J. Korean Math. Soc. **34**(1997), No. 1, pp. 23-42

HYPERSURFACES IN A 6-DIMENSIONAL SPHERE

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1. Introduction

A 6-dimensional sphere considered as a homogeneous space $G_2/SU(3)$ where G_2 is the group of automorphisms of the octonians $\mathbf{O}$. From this representation, we can define an almost comlex structure on a 6-dimensional sphere by making use of the vector cross product of the octonians. Also it is known that a homogeneous space $G_2/U(2)$ coincides with the Grassmann manifold of oriented 2-planes of a 7-dimensional Euclidean space. By using this representation, we can define an almost contact metric structure on any 5-dimendional oriented submanifold immersed in the 7-dimensional Euclidean space. In this paper, we study hypersurface geometry in a 6-dimensional sphere. First, we show that there exists a globally defined almost quaternionic structure on the contact distribution of a hypersurface in a 6-dimensional sphere. Precisely, we can define global 3-almost complex structures on the contact distribution which satisfy the quaternionic relation. By using this structure, we can calculate the characteristic classes of the hypersurfaces, and we also prove that both the Chern classes of the contact distribution and 1st Pontrjagin class vanish. Next, we write down the fundamental structure equations by using the almost quaternion structure in the sense of R.L.Bryant([1]). Also we show that if the almost quaternionic structure on the contact distribution is parallel and flat, then the immersion is totally geodesic porvided that the immersion is complete (Theorem 5.5).

2. Preliminaries

2.1 Notations

Received March 3, 1995. Revised March 9, 1996.

We denote by $M_{p\times q}(\mathbf{C})$ the set of $p \times q$ complex matrices and $[a] \in M_{3\times 3}(\mathbf{C})$ is given by

$$[a] = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix},$$

where $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in M_{3\times 1}(\mathbf{C})$. Then we have

$$[a]b + [b]a = 0,$$

where $a, b \in M_{3\times 1}(\mathbf{C})$.

Let $<\ ,\ >$ be the canonical inner product of $\mathbf{O}$. For any $x \in \mathbf{O}$, we denote by $\overline{x}$ the conjugate of x. We remark that the octonians may be regarded as the direct sum $\mathbf{H} \bigoplus \mathbf{H}$ where $\mathbf{H}$ is the quaternions.

2.2 Struture equation of G_2

We recall the structure equations of (Im$\mathbf{O}$, G_2) which is established by R.Bryant ([1]). The Lie group G_2 is defined by

$$G_2 = \{\ g \in GL_8(\mathbf{R}) :\ g(uv) = g(u)g(v)\ for\ any\ u,\ v\ \in \mathbf{O}\}.$$

Now, we set a basis of $\mathbf{C}\otimes_R$ Im$\mathbf{O}$ by $\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$, $E_1 = iN$, $E_2 = jN$, $E_3 = kN$, $\overline{E_1} = i\overline{N}$, $\overline{E_2} = j\overline{N}$ and $\overline{E_3} = k\overline{N}$, where $N = (1 - \sqrt{-1}\varepsilon)/2$, $\overline{N} = (1 + \sqrt{-1}\varepsilon)/2 \in \mathbf{C} \otimes_R \mathbf{O}$ and $\{1, i, j, k\}$ is the canonical basis of $\mathbf{H}$. A basis $(\ u,\ f,\ \overline{f}\ )$ of $\mathbf{C}\otimes_R$ Im$\mathbf{O}$ is said to be admissible, if there exists $g \in G_2 \subset M_{7\times 7}(\mathbf{C})$ such that $(\ u,\ f,\ \overline{f}\ ) = (\ \varepsilon,\ E,\ \overline{E}\ )g$. We identify the element of G_2 with corresponding admissible basis. Then we have

PROPOSITION 2.1. *There exist left invariant 1-forms κ and θ on G_2; θ with values in $M_{3\times 1}(\mathbf{C})$ and $\kappa = (\kappa_j{}^i)$, $1 \leq i, j \leq 3$, with values in the 3×3 skew Hermitian matrices which satisfy $tr\kappa = 0$, and*

(1)

$$\begin{aligned} d(\ u,\ f,\ \overline{f}\ ) &= (\ u,\ f,\ \overline{f}\ ) \begin{pmatrix} 0 & -\sqrt{-1}\ {}^t\overline{\theta} & \sqrt{-1}\ {}^t\theta \\ -2\sqrt{-1}\ \theta & \kappa & [\overline{\theta}] \\ 2\sqrt{-1}\ \overline{\theta} & [\theta] & \overline{\kappa} \end{pmatrix} \\ &= (\ u,\ f,\ \overline{f}\ )\Phi. \end{aligned}$$

Then Φ *satisfies* $d\Phi = -\Phi \wedge \Phi$, *or equivalently,*

$$d\theta = -\kappa \wedge \theta + [\overline{\theta}] \wedge \overline{\theta}. \tag{2}$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\overline{\theta} - ({}^t\theta \wedge \overline{\theta})\, I_3. \tag{3}$$

Let $F = \mathrm{Im}\mathbf{O} \times G_2$ *and* $x : F \rightarrow \mathrm{Im}\mathbf{O}$ *denotes the projection into the first factor. Then we can regarded* F *as the space of pairs* $(\, y;\ (u,\ f,\ \overline{f})\,)$ *consisting of a base point* $y \in \mathrm{Im}\mathbf{O}$ *and admissible basis at* y. *Then we have*

PROPOSITION 2.2 ([4]). *There exists the dual basis* $(\, \eta,\ \omega,\ \overline{\omega}\,)$ *with respect to* $(\, u,\ f,\ \overline{f}\,)$ *on* F *such that*

$$dx = (\, u,\ f,\ \overline{f}\,) \begin{pmatrix} \eta \\ \omega \\ \overline{\omega} \end{pmatrix} = (\, u,\ f,\ \overline{f}\,)\psi. \tag{4}$$

