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CIRCULAR DISTORTION AND THE DOUBLE DISK PROPERTY OF CURVES

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1. Introduction

Suppose that D is a domain in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For each $z_0 \in \mathbb{C}$ and $0 < r < \infty$, we let $\mathbb{B}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $S(z_0, r) = \partial\mathbb{B}(z_0, r)$. For non-empty sets A, $B \subset \overline{\mathbb{C}}$, $diam(A)$ is the diameter of A and $d(A, B)$ is the distance of A and B.

A domain D in $\overline{\mathbb{C}}$ is a *K-quasidisk*, $1 \leq K < \infty$, if it is the image of the unit disk under a K- quasiconformal self mapping f of $\overline{\mathbb{C}}$. The boundary C of a K-quasidisk D is said to be a *K-quasicircle*. Thus C is a 1-quasicircle if and only if C is a line or circle [A], [LV].

Next we say that a Jordan curve C in $\overline{\mathbb{C}}$ has *circular distortion* c, $1 \leq c < \infty$, if for each Möbius transformation ϕ, either $\phi(C)$ separates the boundary circles of an annulus

$$A = A(z_0; r, s) = \{z \in \mathbb{C} : r \leq |z - z_0| \leq s\}$$

with radii ratio $\frac{s}{r} = c$ or $\phi(C)$ contains the point ∞. The circular distortion is a Möbius invariant which measures how far a Jordan curve differs from being a circle or line. In particular, C has circular distortion 1 if and only if it is a circle or line.

R. Kühnau established the following relation between these two concepts.

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LEMMA 1.1. [Ku] *If C is a K-quasicircle in $\overline{\mathbb{C}}$, then C has circular distortion c, where c depends only on K.*

R. Kühnau found sharp bounds for the constant c in terms of K and then asked if the converse of Lemma 1.1 is true, that is, if each curve C with circular distortion c is a K-quasicircle where K depends only on c. F. W. Gehring and C. Pommerenke answered this question as follows.

LEMMA 1.2. [GP] *If C is a Jordan curve in $\overline{\mathbb{C}}$ with circular distortion $c < \sqrt{2}$, then C is a K-quasicircle where K depends only on c. Also there is a Jordan curve C in $\overline{\mathbb{C}}$ with circular distortion $c = 5$ which is not a quasicircle. (That is, a curve with circular distortion $c \geq 5$ need not be a quasicircle.)*

The bound $c < \sqrt{2}$ in Lemma 1.2 is not sharp [GP]. While we look for the sharp bound of c and another example of a Jordan curve C in $\overline{\mathbb{C}}$ with finite circular distortion c which is not a quasicircle, we find a more geometric condition, so-called the double disk property, which is in between a quasicircle and circular distortion.

We say that a Jordan curve C in $\mathbb{C}$ has *the double disk property* if there exists a constant b, $1 \leq b < \infty$, such that for each $z_0 \in C$ and $0 < r \leq \operatorname{diam}(C)$, there exist open disks B_i and B_e with

$$B_i \subset \operatorname{int}(C), \quad B_e \subset \operatorname{ext}(C), \quad B_i \cup B_e \subset \mathbb{B}(z_0, r),$$

$$(1.3) \qquad b \operatorname{diam}(B_i) \geq r, \quad b \operatorname{diam}(B_e) \geq r,$$

where int(C) and ext(C) are interior and exterior of C, respectively. This property is same as requiring that $\operatorname{int}(C)$ and $\operatorname{ext}(C)$ satisfy the corkscrew condition [JK] or the plumpness condition in [MV]. It is also equivalent to asking that for $0 < r \leq \operatorname{diam}(C)$, each point z of C should subtend disks of a fixed visual angle in each complementary domain of C within distance r of z. Again the constant b in (1.3) measures how far C differs from being a circle. In particular, $b = 1$ if and only if C is a circle.

The main purpose of this paper is to find relations between the double disk property, quasicircle and circular distortion. In section

2, we show that K-quasicircle C in $\mathbb{C}$ has the double disk property with constant b, where b depends only on K. Also we show that if C has the double disk property with constant b, then C has circular distortion $c = 16\,b$. But the converses are not true and we construct counterexamples.

