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MYRBERG-AGARD DENSITY POINTS AND SCHOTTKY GROUPS

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1. Introduction

Let Γ be a discrete subgroup of hyperbolic isometries acting on the Poincaré disc $B^m, m \geq 2$. The discrete group Γ acts properly discontinously in B^m, and acts on ∂B^m as a group of conformal homemorphisms, but need not act properly discontinously on ∂B^m. The action of Γ divides ∂B^m into two sets. The ordinary set $\Omega(\Gamma)$ is the largest open subset of ∂B^m on which Γ acts discontinously. The complement of $\Omega(\Gamma)$ in ∂B^m is the limit set, denoted by $\Lambda(\Gamma)$ or simly Λ. The limit set $\Lambda(\Gamma)$ is the set of accumulation points of the orbit $\Gamma(x)$ for one, hence for every, point $x \in B^m$. Equivalently, the limit set is the smallest nonempty closed set in ∂B^m. If Λ contains two or fewer points, Γ is elementary, and contains a free abelian subgroup of finite index. Otherwise, Γ is nonelementary. In this paper, we always assume that Γ is nonelementary.

It is easy to see that $\Lambda(\Gamma) = \Lambda(\Gamma')$ for any nontrivial normal subgroup Γ' of Γ. Also, if x is any point of ∂B^m, then the accumulation points of any orbit of x under Γ lie in $\Lambda(\Gamma)$. For a nonelementary group Γ, define $CH(\Lambda)$ to be the smallest nonempty convex set in B^m which is invariant under the action of Γ; this is the convex hull of Γ. The boundary at infinity of $CH(\Lambda)$ is precisely Λ, and so $CH(\Lambda)$ contains every geodesic line in B^m both of whose endpoints at infinity are in Λ.

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The limit point $p \in \Lambda(\Gamma)$ is said to be a conical limit point (or a point of approximation) for Γ if for every $x \in B^m$ there is a sequence $\{\gamma_n\} \subset \Gamma$ on which the sequence $\dfrac{|p - \gamma_n(x)|}{1 - |\gamma_n(x)|}$ remains bounded. We will give a geometric characterization of conical limit points due to Beardon-Maskit in section 2. A limit point p is called a controlled concentration point if it has a neighborhood U such that for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ so that $p \in \gamma(U)$ and $\gamma(U) \subset V$. Three characterizations of controlled concentration points and its relation with recurrent geodesics are given in [**AHM**]. From each characterization, it is not hard to see that every controlled concentration point is a conical limit point.

When we study controlled concentration points via recurrent geodesics, we come up with a new kind of limit points which are called Myrberg-Agard density points. That is a mild modification of "density points" described in S. Agard's paper (see [**A**]) in which he proved the set of "density points" has full Hausdorff measure in the limit set if the given group is of divergence type which means the Poincarè series of the group Γ diverges at exponent $m - 1$. Analogously, we proved the set of Myrberg-Agard density points has full Patterson-Sullivan measure in the limit set if the group is of divergence type in general sense. Namely, the Poincarè series of the group diverges at the critical exponent. (See [**H**]).

In this paper we give a topological characterization of Myrberg-Agard density points by using admissible pairs of open neighborhoods and give an example for Schottky groups. This together with a topological characterization of conical limit points might give an overall description of topological characterizations of certain limit points. By a neighborhood of p, we will always mean an open neighborhood of p in ∂B^m.

Now we need to state some definitions for a topological characterization of Myrberg-Agard density points.

DEFINITION 1.1. One says that a pair of open sets (U_1, U_2) in ∂B^m is an *admissible pair* if

(1) $\overline{U_2} \subset U_1$,
(2) $U_2 \cap \Lambda \neq \emptyset$ and
(3) $\Lambda \not\subset \overline{U_1}$.

An admissible pair (U_1, U_2) is called an *admissible pair at* p if both U_1 and U_2 are neighborhoods of p.

DEFINITION 1.2. One says that an admissible pair (U_1, U_2) *can be concentrated* at p if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U_2) \subset \gamma(U_1) \subset V$.