Then ψ *satisfies*

$$d\psi = -\Phi \wedge \psi. \tag{5}$$

2.3 Grassmann manifold of oriented 2-planes in Im**O**

We recall the following representation Grassmann manifold $G(2,\, \mathrm{Im}\mathbf{O})$ of oriented 2-planes in Im**O**. If we define the map $\nu : G_2 \rightarrow G(2,\, \mathrm{Im}\mathbf{O})$ by $\nu(g) = -2\sqrt{-1}\, f_1 \wedge \overline{f_1}$. Then we have

PROPOSITION 2.3 ([4]).

$$G(2,\ \mathrm{Im}\mathbf{O}) \simeq G_2/U(2).$$

3. Fundamental equations of Hypersurface

3.1 Induced almost contact metric structure

Let $M^5 = (\, M^5,\ <\ ,\ >)$ be an oriented hypersurface in a 6-dimensional sphere S^6. We denote by x and ξ the position vector and unit normal vector field of M^5 in S^6, respectively. We define the global unit tangent vector field u as $u = x \times \xi$ and η is the dual 1-form of u, where $\times$ is the exterior product of the octonians which is defined by $y \times z = (\overline{z}y - \overline{y}z)/2$ for any $y, z \in \mathbf{O}$. Let φ be a (1,1) tensor field defined by $\varphi(X) = X \times u$ for any $X \in T_pM$. Then $(\ \varphi,\ u,\ \eta,\ <\ ,\ >\ )$ is an almost contact metric structure on M^5. In fact, we see that

$$\varphi^2(X) = -X + \eta(X)u,$$
$$< \varphi(X),\ \varphi(Y) > = < X,\ Y > \ - \eta(X)\eta(Y),$$

3.2 Structure equations of hypersurfaces

We give the structure equations of hypersurface M^5. We set

$$F_x(M^5) = \{(\ p; (\ u,\ f,\ \overline{f}\ )) \in F \ \mid -2\sqrt{-1} f_1 \wedge \overline{f_1} = x \wedge \xi\}.$$

Then $\pi : F_x(M^5) \to M^5$ is the $U(2)$ principal right bundle over M^5 with the natural projection π. We call this an adapted frame bundle of M^5. By Proposition 2.2, we have the following structure equations on M^5.

$$\omega^1 = \overline{\omega^1} = 0 \ on\ F_x(M^5), \tag{6}$$

$$dx = u \otimes \eta + \sum_{i=2}^{3} \{f_i \otimes \omega^i + \overline{f_i} \otimes \overline{\omega^i}\}, \tag{7}$$

$$du = \sum_{\alpha=1}^{3} \{f_\alpha(-2\sqrt{-1}\theta^\alpha) + \overline{f_\alpha}(2\sqrt{-1}\ \overline{\theta^\alpha})\}, \tag{8}$$

$$df_2 = u(-\sqrt{-1}\sqrt{\theta^2}) + \sum_{\alpha=1}^{3} f_\alpha\ \kappa_2{}^\alpha - \overline{f_3}\ \theta^1 + \overline{f_1}\ \theta^3, \tag{9}$$

$$df_3 = u(-\sqrt{-1}\ \overline{\theta^3}) + \sum_{\alpha=1}^{3} f_\alpha\ \kappa_3{}^\alpha - \overline{f_1}\ \theta^2 + \overline{f_2}\ \theta^1, \tag{10}$$

$$df_1 = u(-\sqrt{-1}\ \overline{\theta^1}) + \sum_{\alpha=1}^{3} f_\alpha\ \kappa_1{}^\alpha - \overline{f_2}\ \theta^3 + \overline{f_3}\ \theta^2, \tag{11}$$

$$d\omega^2 = 2\sqrt{-1}\theta^2 \wedge \eta - \sum_{i=2}^{3} {\kappa_i}^2 \wedge \omega^i - \overline{\theta^1} \wedge \overline{\omega^3}, \tag{12}$$

$$d\omega^3 = 2\sqrt{-1}\theta^3 \wedge \eta - \sum_{i=2}^{3} {\kappa_i}^3 \wedge \omega^i - \overline{\theta^1} \wedge \overline{\omega^2}, \tag{13}$$

$$d\eta = \sqrt{-1}\sum_{i=2}^{3}(\overline{\theta^i} \wedge \omega^i - \theta^i \wedge \overline{\omega^i}), \tag{14}$$

$$\varphi^\flat = -(\sqrt{-1}/2)\sum_{i=2}^{3}(\omega^i \wedge \overline{\omega^i}), \tag{15}$$

where we have put $\varphi^\flat(X,Y) = < X, \varphi(Y) >$ for any $X,\ Y\ \in \mathbf{X}(M^5)$. By (6) and Cartan's lemma, there exist $\mathbf{C}$-valued function a, $M_{2\times1}(\mathbf{C})$-valued functions b, c and $M_{2\times2}(\mathbf{C})$-valued functions A, B, C on $F_x(M^5)$ satisfying

$$ {}^tA = A,\ {}^tC = C,$$

$$\begin{pmatrix} \theta^1 \\ \zeta \\ \Theta \end{pmatrix} = \begin{pmatrix} a & b & c \\ -2\sqrt{-1}\ {}^tb & A & B \\ 2\sqrt{-1}\ {}^t\overline{c} & C & {}^t\overline{B} \end{pmatrix} \begin{pmatrix} \eta \\ \mu \\ \overline{\mu} \end{pmatrix}, \tag{16}$$

where $\zeta = \begin{pmatrix} {\kappa_2}^1 \\ {\kappa_3}^1 \end{pmatrix}$, $\Theta = \begin{pmatrix} \theta^3 \\ -\theta^2 \end{pmatrix}$ and $\mu = \begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix}$. By (16), the second fundamental form $\mathbf{I}I$ is given by

$$\mathbf{I}I = -2\mathbf{R}e\ \{\ (\ 2\sqrt{-1}\eta \circ \theta^1 -{}^t \mu \circ \zeta -{}^t \overline{\mu} \circ \overline{\Theta}\ )f_1\}, \tag{17}$$

where $\circ$ denotes the symmetric product. Hence we have the following splitting:

$$\begin{aligned} \mathbf{I}I^{(2,0)} &= ({}^t\mu \circ A\mu)\ f_1, \\ \mathbf{I}I^{(1,1)} &= ({}^t\mu \circ B\overline{\mu} +{}^t \overline{\mu} \circ{}^t B\mu)\ f_1, \\ \mathbf{I}I^{(0,2)} &= ({}^t\overline{\mu} \circ C\overline{\mu})\ f_1. \end{aligned}$$

