \int_{-\infty}^{\infty} K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3 b(\xi))d\xi\,,\\
\|u\| &= \|u\|_\infty = \sup_{x\in\Re} |\eta_2(x)|\,,\\
H &= \{u \mid u \in C(\Re), \|\exp(\sqrt{A_2}|x|)u\| < \infty\}\,,\\
B_M &= \{u \mid u \in H, \|u\| \leq M, 0 < M < \infty\}\,.
\end{aligned}$$

Then clearly H is a complete metric space and B_M is a closed ball in H.

LEMMA 1. *$T : B_M \to B_M$ if $A_1M^2/3 + \|A_3b(x)\|/M \leq A_2$.*

Proof.

$$\begin{aligned}
\|T(\eta_2)\| \leq& \|A_1\eta_2^3/3 - A_3b\| \sup_{x\in\Re} \int_{-\infty}^{\infty} \exp(-\sqrt{A_2}|x-\xi|)/(2\sqrt{A_2})d\xi\\
\leq& (A_1M^3/3 + \|A_3b\|)/A_2\\
\leq& M \text{ if } A_1M^2/3 + \|A_3b(x)\|/M\\
\leq& A_2.
\end{aligned}$$

Next we show that $T(\eta_2)$ decays rapidly, so that we may consider the behavior of $\exp(\sqrt{A_2}|x|)$ $|T(\eta_2)(x)|$ when $|x|$ is large.

LEMMA 2. $\sup_{x\in\Re}\exp(\sqrt{A_2}|x|)a_2)(x)| < \infty$ *for* $\eta_2 \in B_M$.

Proof. It suffices to prove for $x > 0$.
Let $N = \max_{\xi\in\Re}|-A_3b(\xi)|$.

$$
\begin{aligned}
&\exp(\sqrt{A_2}x)|T(\eta_2)(x)| \\
&\quad =\Big|\int_{-\infty}^{x}\exp(\sqrt{A_2}\xi)(A_1\eta_2^3(\xi)/3 - A_3b(\xi))d\xi \\
&\qquad + \int_{x}^{\infty}\exp(\sqrt{A_2}(2x-\xi))(A_1\eta_2^3(\xi)/3 - A_3b(\xi))d\xi\Big|/(2\sqrt{A_2}) \\
&\quad \le \exp(-\sqrt{A_2}x)\sup_{x\in\Re}(\eta_2(x)\exp(\sqrt{A_2}x)^3/(6A_2) \\
&\qquad + \int_{suppb(x)} N\exp(\sqrt{A_2}\xi)d\xi/(2\sqrt{A_2}) < \infty
\end{aligned}
$$

since $\eta_2 \in H$. Hence $\sup_{x>0}\exp(\sqrt{A_2}x)|T(\eta_2)(x)| < \infty$ since $\eta_2 \in B_M$.
Similarly one can show $\sup_{x\le 0}\exp(-\sqrt{A_2}x)|T(\eta_2)(x)| < \infty$

THEOREM 1. (30) *has a solution in* $C^3(\Re)$ *which decays exponentially at* $|x| = \infty$ *if* A_2 *is sufficiently large.*

Proof. $\|T(\eta_1)-T(\eta_2)\| \le \sup_{x\in\Re} A_1\int_{-\infty}^{\infty}K(x,\xi)|\eta_1^2+\eta_1\eta_2+\eta_2^2||\eta_1-\eta_2|d\xi \le A_1M^2\|\eta_1-\eta_2\|/A_2$. Hence, by the contraction mapping theorem, Lemma 1, and Lemma 2, $\eta_2 = T(\eta_2)$ has the unique solution inB_M. Now $\eta_{2xx}(x) = \int_{-\infty}^{\infty}K_{xx}(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3b(\xi))d\xi = \int_{-\infty}^{\infty}A_2K(x,\xi)(A_1\eta_2^3(\xi)/3 - A_3b(\xi))d\xi - A_1\eta_2^3(x)/3 + A_3b(x) = A_2\eta_2(x) - A_1\eta_2^3(x) + A_3b(x)$ where $K(x,\xi) - K_{xx}(x,\xi) = \delta(x,\xi)$. Hence $\eta_2 \in C^2(\Re)$, and it follows from the right side of the above equation that $\eta_2 \in C^3(\Re)$.

We have shown that (28) has an exponentially decaying solution as x tends to $\pm\infty$. In the following we use numerical computation to find symmetric soliton-like solutions of (28) when the obstruction $b(x)$ is given by $b(x) = R(1-x^2)^{1/2}$ for $|x| \le 1$ and $b(x) = 0$ for $|x| > 1$, where R is a given constant.