DEFINITION 1.3. A geodesic λ is called a geodesic for Γ if both endpoints of λ are limit points of Γ. The limit point p is called a *Myrberg-Agard density point* for Γ if whenever μ is an oriented geodesic for Γ and α is a geodesic ray ending at p in $CH(\Lambda)$ (convex hull of Λ), there is a sequence of elements $\{\gamma_i\}$ such that $\{\gamma_i(\alpha)\}$ converge to μ in an oriented sense.

It is not difficult to see that the set of Myrberg-Agard desity points is Γ-invariant and every Myrberg-Agard density point is a controlled concentration point. (See [**A-H-M**]).

We will assume familiarity with the basic concepts of Möbius groups as exposed, for example, in [**B**]. Of particular importance is the following result, proven on pp. 97-98 of [**B**].

DOUBLE DENSITY THEOREM. Let Γ be a nonelementary group of Möbius transformations of ∂B^m, let V_1 and V_2 be open sets both meeting the limit set of Γ. Then there exists a loxodromic element of Γ with a fixed point in V_1 and a fixed point in V_2.

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2. Conical limit points

In this section, we will examine two geometric characterizations of conical limit points for nonelementary groups and we will give a topological characterization of conical limit points using geometric ones.

PROPOSITION 2.1. *The point $p \in \partial B^m$ is a conical limit point for Γ if and only if there is a geodesic σ ending at p such that for any point $x \in B^m$ there are infinitely many Γ-images of σ within a bounded hyperbolic distance of x.*

Proof. See Theorem 2.4.1 in [**N**].

REMARK. If the limit point p is a conical limit point and if we fix $x = 0$, then the above Proposition implies that there is a compact ball K in B^m centered at 0 so that infinitely many Γ- images of σ (a geodesic ending at p) meet the ball K.

PROPOSITION 2.2. *The point $p \in \partial B^m$ is a conical limit point for Γ if and only if there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ such that $\gamma_n(p)$ converges to q and $\gamma_n(0)$ converges to r where $r \neq q$.*

Proof. It is an immediate consequence of remark after Proposition 2.1 or see VI.B.4 in [**M**].

Now we give a topological characterization of conical limit points using admissible pairs at the limit point.

THEOREM 2.3. *The limit point p is a conical limit point for Γ if and only if there exists an admissible pair (U_1, U_2) at p which can be concentrated at p.*

Proof. Suppose that there exists an admissible pair (U_1, U_2) at p which can be concentrated at p. Then there exist a neighborhood V_n of p and a sequence $\{\gamma_n\}$ of Γ such that

$$\mathrm{diam} V_n \to 0, \quad \gamma_n(U_1) \subset V_n, \quad \text{and } p \in \gamma_n(U_2).$$

Without loss of generality, we may assume that every V_n is "round" disk and 0 is not in the half space HS_1 bounded by hyperplane determined by V_1. That means $\overline{HS_1} \cap \partial B^m = \overline{V_1}$. Then for the geodesic ray α from 0 to p, $\gamma_n^{-1}(\alpha)$ crosses U_1 and U_2. Because $\gamma_n^{-1}(p) \in U_2$, $U_1 \subset \gamma_n^{-1}(V_n)$ and $\overline{U_2} \subset U_1$, it follows that $\gamma_n^{-1}(0)$ converges to r and $\gamma_n^{-1}(p)$ converges to q where $q \neq r$. This shows that p is a conical limit point from Proposition 2.2.

Conversely, suppose that p is a conical limit point. Then by Proposition 2.2, there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n^{-1}(0)$ converges to r and $\gamma_n^{-1}(p)$ converges to q where $r \neq q$. Let λ denote the geodesic ray from 0 to p and μ denote the geodesic from r to q.