The mean curvature vector field $\mathbf{H}$ is given by

$$\mathbf{H} = -(4/5)\mathbf{R}e\{(\sqrt{-1}a - 2\mathrm{tr}B)\ f_1\}. \tag{18}$$

3.3 Quaternion structure on the contact distribution

Let $\mathbf{D} = \{X \in T\ (M^5)|\eta(X) = 0\}$ be the contact distribution of M^5. We define two almost complex structures on $\mathbf{D}$ as follows. For any $X \in T_p(M^5)$, we set

$$\varphi_1(X) = (X \times x)^\wedge,\ \varphi_2(X) = (X \times \xi)^\wedge \tag{19}$$

where $\wedge$ is the projection from TM to D. By the direct calculations, we get

$$\varphi_i{}^2(X) = -X + \eta(X)u,$$
$$< \varphi_i(X),\ \varphi_i(Y) > = < X,\ Y > \ - \eta(X)\eta(Y)$$

for $i = 1, 2$. Therefore if we restrict to this globally defined (1,1) tensor field to the contact distribution $\mathbf{D}$, then they are orthogonal almost complex structures on $\mathbf{D}$. Moreover we have

$$\begin{aligned} \varphi_0 \cdot \varphi_1 &= \varphi_2 = -\varphi_1 \cdot \varphi_0, \\ \varphi_1 \cdot \varphi_2 &= \varphi_0 = -\varphi_2 \cdot \varphi_1, \\ \varphi_2 \cdot \varphi_0 &= \varphi_1 = -\varphi_0 \cdot \varphi_2, \end{aligned} \tag{20}$$

where $\varphi_0 = \varphi$. Hence we can define an almost quaternion structure on the contact distribution $\mathbf{D}$. We may remark here that the vector space $span_{\mathbf{R}}\{u, x, \xi\}$ is closed under the vector cross product.

THEOREM 3.1. *Let M^5 be an oriented hypersurface in a 6-dimensio nal sphere. Then M^5 has an induced alomost contact metric structure $(\varphi,\ u,\ \eta,\ < \ ,\ >)$. Moreover, the contact distridution has three orthogonal almost complex structures which satisfy the relations (20). The structure group of the tangent bundle reduces to $1 \times Sp(1) \simeq 1 \times SU(2)$*

THEOREM 3.2. *Let ι and ι' be two isometric imersions from M^5 to a 6-dimensional sphere. If ι and ι' are G_2 congruent, then the corresponding almost quaternion structures coincide.*

Proof. From the definition of G_2 and the quaternion structure, we get the conclusion. □

COROLLARY 3.4. *Let M^5 be an oriented hypersurface in a 6-dimen sional sphere and R the cuvature tensor of M^5. Then*

$$< R(X, \varphi_i(X))\varphi_i(X), X >,\ < R(X, \varphi_i(X))\varphi_j(X), \varphi_k(X) >,$$

are G_2 invariants.

4. Chracterisric classes of hypersurfaces

4.1 1st Chern class of the contact distribution D

By Theorem 3.1, the structure group of the contact distribution D reduces to $SU(2)$. Then we have the following

PROPOSITION 4.1. *The 1st Chern class* $c_1(\mathbf{D}^{1,0})$ *of the contact distribution vanishes.*

4.2 1st Pontrjagin class of hypersurface

In this secion, we prove the following:

THEOREM 4.2. *Let* M^5 *be an oriented hypersurface in a 6-dimensional sphere. Then the both 1st Pontrjagin class of the tangent bundle and the 2nd Chern class of the contact distribution (with respect to any* φ_i*) vanish.*

Since the 1st Pontrjagin class is conformal invariants, by using the stereographic projection and Theorem 4.2, we have

COROLLARY 4.3. *Let* M^5 *be an oriented hypersurface in a 6-dimensional Euclidean space. Then the 1st Pontrjagin class of the tangent bundle vanishes.*

In order to prove Theorem 4.2, we prepare the following lemma.

LEMMA 4.4. *Let* $id \times x : R \times M^5 \to R \times S^6 \subset R \oplus \mathrm{Im}\mathbf{O} = \mathbf{O}$ *be the product immersion from* $R \times M^5$ *to the octonians* $\mathbf{O}$. *Then the induced almost complex structure* J *is given by*

$$J((\frac{d}{dt}, 0)) = (0, -u) \in TR \oplus TM^5,$$

$$J((0, X)) = (\eta(X)\frac{d}{dt}, \varphi(X))$$

and we have $(T R \oplus TM^5) = T R \oplus span_{\mathrm{R}}(u) \oplus \mathbf{D}$ *and* $TR \oplus span_{\mathrm{R}}(u)$ *is the* J*-invariant 2-dimensional distribution on* $R \times M^5$.

