Pusan Kyŏngnam Math. J. 13(1997), No. 1, pp. 131–138

EMBEDDING OF ORBITAL LYAPUNOV STABILITY IN FLOWS

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By a C^r flow, $0 \leq r \leq \infty$, on a C^r manifold M we mean a C^r map $\phi : M \times \mathbb{R} \to M$ such that

(1) $\phi(p, 0) = p$, for $p \in M$;

(2) $\phi(\phi(p,s),t) = \phi(p, s+t)$, for $p \in M$, and $s, t \in \mathbb{R}$

We can easily see that for each $t \in \mathbb{R}$ the *transition map* $\phi^t : M \to M$ given by

$$\phi^t(p) = \phi(p,t), \quad p \in \mathbb{M},$$

is a C^r diffeomorphism, and for each $p \in \mathbb{M}$, the *orbit map* $\phi_p : \mathbb{R} \to M$ given by

$$\phi_p(t) = \phi(p,t), \quad p \in \mathbb{R},$$

is a C^r map. If $r = 0$ then ϕ corresponds to a (continuous) flow on M and each transition map ϕ^t corresponds to a homeomorphism on M.

For any $p \in \mathbb{M}$ the *orbit* of ϕ through p will be denoted by the set

$$O(p) \equiv \{\phi(p,t) : t \in \mathbb{R}\}.$$

A point $p \in \mathbb{M}$ is said to be *fixed* under a C^r flow ϕ if $\phi(p,t) = p$ for any $t \in \mathbb{R}$, and $p \in M$ is called *regular* if it is not fixed. We can see that each orbit $O(p)$, $p \in M$, is a 1-dimensional immersed submanifold of M if p is regular and $r \geq 1$. For any $p \in M$, we let

$$L^+(p) \equiv \{q \in M : \phi(p,t_n) \to q, \text{ for some } t_n \to \infty\} \quad \text{and}$$
$$L^-(p) \equiv \{q \in M : \phi(p,t_n) \to q, \text{ for some } t_n \to -\infty\}.$$

Elaydi and Farran [4] introduced the notions of Lipschitz stable dynamical systems and Lyapunov stable dynamical systems on Riemannian manifold, and Chen, Chu and Lee studied the systems in [1,

Received Apr. 7, 1997. Revised May 25, 1997.
This work was supported by a grant from the Ministry of Education, 1996.

2, 3]. Moreover Kim, Kye and Lee [5] analysed the concepts of orbital Lipschitz stability in variation of C^r flow on a Riemannian manifold.

In this paper we will introduce the notion of orbital Lyapunov stability in variation of C^r flows(or C^r diffeomorphisms) on a Riemannian manifold M, using the norm on the tangent bundle TM of M, and will study the embedding problem of the orbital Lyapunov stability and orbital Lyapunov stability in variation of C^r diffeomorphisms in C^r flows.

Throughout the paper we let M denote a Riemannian manifold with a Riemannian metric g on M, and let $\|\cdot\|$ be the norm on the tangent bundle TM of M induced by g. A C^1 flow ϕ on M is said to be *Lyapunov stable* at $p \in M$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(\phi^t(p), \phi^t(q)) < \epsilon$$

for any $t \in \mathbb{R}$ and any $q \in M$ with $d(p,q) < \delta$. We say that ϕ is Lyapunov stable in variation at $p \in M$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|D\phi^t(V)\| < \epsilon$$

for all $t \in \mathbb{R}$ and $V \in T_pM$ with $\|V\| < \delta$, where $D\phi^t$ denotes the derivative map of the transition map $\phi^t : M \to M$. ϕ is said to be *Lyapunov stable*(or *Lyapunov stable in variation*) in a subset A of M if ϕ is Lyapunov stable(or Lyapunov stable in variation) at every point of A, respectively, and one can choose $\delta > 0$ independently of the points in A. If ϕ is Lyapunov stable(or Lyapunov stable in variation) in M, then we say that ϕ is Lyapunov stable (or Lyapunov stable in variation), respectively.

A flow ϕ on M is said to be *orbitally Lyapunov stable at* $p \in M$ if for any $\epsilon > 0$ there exist $\delta = \delta(\epsilon, p) > 0$ such that

$$d(\phi^t(x), \phi^t(y)) < \epsilon$$

for all $t \in \mathbb{R}$ and any $x, y \in O(p)$ with $d(x,y) < \delta$. We say that ϕ is *orbitally Lyapunov stable* in a subset A of M if ϕ is orbitally Lyapunov stable at every point of A and we can choose $\delta > 0$ independently of the points of A.

Now we introduce the notion of orbital Lyapunov stability in variation of C^1 flows on M.

DEFINITION 1. A C^1 flow ϕ on M is said to be *orbitally Lyapunov stable in variation* at $p \in M$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$||D\pi^t(V)|| < \epsilon$$

for any $t \in \mathbb{R}$ and any $V \in TO(p)$ with $||V|| < \delta$, where $TO(p) = \bigcup_{x\in p[-1,1]} T_xO(p)$ and $p([-1,1]) = \{\phi(p,t) : -1 \le t \le 1\}$.

We say that ϕ is *orbitally Lyapunov stable in variation* if ϕ is orbitally Lyapunov stable in variation at every point of M, and one can choose $\delta > 0$ independently of the points of M.