Next we say that a simply connected domain D in $\mathbb{C}$ is called an (α,β)-*John disk*, $0 < \alpha \leq \beta < \infty$, if there is $z_0 \in D$ such that each $z \in D$ has a rectifiable curve $\gamma : [0,\ell] \to D$, with arc length as parameter, such that $\gamma(0) = z$, $\gamma(\ell) = z_0$ and

$$\begin{aligned} &\ell \leq \beta \\ &d(\gamma(t), \partial D) \geq \frac{\alpha}{\ell} t, \end{aligned} \tag{1.4}$$

for all $t \in [0,\ell]$. We call z_0 a *John center* (see [MS]).

John disks can be thought of as "one-sided quasidisk" [K], [NV]. For example, a Jordan domain in the plane is a quasidisk if and only if D and $D^* = \overline{\mathbb{C}} \setminus \overline{D}$ are John disks [NV]. In section 3, We show that a John disk satisfies the one-sided analogue of the double disk property of quasidisks and construct a counterexample to show that the converse is not true.

2. Quasicircle, the double disk property and circular distortion

THEOREM 2.1. *If C is a K-quasicircle in $\mathbb{C}$, then C has the double disk property with constant b, where b depends only on K.*

Proof of Theorem 2.1. Fix $z_0 \in C$ and $0 < r \leq \operatorname{diam}(C)$. By hypothesis, there exists a K- quasiconformal self mapping f of $\overline{\mathbb{C}}$ which maps C onto a circle C'. By composing f with an auxiliary Möbius transformation we may further assume that $f(\infty) = \infty$ and hence that

$$f(\operatorname{int}(C)) = \operatorname{int}(C').$$

Let $w_0 = f(z_0)$, $B' = f(\mathbb{B}(z_0, r))$ and let w_i, w_e and t_i, t_e denote the centers and radii of the largest disks in $B' \cap \operatorname{int}(C')$, $B' \cap \operatorname{ext}(C')$ which are tangent to C' at w_0, respectively. Next set

$$z_i = g(w_i), \qquad z_e = g(w_e),$$

where $g = f^{-1}$ and let

$$s_i = \max_{|w-w_i|=t_i} |g(w) - g(w_i)|, \qquad r_i = \min_{|w-w_i|=t_i} |g(w) - g(w_i)|,$$

$$s_e = \max_{|w-w_e|=t_e} |g(w) - g(w_e)|, \qquad r_e = \min_{|w-w_e|=t_e} |g(w) - g(w_e)|.$$

Then by [LV, Theorem 9.3] we have

$$(2.2) \qquad s_i \le \lambda(K)\, r_i, \qquad s_e \le \lambda(K)\, r_e,$$

where $\lambda(K)$ is an increasing function of K with $\lambda(1) = 1$. Finally let $B_i = \mathbb{B}(z_i, r_i)$ and $B_e = \mathbb{B}(z_e, r_e)$. Then by (2.2)

$$\lambda(K)\, \mathrm{diam}(B_i) \ge 2\, s_i \ge r, \qquad B_i \subset \mathrm{int}(C) \cap \mathbb{B}(z_0, r),$$

$$\lambda(K)\, \mathrm{diam}(B_e) \ge 2\, s_e \ge r, \qquad B_e \subset \mathrm{ext}(C) \cap \mathbb{B}(z_0, r),$$

and hence B_i and B_e satisfy (1.3) with $b = \lambda(K)$. □

THEOREM 2.3. *If C has the double disk property with constant b, then C has circular distortion $c = 16\, b$.*

Now we give a proof of Theorem 1.5.

Proof of Theorem 2.3. Suppose that ϕ is any Möbius transformation for which $C' = \phi(C)$ does not contain ∞. We shall show there exists a disk B' such that

$$(2.4) \qquad B' \subset \mathrm{int}(C'), \qquad \mathrm{diam}(C') \le 8b\, \mathrm{diam}(B').$$

This, in turn, will imply that C' separates the boundary circles of an annulus with radii ratio $16\, b$ and hence that C has circular distortion $c = 16\, b$.

Suppose first that ϕ is a Euclidean similarity and let $z_0 \in C$ and $r = \mathrm{diam}(C)$. By (1.3) there exists a disk $B = B_i$ such that

$$(2.5) \qquad B \subset \mathrm{int}(C), \qquad \mathrm{diam}(C) \le b\, \mathrm{diam}(B),$$

and (2.4) follows immediately with $B' = \phi(B)$ and b in place of $8b$.