Let H_1 denote the hyperplane in B^m that passes through p_1 and be perpendicular to λ at p_1 where p_1 is a point on $\gamma_1(K) \cap \lambda$. Here K

denotes the compact ball as in the remark after Proposition 2.1. Let U_1 be the component of $\partial B^m - \partial H_1$ containing p. Since $\gamma_i(K) \cap \lambda \neq \emptyset$, for each i, we choose $p_i \in \gamma_i(K) \cap \lambda$. Let v_i denote the unit tangent vector to λ at p_i. Since the unit tangent bundle $T_1(K)$ of K is compact, the sequence $\{d\gamma_i^{-1}(v_i)\}$ converges to w_0 for some $w_0 \in T_1(K)$. Hence by passing to a subsequence (choose a new H_1 and a new U_1 if it is necessary), we may have that $\gamma_n\gamma_1^{-1}(U_1)$ meets λ with its normal vector at the intersection points making angles within small enough of $\pi/2$ with the tangent vector to λ. Then for each n, $\gamma_n\gamma_1^{-1}(U_1)$ contains p and the diameters limit to 0. Hence $\gamma_n\gamma_1^{-1}(U_1)$ forms a local base at p. Let H_2 denote the hyperplane that passes through p_2 and be perpendicular to λ at p_2 where p_2 is a point on $\gamma_2(K) \cap \lambda$. Let U_2 be the component of $\partial B^m - \partial H_2$ containing p with $\overline{U_2} \subset U_1$. Again by passing to a subsequence of the given sequence if it is necessary, we see that $q \in \gamma_1^{-1}(U_2)$ because $\gamma_n^{-1}(\lambda)$ converges to μ and $\gamma_1^{-1}(U_2)$ is orthogonal to $\gamma_1^{-1}(\lambda)$ at $\gamma_1^{-1}(p_2)$. (See Figure 1 and 2.) Since $\{\gamma_n\gamma_1^{-1}(U_1)\}$ forms a local base at p and $p \in \gamma_n\gamma_1^{-1}(U_2)$, (U_1, U_2) is an admissible pair at p which can be concentrated at p. This completes the proof of Theorem 2.3.

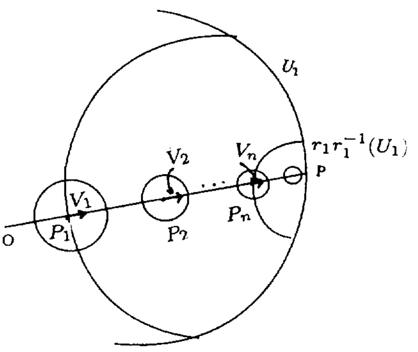

FIGURE 1

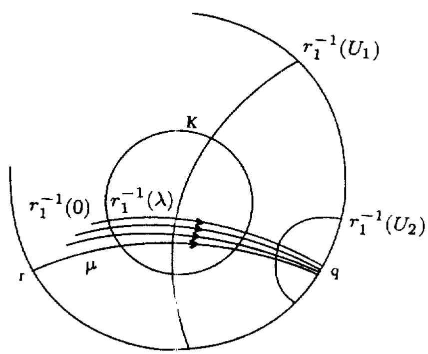

FIGURE 2

From the above characterization, we can deduce the following corollary.

COROLLARY 2.4. *Every controlled concentration point is a conical limit point.*

Proof. Suppose that p is a controlled concentration point. Then there exists a neighborhood U which can be concentrated at p. Hence if we choose $U_1 = U$ and U_2 is any neighborhood of p with $\overline{U_2} \subset U_1$. Here U_2 takes the role of a neighborhood V of p in the definition of controlled concentration points. Then (U_1, U_2) is an admissible pair which can be concentrated at p. This implies that p is a conical limit point from Theorem 2.3. This completes the proof of Corollary 2.4.

3. Myrberg-Agard density points and an example

Now we give one of topological characterization of Myrberg-Agard density points and give an example for the case that Γ is a two generator Schottky group.

THEOREM 3.1. *A limit point p is a Myrberg-Agard density point for Γ if and only if every admissible pair (U_1, U_2) can be concentrated at p.*

Proof. Let λ be a geodesic ending at p, whose other end point is a limit point different from p. Let μ be any geodesic for Γ. Construct a sequence of neighborhoods W_i and Z_i of the endpoints of μ such that μ is oriented from W_i to Z_i and such that for each i, $\overline{W_i} \cap \overline{Z_i} = \emptyset$ and $\text{diam}W_i \to 0$, $\text{diam}Z_i \to 0$. Observe that each $(\partial B^m - \overline{W_i}, Z_i)$ is an admissible pair. Let V be a neighborhood of p whose closure does not contain the other endpoint of λ. Choose $\gamma_i \in \Gamma$ so that $p \in \gamma_i^{-1}(Z_i) \subset \gamma_i^{-1}(\partial B^m - \overline{W_i}) \subset V$. Then $\gamma_i(\lambda)$ runs from W_i to Z_i. Therefore $\gamma_i(\lambda) \to \mu$ in an oriented sense. By the construction of γ_i, for every geodesic ray α (ending at p) of λ, $\gamma_i(\alpha)$ converges to μ.