Proof. The induced almost complex structure is defined by

$$J(Z) = Z \times u$$

where $Z \in T(\mathrm{R} \times M^5)$. By the direct calculations, we get the desired results. □

Now we recall the following result:

COROLLARY 1. (in [3]) *Let M^6 be a 6-dimensional almost Hermitian submanifold immersed in the octonians* **O** *with flat normal connection. Then we have*

$$c_1(T^{(1,0)}) = 0 \text{ in } H^2(M^6; \mathbf{Z})$$

and

$$p_1(TM^6) = 0 \text{ and } c_2(T^{(1,0)}) = 0 \text{ in } H^4(M^6; \mathbf{Z}).$$

Proof of Theorem 4.1. By Lemma 4.4, we have

$$T\mathrm{R} \oplus TM^5 = T\mathrm{R} \oplus span_{\mathrm{R}}(u) \oplus \mathbf{D}.$$

By Corollary 1 in [3], we get

$$p_1(T\mathrm{R} \oplus TM^5) = p_1(TM^5) = 0 \qquad \square$$

Next we prove the second Chern class vanises. First we remark that

$$\begin{aligned}\mathbf{C} \otimes (T\mathbf{R} \oplus TM^5) &= \mathbf{C} \otimes (T\mathrm{R} \oplus span_{\mathrm{R}}(u) \oplus \mathbf{D}) \\ &= \mathbf{C} \otimes (\, T\mathrm{R} \oplus span_{\mathrm{R}}(u)) \oplus (\mathbf{C} \otimes \mathbf{D}).\end{aligned}$$

Hence we have

$$\mathbf{C}^{(1,0)}(T\mathbf{R} \oplus TM^5) = \mathbf{C}^{(1,0)}(\, T\mathrm{R} \oplus span_{\mathrm{R}}(u)) \oplus \mathbf{D}^{(1,0)}.$$

Since the subbundle $T\mathrm{R} \oplus span_{\mathrm{R}}(u)$ is a trivial bundle, we have

$$c(\mathbf{D}^{(1,0)}) = c(\mathbf{C}^{(1,0)}(T\mathbf{R} \oplus TM^5)),$$

where $c(\mathbf{D}^{(1,0)})$ is the total Chern class of $\mathbf{D}^{(1,0)}$. By Proposoton 4.1 and Corollary 1 in [3], we get the desired result.

5. Fundamental relations of the second fundamental form and the quaternion structure

Let $M^5 = (M^5, <\ ,\ >, x)$ be an oriented hypersurface in the unit 6-dimensional sphere S^6. We denote by x and ξ the position vector and unit normal vector field of M^5 in S^6, respectively. We denote by A_ξ the shape operator of the immersion x. Let e_1 be a local unit tangent vector field belonging to $\mathbf{D}$. Then $\{u,\ e_1,\ \varphi_0(e_1),\ \varphi_1(e_1),\ \varphi_2(e_1)\}$ is an orthonormal frame of M^5. We may put the complexified normal vector field and complexified orthonormal frame field such that

(21) $$f_1 = \frac{1}{2}(\xi - \sqrt{-1}x),$$

(22) $u,\ f_2 = \frac{1}{2}(e_1 - \sqrt{-1}\varphi_0(e_1)),\ f_3 = \frac{1}{2}(\varphi_1(e_1) - \sqrt{-1}\varphi_2(e_1)).$

Then we have

$$\varphi_0(f_2) = \sqrt{-1}f_2,\ \varphi_0(f_3) = \sqrt{-1}f_3,$$
$$\varphi_1(f_2) = \overline{f_3},\ \varphi_1(f_3) = -\overline{f_2},$$
$$\varphi_2(f_2) = -\sqrt{-1}\ \overline{f_3},\ \varphi_2(f_3) = \sqrt{-1}\ \overline{f_2}.$$

By (8) $\sim$ (11), we have the following relations:

(23)
$$\theta^1(X) = -\sqrt{-1} < d\overline{f_1}(X),\ u >= \frac{1}{2}\{\eta(X) + \sqrt{-1} < A_\xi(X),\ u >\},$$

(24)
$$\begin{aligned}\theta^2(X) &= 2 < df_1(X),\ f_3 > \\ &= -\frac{1}{2}\{< A_\xi(X),\ \varphi_1(e_1) > + < X,\ \varphi_2(e_1) > \\ &\quad + \sqrt{-1}(< X,\ \varphi_1(e_1) > - < A_\xi(X),\ \varphi_2(e_1) >)\},\end{aligned}$$

(25)
$$\begin{aligned}\theta^3(X) &= -2 < df_1(X),\ f_2 > \\ &= \frac{1}{2}\{< A_\xi(X),\ e_1 > + < X,\ \varphi_0(e_1) > \\ &\quad + \sqrt{-1}(< X,\ e_1 > - < A_\xi(X),\ \varphi_0(e_1) >)\},\end{aligned}$$

(26) $$\kappa_1{}^1 = 0,$$

(27)
$$\kappa_2{}^1\ (X) = -2 < d\overline{f_1}(X),\ f_2 >$$
$$= \frac{1}{2}\{< A_\xi(X),\ e_1 > - < X,\ \varphi_0(e_1) >$$
$$- \sqrt{-1}(< X,\ e_1 > + < A_\xi(X),\ \varphi_0(e_1) >)\},$$
$$\kappa_3{}^1\ (X) = -2 < d\overline{f_1}(X),\ f_3 >$$
$$= \frac{1}{2}\{< A_\xi(X),\ \varphi_1(e_1) > - < X,\ \varphi_2(e_1) >$$
$$- \sqrt{-1}(< X,\ \varphi_1(e_1) > + < A_\xi(X),\ \varphi_2(e_1) >)\},$$