Suppose next that $\phi(z) = \frac{1}{z}$ and choose z_0, $z_1 \in C$ and $t \geq 1$ so that

$$|z_0| = \min_{z \in C} |z| = d, \qquad |z_1| = \max_{z \in C} |z| = td.$$

By hypothesis, $0 \notin C$ and $d > 0$, while we have $(t-1)\,d \leq \operatorname{diam}(C)$.

We consider several cases depending on whether $t \leq 2$ or $t > 2$ and $0 \in \operatorname{ext}(C)$ or $0 \in \operatorname{int}(C)$.

(Case 1). Suppose first that $t \leq 2$.

If $0 \in \operatorname{ext}(C)$, let $B' = \phi(B)$ where B is the disk given in (2.5). Then

$$B \subset \mathbb{B}(0, 2d), \qquad B' \subset \operatorname{int}(C').$$

Let w_2, $w_3 \in C'$ with $\operatorname{diam}(C') = |w_2 - w_3|$. Then there are z_2, $z_3 \in C$ such that

$$\phi(z_2) = w_2, \qquad \phi(z_3) = w_3.$$

Thus

$$\tag{2.6} \begin{aligned} \operatorname{diam}(C') &= |w_2 - w_3| = |\frac{1}{z_2} - \frac{1}{z_3}| \\ &\leq \frac{1}{d^2}\,|z_3 - z_2| \leq \frac{1}{d^2}\operatorname{diam}(C) \\ &\leq \frac{1}{d^2}\,b\operatorname{diam}(B) = 4b\,\frac{1}{(2d)^2}\operatorname{diam}(B). \end{aligned}$$

Now since $|z| < td < 2d$ for all $z \in \operatorname{int}(C)$, for some z, $z' \in \partial B$

$$\tag{2.7} \begin{aligned} \frac{1}{(2d)^2}\operatorname{diam}(B) &= \frac{1}{(2d)^2}|z - z'| \leq \frac{|z - z'|}{|z|\,|z'|} \\ &= |\frac{1}{z} - \frac{1}{z'}| = |w - w'| \leq \operatorname{diam}(B'), \end{aligned}$$

where $w = \phi(z)$, $w' = \phi(z')$. Hence by (2.6) and (2.7) we have

$$\operatorname{diam}(C') \leq 4b\operatorname{diam}(B').$$

Therefore we obtain a disk B' such that

$$\tag{2.8} B' \subset \operatorname{int}(C'), \qquad \operatorname{diam}(C') \leq 4b\operatorname{diam}(B').$$

Now if $0 \in \mathrm{int}(C)$, let $B' = \mathbb{B}(0, \frac{1}{2d})$. Since

$$\frac{1}{2d} < \frac{1}{td} \le \frac{1}{|z|} = |\phi(z)| < \frac{1}{d},$$

we have

$$B' \subset \mathrm{int}(C'), \qquad C' \subset \mathbb{B}(0, \frac{1}{d})$$

Therefore we obtain a disk B' such that

$$B' \subset \mathrm{int}(C'), \qquad \mathrm{diam}(C') \le \frac{2}{d} = 2\,\mathrm{diam}(B'). \tag{2.9}$$

(Case 2). Suppose next that $t > 2$.

Since $(t-1)\,d \le \mathrm{diam}(C)$, we have $d \le \mathrm{diam}(C)$. Let B_i and B_e be the disks corresponding to z_0 and $r = d$ which satisfy the double disk property (1.3).

If $0 \in \mathrm{ext}(C)$, let $B = B_i$ and $B' = \phi(B)$. Since $B \subset \mathbb{B}(z_0, r) = \mathbb{B}(z_0, d)$ and $|z_0| = d$, by triangle inequality,

$$B \subset \mathbb{B}(0, 2d), \qquad B' \subset \mathrm{int}(C'), \qquad B' \subset \overline{\mathbb{B}}(0, \frac{1}{d}).$$

Hence we have

$$\begin{aligned}\mathrm{diam}(C') &\le \frac{2}{d} = \frac{8}{d^2}\,\frac{d}{4} \\ &\le \frac{8}{d^2}\,\frac{1}{4}\,b\,\mathrm{diam}(B) = 8b\,(\frac{1}{2d})^2\,\mathrm{diam}(B)\end{aligned}$$

Now since $|z| < 2d$ for all $z \in \partial B$, we have the same inequality as (2.7) for some z, $z' \in \partial B$. Therefore we get

$$\mathrm{diam}(C') \le 8b\,\mathrm{diam}(B'). \tag{2.10}$$

Next if $0 \in \mathrm{int}(C)$, let $B = B_e$ and $B' = \phi(B)$. By exactly same argument as above, we obtain

$$\mathrm{diam}(C') \le 8b\,\mathrm{diam}(B'). \tag{2.11}$$

Hence inequalities (2.8), (2.9), (2.10) and (2.11) imply that (2.4) holds for the case where $\phi = \frac{1}{z}$.