Conversely, we assume that p is a Myrberg-Agard density point and (U_1, U_2) is any admissible pair. Consider the following two cases.

Case 1. $p \in U_2$. Choose $q \in \Lambda - \overline{U_1}$. Let λ be the geodesic from q to p and let α be a geodesic ray (ending at p) of λ. Because p is a Myrberg-Agard density point, there is a sequence $\{\gamma_n\}$ such that $\{\gamma_n^{-1}(\alpha)\}$ converge to λ in an oriented sense. If we denote α by $[x_0, p)$ then the above convergence implies $\gamma_n^{-1}(x_0) \to q$ and $\gamma_n^{-1}(p) \to p$. We may choose large enough n so that $p \in \gamma_n(U_2)$, because $\gamma_n^{-1}(p)$ converge to p, so for large enough n, $\gamma_n^{-1}(p) \in U_2$. Since we may choose α as we wish, by taking the starting point x_0 of α as close to p as we want if it is necessary, we may assume that $\{\gamma_n(U_1)\}$ is contained in a local base at p. Therefore for a neighborhood V of p, we may choose large enough n so that $p \in \gamma_n(U_2) \subset \gamma_n(U_1) \subset V$.

Case 2. p is not in U_2. Let V be any neighborhood of p. Because the Myrberg-Agard density points is Γ-invariant, and the orbit of any limit point is dense in the limit set, there exists $\tau \in \Gamma$ so that $\tau(p) \in U_2$. By case 1, there exists γ so that $\tau(p) \in \gamma(U_2) \subset \gamma(U_1) \subset \tau(V)$, so $p \in \tau^{-1}\gamma(U_2) \subset \tau^{-1}\gamma(U_1) \subset V$. This completes the proof of Theorem 3.1.

As an immediate consequence, we have the following corollary.

COROLLARY 3.2. *Every Myrberg-Agard density point is a controlled concentration point.*

Proof. Let p be a Myrberg-Agard density point and fix a neighborhood U_1 of p. For a neighborhood V of p, choose a neighborhood U_2 of p so that $U_2 \subset V$ and $\overline{U_2} \subset U_1$. Then (U_1, U_2) is an admissible pair at p. Since p is a Myrberg-Agard density point, (U_1, U_2) can be concentrated at p by Theorem 2.1. Therefore there exists an element γ such that $p \in \gamma(U_2)$ ($\subset \gamma(V)$) and $\gamma(U_1) \subset V$. This implies that p is a controlled concentration point.

An example: Schottky groups

Now we will obtain examples of Myrberg-Agard density points. For simplicity, we will work with a 2-generator m-dimensional Schottky group Γ, although it will be apparent that the same phenomena occur for other examples (in particular, with more generators). The limit set of Γ is a Cantor set which can be understood quite explicitly using the sequence of crossings of a geodesic ray (ending at the limit point) with the translates of two fixed sides of a fundamental domain.

To define Γ, we work in the Poincaré unit disc B^m. Let a and a' be the geodesic hyperplanes in B^m which lie in the spheres in $\mathbb{R}^m$ with centers at the points (1.1,0,...,0) and (-1.1,0,...0), say. Similarly, let b and b' lie in the spheres with centers at the points (0,...,0,1.1) and (0,...,0,-1.1). Choose a, a', b, b' so that they are mutually disjoint. As the generators of Γ, select two orientation-preserving hyperbolic isometries: one carrying a to a' and one carrying b to b'. Fix one of the direction normal to a as the positive direction. It determines a positive normal direction for each translate of a. Similarly, we label b and its translates. A crossing of an oriented geodesic of geodesic ray in B^m with a translate of a or b will be called a *positive* crossing when it agrees with the selected direction; otherwise it will be called a *negative* crossing.