By (16) and above equations, we have

$$a = \theta^1(u) = \frac{1}{2}(1 + \sqrt{-1}\ < A_\xi(u), u >),$$
$$b^1 = \theta^1(f_2) = -\frac{1}{4}\{ < A_\xi(u), \varphi_0(e_1) > + \sqrt{-1} < A_\xi(u), e_1 >\},$$
$$b^2 = \theta^1(f_3) = -\frac{1}{4}\{ < A_\xi(u), \varphi_2(e_1) > + \sqrt{-1} < A_\xi(u), \varphi_1(e_1) >\},$$
$$c^1 = \theta^1(\overline{f_2}) = \frac{1}{4}(< A_\xi(u), \varphi_0(e_1) > - \sqrt{-1} < A_\xi(u), e_1 >),$$
$$c^2 = \theta^1(\overline{f_3}) = \frac{1}{4}(< A_\xi(u), \varphi_2(e_1) > - \sqrt{-1} < A_\xi(u), \varphi_1(e_1) >),$$
$$A_1{}^1 = \kappa_2{}^1(f_2)$$
$$= \frac{1}{4}\{< A_\xi(e_1), e_1 > - < A_\xi(\varphi_0(e_1)), \varphi_0(e_1) >$$
$$- 2\sqrt{-1} < A_\xi(e_1), \varphi_0(e_1) >\},$$
$$A_1{}^2 = A_2{}^1 = \kappa_2{}^1(f_3) = \kappa_3{}^1(f_2)$$
$$= \frac{1}{4}\{< A_\xi(\varphi_1(e_1)), e_1 > - < A_\xi(\varphi_2(e_1)), \varphi_0(e_1) >$$
$$- \sqrt{-1}(< A_\xi(\varphi_2(e_1)), e_1 > + < A_\xi(\varphi_1(e_1)), \varphi_0(e_1) >)\},$$
$$A_2{}^2 = \kappa_3{}^1(f_3)$$
$$= \frac{1}{4}\{< A_\xi(\varphi_1(e_1)), \varphi_1(e_1) > - < A_\xi(\varphi_2(e_1)), \varphi_2(e_1) >$$
$$- 2\sqrt{-1} < A_\xi(\varphi_1(e_1)), \varphi_2(e_1) >\},$$

$$
\begin{aligned}
B_1{}^1 &= \kappa_2{}^1(\overline{f}_2)\\
&= \frac{1}{4}\{< A_\xi(e_1), e_1 > + < A_\xi(\varphi_0(e_1)), \varphi_0(e_1) > -2\sqrt{-1}\},\\
B_2{}^1 &= \kappa_2{}^1(\overline{f}_3)\\
&= \frac{1}{4}\{< A_\xi(e_1), \varphi_1(e_1) > + < A_\xi(\varphi_2(e_1)), \varphi_0(e_1) >\\
&\quad + \sqrt{-1}(< A_\xi(e_1), \varphi_2(e_1) > - < A_\xi(\varphi_1(e_1)), \varphi_0(e_1) >)\},\\
B_1{}^2 &= \kappa_3{}^1(\overline{f}_2)\\
&= \frac{1}{4}\{< A_\xi(e_1), \varphi_1(e_1) > + < A_\xi(\varphi_2(e_1)), \varphi_0(e_1) >\\
&\quad - \sqrt{-1}(< A_\xi\varphi_0(e_1), \varphi_1(e_1) > + < A_\xi(e_1), \varphi_2(e_1) >)\},\\
B_2{}^2 &= \kappa_3{}^1(\overline{f}_3)\\
&= \frac{1}{4}\{< A_\xi(\varphi_1(e_1)), \varphi_1(e_1) > + < A_\xi(\varphi_2(e_1)), \varphi_2(e_1) > -2\sqrt{-1}\},\\
C_i{}^j &= A_i{}^j
\end{aligned}
$$

for any i,j = 1,2.

Next we calculate the covariant derivatives of three almost complex structures on **D**. Let ∇ be a Riemannian connection on M^5.

PROPOSITION 5.1. *Let $M^5 = (\ M^5,\ <\ ,\ >, \iota\ )$ be an oriented hypersurface in a 6-dimensional sphere S^6. Then we have*

$$\nabla_X u = \varphi_1(A_\xi(X)) + \varphi_2(X), \tag{29}$$

$$
\begin{aligned}
(\nabla_X\varphi_0)Y = (- < X, \varphi_1(Y) > + < \varphi_2(A_\xi(X)), Y >)u\\
- \eta(X)\varphi_1(Y) + < A_\xi(X), u > \varphi_2(Y),
\end{aligned}
\tag{30}
$$

$$(\nabla_X\varphi_0)u = - \varphi_2(A_\xi(X)) + \varphi_1(X), \tag{31}$$

$$(\nabla_X\varphi_1)Y = -(< X, \varphi_0(Y) > + < A_\xi(X), Y >)u + \eta(X)\varphi_0(Y), \tag{32}$$

$$(\nabla_X\varphi_1)u = A_\xi(X) - \varphi_0(X) - < A_\xi(X), u > u, \tag{33}$$

$$
\begin{aligned}
(\nabla_X\varphi_2)Y = - (< X, Y > + < A_\xi(X), \varphi_0(Y) >)u\\
- < A_\xi(X), u > \varphi_0(Y),
\end{aligned}
\tag{34}
$$

$$(\nabla_X\varphi_2)u = \varphi_0(A_\xi(X)) + X - \eta(X)u, \tag{35}$$

where $Y \in \mathbf{D}$ *and* $X \in TM^5$.

Proof. For any $X \in TM^5$, we have

$$\begin{aligned}(36)\qquad \nabla_X u =& \nabla_X(x\xi) = (X\xi + A_\xi(X)x)^\top \\ =& \varphi_1(A_\xi(X)) + \varphi_2(X).\end{aligned}$$

where we denote by $\top$ is the projection from T_xS^6 to T_xM^5. Let D be the canonical connection of $\mathrm{Im}\mathbf{O}$. For any $X \in TM^5$ and $Y \in \mathbf{D}$, we calculate

(37)

$$\begin{aligned}(\nabla_X\varphi_0)Y &= \nabla_X(\varphi_0(Y)) - \varphi_0(\nabla_X Y) \\ &= \nabla_X(Y \times u) - (\nabla_X Y) \times u \\ &= ((D_X Y) \times u + Y \times (D_X u))^\top - (\nabla_X Y) \times u \\ &= ((\nabla_X Y + \mathbf{I}I(X,Y)) \times u)^\top \\ &\quad + (Y \times (\nabla_X u + \mathbf{I}I(X,u))^\top - (\nabla_X Y) \times u \\ &= (Y \times (\nabla_X u))^\top \\ &\quad + (Y \times (< \mathbf{I}I(X,u), \xi > \xi + < \mathbf{I}I(X,u), x > x))^\top.\end{aligned}$$