Now let ϕ be any Möbius transformation for which $C' = \phi(C)$ does not contain ∞. Then

$$\phi(z) = \phi_3 \circ \phi_2 \circ \phi_1(z),$$

where ϕ_1 and ϕ_3 are Euclidean isometries and ϕ_2 is an inversion. By hypothesis C has the double disk property. Since the double disk property is invariant with respect to Euclidean isometries, $\phi_1(C)$ has the double disk property. Now since $\infty \notin C'$, we have $\infty \notin \phi_2 \circ \phi_1(C)$. Hence by what we proved above, $\phi_2 \circ \phi_1(C)$ satisfies the condition (2.4). But the condition (2.4) is invariant with respect to Euclidean isometries and hence $C' = \phi_3 \circ \phi_2 \circ \phi_1(C)$ satisfies the condition (2.4). Therefore (2.4) holds any Möbius transformation ϕ. This completes the proof of Theorem 2.3. □

Now we construct a counterexample to show that the converse of Theorem 2.1 is false.

THEOREM 2.12. *There exists a Jordan curve C in $\mathbb{C}$ which has the double disk property and is not a quasicircle.*

Proof of Theorem 2.12. For each a, $b \in \mathbb{R}$, let $D(a, b)$ denote open square which has the segment (a, b) as a diameter. Next For $j = 1, 2, \ldots$, let

$$D_j = D(2^{-j} - 2^{-2j},\ 2^{-j+1}), \qquad z_j = 2^{-j} + i\,2^{-2j},$$

and set

$$D = \cup_{j=1}^{\infty} D_j, \qquad C = \partial D.$$

Then C is a Jordan curve in $\mathbb{C}$, z_j, $\overline{z_j} \in C$ for each j and

$$\begin{aligned} \min\left(\operatorname{diam}(C_{1,j}), \operatorname{diam}(C_{2,j})\right) > 2^{-j} &= 2^{j-1} \cdot 2^{-2j+1} \\ &= 2^{j-1}|z_j - \overline{z_j}|, \end{aligned}$$

where $C_{1,j}$, $C_{2,j}$ denote the components of $C \setminus \{z_j, \overline{z_j}\}$. Hence C is not a quasicircle by Ahlfors' well known criterion, the so-called three point property [A], [G].

Next fix $z_0 \in C$ and $0 < r < \operatorname{diam}(C)$. We shall show that there exist open disks B_i and B_e which satisfy (1.3) with a fixed positive constant b, $1 \le b < \infty$.

For the existence of B_i, choose an integer k so that

$$2^{-k+1} < r \le 2^{-k+2}. \tag{2.13}$$

First if $z_0 = 0$ or $z_0 \in \partial D_j$ for $j \ge k$, let B_i denote the largest open disk contained in D_k. Then by (2.13)

$$4\sqrt{2}\operatorname{diam}(B_i) \ge 4\operatorname{diam}(D_k) = 4 \cdot 2^{-k}(1 + 2^{-k}) \ge 2^{-k+2} \ge r$$

and

$$B_i \subset D_k \cap \mathbb{B}(z_0, r).$$

Next if $z_0 \in \partial D_j$ for $j < k$, then also by (2.13)

$$\operatorname{diam}(\partial D_j) = 2^{-j}(1 + 2^{-j}) \ge 2^{-k}(1 + 2^{-k}) \ge \frac{r}{4}(1 + \frac{r}{4}) \ge \frac{r}{4},$$

and because D_j is square, an elementary calculation shows that there exists an open disk B_i such that

$$4\sqrt{2}\operatorname{diam}(B_i) \ge r, \qquad B_i \subset D_j \cap \mathbb{B}(z_0, \frac{r}{4}).$$

Thus in either case B_i satisfies (1.3) with $b = 4\sqrt{2}$.