Suppose α is a geodesic ray in B^m, which does not lie in a translates of a and b. Then α crosses a sequence (finite or infinite, possibly of length 0) of translates of a and b. (When a geodesic ray starts in a translate, we count that intersection as a crossing.) To α, we associate a sequence $S(\alpha) = x_1x_2x_3$...of elements in the set $\{a, \bar{a}, b, \bar{b}\}$ in the following way. If the nth crossing of α with the union of the translates of a and b is a positive crossing with a translate of a, then $x_n = a$. If the nth crossing is a negative crossing with a translate of a, then $x_n = \bar{a}$. For crossings with translates of b, the elements b and $\bar{b}$ are assigned similarly. Note that $S(\alpha)$ is an infinite sequence if and only if α ends at a limit point of Γ, and note that, for each sequence $S = x_1x_2x_3$... of elements of the set $\{a, \bar{a}, b, \bar{b}\}$ (with the property that for no n is x_nx_{n+1} in the set $\{a\bar{a}, \bar{a}a, b\bar{b}, \bar{b}b\}$), there exists a geodesic ray α with $S(\alpha) = S$. Using these sequences, the Myrberg-Agard density points of Γ can be characterized as follows.

THEOREM 3.3. *The endpoint p of a geodesic ray α is a Myrberg-Agard density point for Γ if and only if $S(\alpha) = x_1x_2x_3\ldots$ has the following property. For all positive n and all positive k, there exists arbitrary large m such that $x_{n+i} = x_{m+i}$ for every i with $0 \leq i \leq k$.*

Proof. Denote by c_n the translate of a or b whose crossing with α determines x_n, and by W_n the neighborhood of p determined by c_n. Suppose the condition in the theorem hold and (U_1, U_2) is an admissible pair. First of all, if (U_1, U_2) is an admissible pair at p and U_1 is contained in W_1, then for a neighborhood V of p choose l and k such that W_l contains U_1 and W_{l+k} is contained in U_2. Since diam$\{W_i\}$ converges to 0, we may assume that for large enough m,

W_m is contained in V. Now by hypothesis, we have $x_{l+i} = x_{m+i}$ for $0 \leq i \leq k$. Let $\gamma \in \Gamma$ be the element that translates c_l to c_m then $p \in \gamma(U_2) \subset \gamma(U_1) \subset V$. If W_1 is contained in U_1, then by using Double Density Theorem, choose a loxodromic element γ_0 whose attracting fixed point is sufficiently close to p so that $\gamma_0^{-1}(p) \in U_2$ and repelling fixed point is in $\partial B^m - \overline{U_1}$. For such a loxodromic element γ_0, we may assume that $p \in \gamma_0(U_2) \subset \gamma_0(U_1) \subset W_1$ by taking some powers of γ_0. Now repeat the same argument for the pair $(\gamma_0(U_1), \gamma_0(U_2))$ as above.

Secondly, if (U_1, U_2) is not an admissible pair at p then because the orbit of any limit point is dense in the limit set, there exists $\tau \in \Gamma$ such that $\tau^{-1}(p) \in U_2$. Therefore $p \in \tau(U_2) \subset \tau(U_1)$ and this implies that $(\tau(U_1), \tau(U_2))$ is an admissible pair at p. Hence we may apply the same argument as in the first case. Then by Theorem 3.1, p is a Myrberg-Agard density point.

Conversely, suppose p is a Myrberg-Agard density point. Then by Theroem 3.1, every admissible pair can be concentrated. Hence for any n, k and any m, there exists $\gamma \in \Gamma$ so that $\gamma(W_n) \subset W_m$ and $p \in \gamma(W_{n+k})$. Here we use triple (W_n, W_{n+k}, W_m) in replace of (U_1, U_2, V) to apply theorem 3.1. This γ must move c_{n+i} onto c_{l+i} for $0 \leq i \leq k$ and some $l \geq m$, c.f. proposition 5.1 of [**AHM**]. Thus the condition of the theorem holds and this completes the proof.

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