Without loss of essentiality, we may assume that the tangent vector Y has a unit length. Then $\{u, Y, \varphi_0(Y), \varphi_1(Y), \varphi_2(Y)\}$ forms an orthonormal frame. So we get

$$\begin{aligned}(38)\qquad (Y \times (\nabla_X u))^\top = & < Y \times (\nabla_X u), u > u \\ & + < Y \times (\nabla_X u), Y > Y \\ & + < Y \times (\nabla_X u), \varphi_0(Y) > \varphi_0(Y) \\ & + < Y \times (\nabla_X u), \varphi_1(Y) > \varphi_1(Y) \\ & + < Y \times (\nabla_X u), \varphi_2(Y) > \varphi_2(Y).\end{aligned}$$

On the other hand, for any tangent vector Z, we have

$$< Y \times (\nabla_X u), Z > = - < \nabla_X u, Y \times Z > .$$

This yields

(39)

$$\begin{aligned}< Y \times (\nabla_X u), u > =& - < \nabla_X u, Y \times u > \\ =& - < X, \varphi_1(Y) > + < \varphi_2(A_\xi(X)), Y >,\end{aligned}$$

$$(40) \qquad \begin{aligned} < Y \times (\nabla_X u), Y > &= < Y \times (\nabla_X u), \varphi_0(Y) > \\ &= < Y \times (\nabla_X u), \varphi_1(Y) > \\ &= < Y \times (\nabla_X u), \varphi_2(Y) > = 0. \end{aligned}$$

By (36) $\sim$ (40), we get (30). Similaly, we have (31) $\sim$ (35). □

Let us now consider the almost quaternion structure $\{\varphi_0, \varphi_1, \varphi_2\}$ on the contact distribution $\mathbf{D}$. First we define the induced connection $\tilde{\nabla}$ on $\mathbf{D}$ by putting $\tilde{\nabla}_X Y = (\nabla_X Y)^\wedge$ where $X \in \mathbf{D}_p$, and Y is a section of the contact distribution $\mathbf{D}$. Then we obtain

$$(41) \qquad (\tilde{\nabla}_X \varphi_0)Y = < A_\xi(u), X > \varphi_2(Y),$$

$$(42) \qquad (\tilde{\nabla}_X \varphi_1)Y = 0,$$

$$(43) \qquad (\tilde{\nabla}_X \varphi_2)Y = - < A_\xi(u), X > \varphi_0(Y)$$

for any $X, Y \in \mathbf{D}$. Also, we may consider each φ_i (restricted to $\mathbf{D}$) is a section of the endomorphism bundle End($\mathbf{D}$). Then we have

$$\tilde{\nabla}_X \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & < A_\xi(u), X > \\ 0 & 0 & 0 \\ - < A_\xi(u), X > & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix}$$

where $X \in \mathbf{D}$. Therefore the matrix

$$\tilde{A}(X) = \begin{pmatrix} 0 & 0 & < A_\xi(u), X > \\ 0 & 0 & 0 \\ - < A_\xi(u), X > & 0 & 0 \end{pmatrix}$$

is a connection 1-form on the subbundle $Span_{\mathbf{R}}(\varphi_0, \varphi_1, \varphi_2)$. The corresponding curvature tensor $\mathbf{R}$ is given by

$$\mathbf{R} = d\tilde{A} - \tilde{A} \wedge \tilde{A} = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ -t & 0 & 0 \end{pmatrix}$$

where we have put $t(X, Y) = < (2A_\xi \cdot \varphi_1 \cdot A_\xi + A_\xi \cdot \varphi_2 + \varphi_2 \cdot A_\xi)(X), Y >$ for any $X, Y \in \mathbf{D}$. Hence we have

PROPOSITION 5.2. *Let* $M^5 = (M^5, <\ ,\ >, x)$ *be an oriented hypersurface in a 6-dimensional sphere* S^6.

(1) *The subbundle* $Span_{\mathbf{R}}\{\varphi_0, \varphi_1, \varphi_2\}$ *is parallel in the endomorphism bundle* **End(D)** *if and only if*

$$< A_\xi(u), X >= 0$$

for any $X \in \mathbf{D}$.

(2) *The subbundle* $Span_{\mathbf{R}}\{\varphi_0, \varphi_1, \varphi_2\}$ *is flat in the endomorphism bundle* **End(D)** *if and only if*

$$< (2A_\xi \cdot \varphi_1 \cdot A_\xi + A_\xi \cdot \varphi_2 + \varphi_2 \cdot A_\xi)(X), Y >= 0$$

for any $X, Y \in \mathbf{D}$. *This condition is equivalent to the following:*

$$A_\xi \cdot \varphi_1 \cdot A_\xi + A_\xi \cdot \varphi_2$$

is a symmetric tensor on the contact distribution **D**.
(*This condition is also equivalent to the following:*

$$< A_\xi(\nabla_X u), Y >=< A_\xi(\nabla_Y u), X >$$

for any $X, Y \in \mathbf{D}$).

Next, we study hypersurfaces which satisfy the condition (1) of Proposition 5.2.

LEMMA 5.3. *Let* $M^5 = (M^5, <\ ,\ >, x)$ *be an oriented hypersurface in a 6-dimensional sphere* S^6. *The following conditions are equivalent;*

(1) $< A_\xi(u), X >= 0$ *for any* $X \in \mathbf{D}$,
(2) $A_\xi(u) = k_1 u$,
(3) $\nabla_u u = 0$,
(4) $L_u \eta = 0$.