Let

$$\eta_2(x) = \pm(6A_2/A_1)^{1/2}\text{sech}((A_2)^{1/2}(x - x_0)), \tag{31}$$

where x_0 is a phase shift. To find a solution in $|x| \leq 1$, we need only consider (31) in $-1 \leq x \leq 0$ subject to $(\eta_2'(x))^2 = -A_1\eta_2^4/6 + A_2\eta_2^2$ at $x = -1$ and $\eta_2(x) = 0$ at $x = 0$. This problem can be solved numerically by a shooting method and the phase shift x_0 is determined by (31) for $x = -1$. The numerical results are given in Fig. 2 and Fig. 3. Four typical soliton-like solutions are given in Fig. 2 and Fig. 3 show the relation between the value of soliton-like solution at $x = 0$ and λ. In both numerical results, we assume $R = 1$.

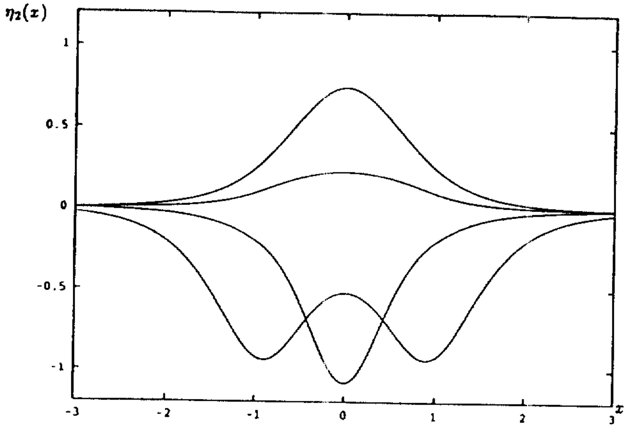

Fig. 2. Four typical soliton-like solutions
$h = 0.98,\ R = 1,\ \lambda = -4$

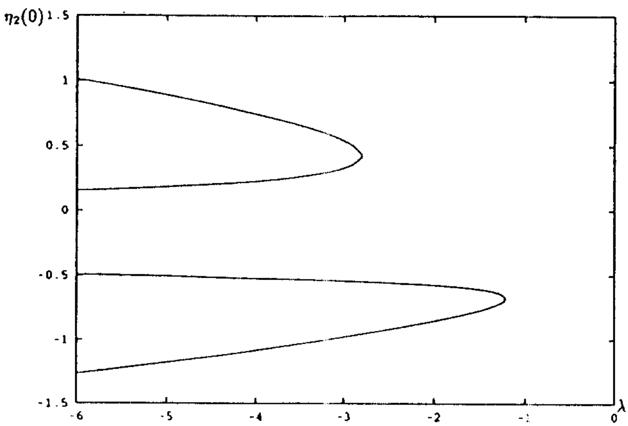

Fig. 3. Relationship between $\eta_2(0)$ and λ
$h = 0.98,\ R = 1$

3.2. Symmetric waves with zero behind and ahead of the obstruction

Similar to the section 3.1, we consider the equation

$$\eta_{2xxx} = -A_1\eta_2^2\eta_{2x} + A_2\eta_{2x} + A_3 b_x, \tag{32}$$

where $A_1 = F_1/F_3 > 0$, $A_2 = -F_2/F_3$, $A_3 = -F_4/F_3$. Integrating (32) from $-\infty$ to x, we obtain

$$\eta_{2xx} = -A_1\eta_2^3/3 + A_2\eta_2 + A_3 b(x), \tag{33}$$

where $b(x)$ is assumed to have compact support and $\eta_2(-\infty) = 0$. We assume $\eta_2 \equiv 0$ in $(-\infty, x_-)$ where $[x_-, x_+]$ is the support of the obstruction. In the following, we show that the solution of (33) exists with initial values $\eta_2(x_-) = \eta_{2x}(x_-) = 0$.

THEOREM 2. *$\eta_{2xx} = -A_1\eta_2^3/3 + A_2\eta_2 + A_3 b(x), A_1 > 0$ with $\eta_2(x_-)$ $= \eta_{2x}(x_-) = 0$ has a C^2 solution for $x_- \leq x \leq x_+$.*

Proof. It suffices to show that η_2 is bounded in $[x_-, x_+]$. For simplicity we assume $x_- = -1$ and $x_+ = 1$. By multiplying η_{2x} to the given equation and integrating it from -1 to $x \leq 1$,

$$\begin{aligned}(\eta_{2x})^2 &= -(A_1/6)\eta_2^4(x) + A_2\eta_2^2(x) - 2\int_{-1}^{x} A_3 b(t)\eta_2'(t)dt \\ &= -(A_1/6)(\eta_2^2 - 3A_2/A_1)^2 + 3A_2^2/(2A_1) - 2\int_{-1}^{x} A_3 b(t)\eta_2'(t)dt.\end{aligned}$$