For the existence of B_e, let

$$W(z_0) = \{z = z_0 + i t e^{i\theta} : 0 < t < \infty, |\theta| < \frac{\pi}{4}\}$$

for $\operatorname{Im}(z_0) \ge 0$ and let

$$W(z_0) = \{z = z_0 - i t e^{i\theta} : 0 < t < \infty, |\theta| < \frac{\pi}{4}\}$$

for $\operatorname{Im}(z_0) < 0$. Then $W(z_0)$ is an open sector of angle $\frac{\pi}{2}$ which lies in $\operatorname{ext}(C)$. It is easy to check that there exists an open disk B_e with

$$\sqrt{2}\operatorname{diam}(B_e) \ge r, \qquad B_e \subset W(z_0) \cap \mathbb{B}(z_0, r).$$

Hence B_e satisfies (1.3) with $b = \sqrt{2}$.

Combining the cases considered above completes the proof of Theorem 2.12 with $b = 4\sqrt{2}$. □

If we combine Theorem 2.3 and Theorem 2.12, we obtain a negative answer to the question posed by Kühnau concerning circular distortion and quasicircles. This example is totally different from Lemma 1.2.

COROLLARY 2.14. *There exists s Jordan curve C in $\mathbb{C}$ with finite circular distortion which is not a quasicircle.*

Next we construct a counterexample to show that the converse of Theorem 2.3 is false.

THEOREM 2.15. *There exists a Jordan curve C in $\overline{\mathbb{C}}$ which has a finite circular distortion, but not have the double disk property.*

Proof of Theorem 2.15. For $j = 1, 2, \ldots$, let α_j and β_j denote the upper and lower semicircles

$$\alpha_j = \{z : |z-1| = 2j-1, \operatorname{Im}(z) \geq 0\},$$

$$\beta_j = \{z : |z+1| = 2j-1, \operatorname{Im}(z) \leq 0\}.$$

Then

$$\alpha_j \cap \beta_k = \begin{cases} \{0\} & \text{if } j = k = 1 \\ \{2j\} & \text{if } j = k-1 \\ \{-2j+2\} & \text{if } j = k+1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence

$$\gamma_1 = \cup_{j=1}^{\infty}(\alpha_{2j-1} \cup \beta_{2j}) \cup \{\infty\}$$

$$\gamma_2 = \cup_{j=1}^{\infty}(\alpha_{2j} \cup \beta_{2j-1}) \cup \{\infty\}$$

are arcs which have only their endpoint 0, ∞ in common and

$$C = \gamma_1 \cup \gamma_2 = \cup_{j=1}^{\infty}(\alpha_j \cup \beta_j) \cup \{\infty\}$$

is a Jordan curve in $\overline{\mathbb{C}}$. In [GP], Gehring and Pommerenke show that C has circular distortion 5.

On the other hand, C does not have the double disk property. To see this, suppose that C has the double disk property. That is, there exists a constant b, $1 \leq b < \infty$, such that for each $z_0 \in C$ and $0 < r \leq \operatorname{diam}(C)$, there exist open disks B_i and B_e which satisfy (1.3). Since $0 \leq \operatorname{diam}(B_i) \leq 2$ and by (1.3), we have

$$\frac{r}{b} \leq \operatorname{diam}(B_i) \leq 2.$$

But $\operatorname{diam}(C) = \infty$, then r can be ∞. This is a contradiction. □

3. John disk and the double disk property

Now we show that a John disk satisfies the one-sided analogue of the double disk property of quasidisks.

LEMMA 3.1. [MV] *Suppose that D is an (α, β)-John domain. If $0 < t \leq \alpha$ and $z_0 \in \partial D$, then*

$$d(z, \partial D) \geq \frac{\alpha}{\beta} t$$

for some $z \in S(z_0, t) \cap D$.

THEOREM 3.2. *If a Jordan curve C in $\mathbb{C}$ is the boundary of a (α, β)-John disk D, then there exists a constant b, $1 \leq b < \infty$, such that for each $z_0 \in C$ and $0 < r \leq \operatorname{diam}(C)$, there is an open disk B with*

$$B \subset \operatorname{int}(C), \quad B \subset \mathbb{B}(w, r), \quad b \operatorname{diam}(B) \geq r, \tag{3.3}$$

where b depends only on α and β.