Proof. By the direct calculations, we get the desired results. □

PROPOSITION 5.4. *Let* $M^5 = (M^5, <\ ,\ >, x)$ *be an oriented hypersurface in a 6-dimensional sphere* S^6 *which satisfies* $A_\xi(u) = k_1 u$. *Suppose that the subbundle* $Span_{\mathbf{R}}\{\varphi_0, \varphi_1, \varphi_2\}$ *is flat in the endomorphism bundle* **End(D)**. *Then* k_1 *is identically zero.*

Proof. We assume that k_1 is not identically zero and derive a contradiction. Let U be a connected component of the set $\{p \in M^5 | k_1(p) \neq 0\}$. We shall show that $dk_1(X) = 0$ for any $X \in \mathbf{D}$. In fact, by the Codazzi equation, we have

$$
\begin{aligned}
dk_1(X) &= < (\nabla_X A_\xi)(u), u > \\
&= < (\nabla_u A_\xi)(u), X > \\
&= u < A_\xi(u), X > - < A_\xi(X), \nabla_u u >= 0.
\end{aligned} \tag{44}
$$

By (2) of Proposition 5.2, 5.4 and the Codazzi equation, we obtain

$$< \nabla_X(A_\xi(u)), Y >=< \nabla_Y(A_\xi(u)), X >$$

on U. From this we get,

$$< \nabla_X u, Y >=< \nabla_Y u, X >$$

for any $X, Y \in \mathbf{D}$ on U. This equation yields

$$< (\varphi_1 \cdot A_\xi + A_\xi \cdot \varphi_1)(X), Y >= -2 < \varphi_2(X), Y > .$$

Let $\{e_i\}$ be principal vectors of the shape operator of A_ξ on $\mathbf{D}$ and let $\{\lambda_i\}$ be principal curvatures of A_ξ on $\mathbf{D}$, respectively. If we put $X = e_i$ and $Y = e_j$ in the above identity, then we have

$$(\lambda_i + \lambda_j) < \varphi_1(e_i), e_j >= -2 < \varphi_2(e_i), e_j >$$

on U. On the other hand, by (2) of Proposition 5.2, we have

$$2\lambda_i \cdot \lambda_j < \varphi_1(e_i), e_j > +(\lambda_i + \lambda_j) < \varphi_2(e_i), e_j >= 0$$

on U. These two equalities follow us to

$$(\lambda_i - \lambda_j)^2 < \varphi_2(e_i), e_j >= 0 \tag{45}$$

for any $i, j = 1, 2, 3, 4$. We may consider the following 4 cases;

Case(1) Four distinct principal curvatures of A_ξ on $\mathbf{D}$.

Case(2) Three distinct principal curvatures of A_ξ on $\mathbf{D}$.

Case(3) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$.

Case(4) $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$.

By (45), we have a contradiction in the cases (1)~(3). The only possibility the case (4). In this case, without loss of generality, we may assume the orthonormsal frame $\{u, e_1, e_2 = \varphi_2(e_1), e_3, e_4 = \varphi_2(e_3)\}$ is consisting of the principal vectors, and the shape operator A_ξ (with respect to this basis) is the following form

$$A_\xi = \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix}.$$

From this and the Codazzi equation , we have

$$< (\nabla_u A_\xi)(e_1), e_2 >= k_1 - \lambda, \ < (\nabla_{e_1} A_\xi)(u), e_2 >= 0.$$

Hence we have $k_1 = \lambda$. By using the same argument with respect to $\{e_3, e_4\}$, we have $k_1 = \lambda = \mu$. Hence the immersion x is totally umbilic on U. By (2) of Proposition 5.2, we have

$$k_1 \cdot \varphi_1 + \varphi_2 = 0$$

This is a contradiction. Hence we have $k_1 \equiv 0$ on M^5. □

THEOREM 5.5. *Let $M^5 = (\ M^5, <\ ,\ >, x\ )$ be an oriented complete hypersurface in a 6-dimensional sphere S^6 which satisfy $A_\xi(u) = k_1 u$. Suppose that the subbundle $Span_{\mathbf{R}}\{\varphi_0, \varphi_1, \varphi_2\}$ is flat in the endomorphism bundle* **End(D)**. *Then the immersion x is totally geodesic.*

Proof. We show that the index of minimum relativity nullity of M^5 is not less than 3. First we assume that the rank of $A_\xi|_{\mathbf{D}}$ is 4. we can take the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $\mathbf{D}$ such that $A_\xi|_{\mathbf{D}}(e_i) = \lambda_i e_i$. By (2) of Proposition 5.2, we have

$$\varphi_0 = 2A_\xi|_{\mathbf{D}} - \varphi_1 \cdot (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_2 \cdot A_\xi|_{\mathbf{D}}, \tag{46}$$

$$\varphi_2 = -\,2A_\xi|_{\mathbf{D}} \cdot \varphi_1 - \ A_\xi|_{\mathbf{D}} \cdot \varphi_2 \cdot (A_\xi|_{\mathbf{D}})^{-1}. \tag{47}$$

By (20), (46), (47) and (2) of Proposition 5.2, we obtain

$$\varphi_1 = A_\xi|_{\mathbf{D}} \cdot \varphi_2 \cdot (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_1 \cdot (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_2.$$

This yields

$$(48)\quad (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_1 \cdot (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_2 = -\varphi_2 \cdot (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_1 \cdot (A_\xi|_{\mathbf{D}})^{-1}.$$

On the other hand, we have

$$\begin{aligned} &< (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_1 \cdot (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_2(e_i), e_j > \\ &\quad = \frac{2}{\lambda_j}\delta_{ij} - \frac{1}{\lambda_i\lambda_j} < \varphi_0(e_i), e_j >, \end{aligned}$$

$$\begin{aligned} &- < \varphi_2 \cdot (A_\xi|_{\mathbf{D}})^{-1} \cdot \varphi_1 \cdot (A_\xi|_{\mathbf{D}})^{-1}(e_i), e_j > \\ &\quad = -\frac{2}{\lambda_i}\delta_{ij} + \frac{1}{\lambda_i\lambda_j} < \varphi_0(e_i), e_j > . \end{aligned}$$

By (48) and the above two equalities, if we take $i = j$, then we have $\frac{1}{\lambda_i} = 0$. This is a contradiction. Hence the rank of $A_\xi|_{\mathbf{D}}$ is less than 4.