Hence

$$\begin{aligned}(\eta_{2x})^2 &\leq 3A_2^2/(2A_1) + 2\int_{-1}^{x} |A_3 b(t)\eta_2'(t)|dt \leq 3A_2^2/(2A_1) \\ &\quad + \int_{-1}^{x} (8|A_3 b(t)|^2 + |\eta_2'(t)|^2/8)dt,\end{aligned}$$

by Young's inequality, where $' = d/dt$. Suppose η_2 is not bounded in $[-1, 1]$. Then there is $x_0 \in [-1, 1]$ such that $|\eta_2| \to \infty$ as x tends to x_0 and $x_0 > x_- + \epsilon$ for some $\epsilon > 0$ by the existence theorem in the theory of ordinary differential equations. Let $x_0 = \inf_\xi\{\xi \in$

$[-1,1]||\lim_{x\to\xi^-}\eta_2(x)| = \infty\}$. Choose δ so that $-1 < \delta < x_0$. Then the solution of the given differential equation exists in $[-1,\delta]$, and by the above inequality, $\eta_{2x}^2 \le (1/8)(1+\delta)\sup_{-1\le t\le\delta}(|\eta_{2x}(t)|^2 + 64M^2) + 3A_2^2/(2A_1)$ for $x \in [-1,\delta]$, where M is $\sup suppA_3b(x)$.

Hence $\sup_{-1\le t\le\delta}|\eta_{2x}|^2 \le 64M^2 + 3A_2^2/(2A_1)/(1-(1+\delta)/8) < 64M^2 + 12A_2^2/(7A_1)$ for every δ with $-1 < \delta < x_0$. Thus $\eta_2'(x)$ is bounded when $x \in [-1, x_0]$, and $\eta_2(x) = \int_{-1}^{x}\eta_2'(t)dt$ is bounded which contradicts to $|\eta_2(x)| \to \infty$ as $x \to x_0$. Therefore, $\eta_2(x)$ is bounded in $[-1,1]$ and the solution of the given equation exists.

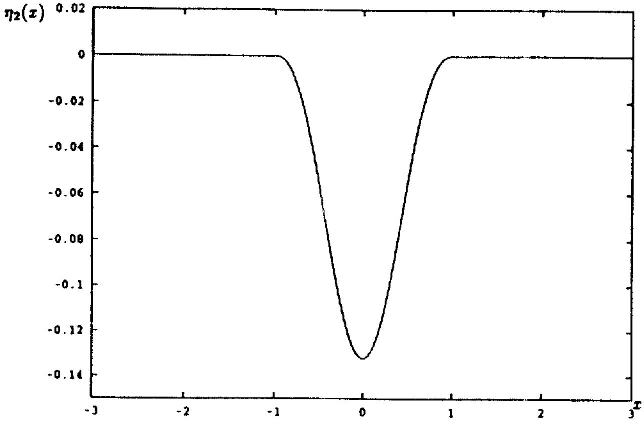

Fig. 4. Symmetric solution with one hump
$h = 0.98$, $R = 1$, $\lambda = 14.000283$

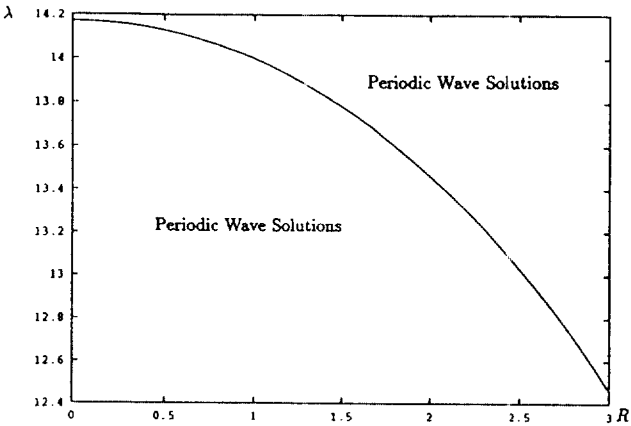

Fig. 5. Solution curve of symmetric solutions with one hump, $h = 0.98$

We have shown that the solution of (33) exists and is bounded. In the following, we use numerical computation to find symmetric wave solution of (33) which is zero behind and ahead of the elliptic obstruction. Similar methods as in section 3.1 is used to find the solutions of this problem. To find a solution in $|x| \leq 1$, we need only consider (33) in $-1 \leq x \leq 0$ subject to $\eta_2'(x) = \eta_2(x) = 0$ at $x = -1$ and $\eta_{2x}(x) = 0$ at $x = 0$. Same assumption as in section 3.1 has been given for obstruction and the numerical results are shown in Fig. 4 and 5. Fig. 4 shows the symmetric solutions for positive values of λ. The relations between R, which represents the hight of the obstruction, and $\eta_2(0)$ are given in Fig. 5. We note that, for a given R, symmetric solution is embedded in periodic solutions.

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