Proof of Theorem 3.2. Let $z_0 \in \partial D = C$. First we consider the case $0 < r \leq \alpha \leq \operatorname{diam}(C)$. Then by Lemma 3.1 with $t = \frac{r}{2}$,

$$d(z, C) \geq \frac{\alpha}{\beta} \cdot \frac{r}{2}$$

for some $z \in S(z_0, \frac{r}{2}) \cap D$. Hence there exists an open disk $B = \mathbb{B}(z, \frac{\alpha}{2\beta} r)$ such that $B \subset D \cap \mathbb{B}(z_0, r)$ and

$$\frac{2\beta}{\alpha} \operatorname{diam}(B) \geq r. \tag{3.4}$$

Secondly if $0 < \alpha \leq r \leq \operatorname{diam}(C)$, then by what we proved above, we can choose an open disk B_α for α such that

$$B_\alpha \subset D \cap \mathbb{B}(z_0, \alpha), \qquad \frac{2\beta}{\alpha} \operatorname{diam}(B_\alpha) \geq \alpha.$$

Then

$$B_\alpha \subset D \cap \mathbb{B}(z_0, r), \qquad \operatorname{diam}(C) \frac{2\beta}{\alpha} \operatorname{diam}(B_\alpha) \geq r\alpha.$$

Since $\operatorname{diam}(C) \leq 2\beta$, we have

$$\left(\frac{2\beta}{\alpha}\right)^2 \operatorname{diam} B_\alpha \geq r. \tag{3.5}$$

Therefore by (3.4) and (3.5), we obtain (3.3) with $b = (\frac{2\beta}{\alpha})^2$. □

Now we construct a counterexample to show that the converse of Theorem 3.2 is not true. For this we introduce more general definition of a John disk [GHM], [K], [NV].

We say that a bounded simply connected domain D in $\mathbb{C}$ is a *c-John disk* if there exist a point $z_0 \in D$ and a constant $c \geq 1$ such that each point $z \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z, z_1)) \leq c\, d(z_1, \partial D),$$

for each $z_1 \in \gamma$. We call z_0 a *John center*. Therefore a (α, β)-John disk is a c-John disk with $c = \frac{\beta}{\alpha}$.

We say that an arc α in a Jordan domain D is a *crosscut* of D if α lies in D except for its two end points. We say also that a crosscut α is *straight* if it is a line segment. Any crosscut of D divides D into two subdomains which are also conformally equivalent to the unit disk.

LEMMA 3.6. [GHM], [NV] *If D is a Jordan domain in $\mathbb{C}$, then the following conditions are equivalent, where the constants in each condition depend on each other, but need not be the same:*

(1) *D is a c-John disk.*
(2) *For every cross cut α of D dividing D into subdomains D_1 and D_2, we have*

$$\min(\operatorname{diam}(D_1), \operatorname{diam}(D_2)) \leq c\operatorname{diam}(\alpha).$$

(3) *Condition (2) holds for all strange cross cut of D.*

THEOREM 3.7. *There exists a Jordan domain D whose boundary satisfies (3.3) and which is not a c-John disk.*

Proof of Theorem 3.7. For each $a, b \in \mathbb{R}$, let $D(a,b)$ denote open square which has the segment (a,b) is a diameter. Next For $j = 1, 2, ...,$ let

$$D_j = D(2^{-j},\ 2^{-j+1} + 2^{-2j})$$

and set

$$D = \cup_{j=1}^{\infty} D_j, \qquad C = \partial D.$$

To show that D is not a c-John disk, let α_j, $j = 2, 3, \ldots$, be a straight crosscut of D joining two points at which D_{j-1} and D_j meet. Let A_j and B_j denote two subdomains of D divided by α_j with $2^{-j} \in A_j$ and $2^{-j+1} + 2^{-2j} \in B_j$ for $j = 2, 3 \ldots$. Then

$$\min(\operatorname{diam}(A_j), \operatorname{diam}(B_j)) = \operatorname{diam}(A_j) = \operatorname{diam}(D_j) = 2^{-j}(1 + 2^{-j})$$

and $\operatorname{diam}(\alpha_j) = 2^{-2j}$. Thus for $j = 2, 3, \ldots,$

$$\frac{\min(\operatorname{diam}(a_j), \operatorname{diam}(B_j))}{\operatorname{diam}(\alpha_j)} = \frac{2^{-j}(1 + 2^{-j})}{2^{-2j}} = 2^j(1 + 2^{-j}) = 2^j + 1.$$

Hence there is no constant c such that

$$\min(\operatorname{diam}(A_j), \operatorname{diam}(B_j)) \le c \operatorname{diam}(\alpha_j).$$

Therefore by Lemma 3.6, D is not a John disk.

Next by the similar argument in the proof of Theorem 2.12, we can show that $C = \partial D$ satisfies (3.3). □

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