Next we assume that the rank of $A_\xi|_{\mathbf{D}}$ is 3. Then we can take an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $\mathbf{D}$ such that $A_\xi|_{\mathbf{D}}(e_i) = \lambda_i\ e_i$ for $i = 1, 2, 3$ and $A_\xi|_{\mathbf{D}}(e_4) = 0$, where $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$. By Proposition 5.2, we obtain

$$\lambda_1 < e_1, \varphi_2(e_4) >= \lambda_2 < e_2, \varphi_2(e_4) >= \lambda_3 < e_1, \varphi_2(e_4) >= 0.$$

Also we have $< e_4, \varphi_2(e_4) >= 0$, this is a contradiction. Therefore the rank of $A_\xi|_{\mathbf{D}}$ is less than 3. By Proposition 5.4, we see that the index of minimum relativity nullity is greater than or equal to 3. Hence we get the desired conclusion by the following.

PROPOSITION 5.6. (Dajczer([2]) pp 99-100) *Let $f : M^n \to S^{n+1}(1)$ be a complete hypersurface in a unit sphere with index of minimum relativity nullity of f $\geq$ n-2. ($n \geq 4$). Then the immersion f is totally geodesic.* □

Here we consider the special case in (2) of Proposition 5.2.

THEOREM 5.7. *Let $M^5 = (M^5, < , >, x)$ be an oriented complete hypersurface in a 6-dimensional sphere S^6.*

$$A_\xi \cdot \varphi_1 \cdot A_\xi + A_\xi \cdot \varphi_2 = 0$$

on $\mathbf{D}$ *(or equivalently $A_\xi(\nabla_X u) = 0$ for any $X \in \mathbf{D}$) if and only if the immersion x is totally geodesic.*

Proof. We shall show that $A_\xi|_{\mathbf{D}}$ is identically zero. If the rank of $A_\xi|_{\mathbf{D}}$ is 4, by (2) of Proposition 5.2, then we have

$$A_\xi|_{\mathbf{D}} = \varphi_0$$

Since $A_\xi|_{\mathbf{D}}$ is symmetric and φ_0 is skew symmetric, we get $A_\xi|_{\mathbf{D}} = 0$. This is a contradiction.

Next, we suppose that the rank of $A_\xi|_{\mathbf{D}}$ is 3. Then we can take an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $\mathbf{D}$ such that $A_\xi|_{\mathbf{D}}(e_i) = \lambda_i e_i$ for $i = 1, 2, 3$ and $A_\xi|_{\mathbf{D}}(e_4) = 0$, where $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$ By Proposition 5.2, we obtain

$$\lambda_1 < e_1, \varphi_2(e_4) >= \lambda_2 < e_2, \varphi_2(e_4) >= \lambda_3 < e_1, \varphi_2(e_4) >= 0.$$

Since we have $< e_4, \varphi_2(e_4) >= 0$, this is a contradiction.

Now, we suppose that the rank of $A_\xi|_{\mathbf{D}}$ is 2. Then we can take an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $\mathbf{D}$ such that $A_\xi|_{\mathbf{D}}(e_i) = \lambda_i e_i$ for $i = 1, 2$, and $A_\xi|_{\mathbf{D}}(e_j) = 0$ for $j = 3, 4$, where $\lambda_1 \cdot \lambda_2 \neq 0$. By Proposition 5.2, we obtain

$$\varphi_2(e_3) = \pm e_4, \ \varphi_2(e_4) = \mp e_3.$$

and

$$\varphi_1 \cdot A_\xi(e_1) + \varphi_2(e_1) \in span_{\mathbf{R}}\{e_3, e_4\}.$$

Using the above two relations, we have

$$\lambda_1 \varphi_0(e_1) + e_1 \in span_{\mathbf{R}}\{e_3, e_4\}.$$

This yields

$$1 =< \lambda_1 \varphi_0(e_1) + e_1, e_1 >=< ae_3 + be_4, e_1 >= 0$$

for some $a,\ b\ \in \mathbf{R}$. We have also a contradiction.

Lastly, we suppose that the rank of $A_\xi|_{\mathbf{D}}$ is 1. Then we can take an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $\mathbf{D}$ such that $A_\xi|_{\mathbf{D}}(e_1) = \lambda_1 e_1$ and $A_\xi|_{\mathbf{D}}(e_j) = 0$ for $j = 2, 3, 4$, where $\lambda_1 \neq 0$. By Proposition 5.2, we obtain $A_\xi|_{\mathbf{D}} \cdot \varphi_1 \cdot (A_\xi|_{\mathbf{D}} - \varphi_0) = 0$. On the other hand, we have

$$A_\xi|_{\mathbf{D}} \cdot \varphi_1 \cdot (A_\xi|_{\mathbf{D}} - \varphi_0)(\varphi_2(e_1)) = -\lambda_1 e_1$$

This is a contradiction. Hence we have $A_\xi|_{\mathbf{D}} = 0$.

From mentioned above observations, the shape operator A_ξ with respect to an orthonormal basis $\{\ u,\ e_1,\ e_2,\ e_3,\ e_4\ \}$ is given by

$$A_\xi = \begin{pmatrix} a & k & l & m & n \\ k & 0 & 0 & 0 & 0 \\ l & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the characteristic polinomial of A_ξ is given by

$$\mid x\mathbf{I} - A_\xi \mid = x^3(x^2 - ax - (k^2 + l^2 + m^2 + n^2)).$$

So, there exists an ortonormal frame $\{\ \tilde{e_1},\ \tilde{e_2},\ \tilde{e_3},\ \tilde{e_4},\ \tilde{e_5}\ \}$ such that

$$A_\xi = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\lambda_1 + \lambda_2 = a =< A_\xi(u), u >$ and $\lambda_1 \cdot \lambda_2 = -(k^2 + l^2 + m^2 + n^2)$. By proposition 5.6, we get the desired result. □

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