THEOREM 2. *A C^1 flow ϕ on M is orbitally Lyapunov stable in variation at $p \in M$ if and only if it is orbitally Lyapunov stable in variation at every point of $O(p)$.*

Proof. Suppose ϕ is not orbitally Lyapunov stable at $q \in O(p)$, and let $q = \phi^\tau(p)$ for some $\tau \in \mathbb{R}$. Then there exists $\epsilon > 0$ such that for any $a > 0$, we can choose $t \in \mathbb{R}$, $x \in \{\phi(q,t) : -1 \le t \le 1\}$, and $V \in T_xO(p)$ such that

$$||V|| < a \ \text{ and } \ ||D\phi^t(V)|| > \epsilon.$$

Since ϕ is orbitally Lyapunov stable at $p \in M$, we can choose $\delta' > 0$ such that if $U \in T_xO(p)$ for any $x \in \{\phi(p,t) : -1 \le t \le 1\}$ with $||U|| < \delta'$ then

$$||D\phi^t(U)|| < \epsilon \ \text{ for all } \ t \in \mathbb{R}.$$

Moreover we can select $\delta > 0$ such that if $V \in T_yO(p)$ for some $y \in \{\phi(q,t) : -1 \le t \le 1\}$ with $||V|| < \delta$, Then

$$||D\phi^{-\tau}(V)|| < \delta'.$$

Since ϕ is not orbitally Lyapunov stable at q, we can choose $s \in \mathbb{R}$ and $V \in T_xO(p)$ for some $x \in \{\phi(q,t) : -1 \le t \le 1\}$, satisfying

$$||V|| < \delta \ \text{ and } \ ||D\phi^s(V)|| > \epsilon.$$

Let $U = D\phi^{-\tau}(V)$. Then we have $U \in T_xO(p)$ for some $x \in \{\phi(p,t) : -1 \leq t \leq 1\}$, and $||U|| < \delta'$. Consequently we obtain

$$||D\phi^s(V)|| = ||D\phi^{s+\tau}(U)|| < \epsilon.$$

This contradiction implies that ϕ is orbitally Lyapunov stable at every point of $O(p)$.

THEOREM 3. *Let M be a complete, connected Riemannian manifold. If a C^1 flow ϕ on M is orbitally Lyapunov stable in variation at $p \in M$ then it is orbitally Lyapunov stable at $p \in M$.*

Proof. Let $\epsilon > 0$ be arbitrary number. Since ϕ is orbitally Lyapunov stable in variation at $p \in M$, we can select $\delta > 0$ such that if $V \subset T_xO(p)$ for some $x \in O(p)$ and $||V|| < \delta$ then

$$||D\phi^t(V)|| < \epsilon \text{ for all } t \in \mathbb{R}.$$

Moreover we can choose $\delta' > 0$ such that if x and y are any two points in $O(p)$ with $d(x,y) < \delta'$ then there exists a path $\alpha : [0,1] \to O(p)$ satisfying

$$\alpha(0) = x, \quad \alpha(1) = y, \quad \text{and} \quad ||\alpha'(s)|| < \delta,$$

for $s \in [0,1]$. For any $t \in \mathbb{R}$, we have

$$\begin{aligned} d(\phi^t(x), \phi^t(y)) &\leq \int_0^1 ||(\phi^t \circ \alpha)'(s)||\, ds \\ &\leq \int_0^1 ||D\phi^t(\alpha'(s))||\, ds \\ &< \epsilon \end{aligned}$$

This means that ϕ is orbitally Lyapunov stable at p.

Here we give an example to show that a flow ϕ need not be orbitally Lyapunov stable at a point $p \in M$ even if the speed of the orbit is constant at every point of $O(p)$.

EXAMPLE 4. Let $M = \{(x,y) \in \mathbb{R}^2 : -3 \le x \le 3\}$ and $D = \{(x,y) \in M : x^2 + y^2 < 1\}$. Consider a C^1 dynamical system ϕ on the space M with the constant speed at every point of $M - D$, and the following properties.

Let $A = \{(x,y) \in M : y = -3 \text{ or } 3\}$. For any $x \in M - (A \cup \overline{D})$, we have

$$L^+(x) = A \text{ and } L^-(x) = S^1.$$

For any $x \in D - \{(0,0)\}$, we get

$$L^+(x) = S^1 \text{ and } L^-(x) = \{(0,0)\}.$$

If $x \in A$, Then we have

$$L^+(x) = L^-(x) = \emptyset$$

Every point of S^1 is periodic and $(0,0)$ is the unique fixed point of ϕ. Then we can see that ϕ is not orbitally Lyapunov stable at p. To show this, we choose two sequences $\{x_n\}, \{y_n\}$ in

$$O(p) \cup \{(0,y) : 1 < y < 3\}$$

which are converging to $(0,3)$, and $x_n \ne y_n$ for each $n = 1, 2, \ldots$. Then we have

$$d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Moreover, we can choose a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that

$$\frac{d(\phi^{t_n}(x_n), \phi^{t_n}(y_n))}{d(x_n, y_n)} \to \infty$$

as $n \to \infty$. This means that ϕ is not orbitally Lyapunov stable at p.

Let f be a homeomorphism on M. If there exists a C^r flow ϕ on M such that $\phi^t = f$ for some $t \in \mathbb{R}^+$ then we say that f is C^r *embedded* in the flow ϕ.

The embedding problem in dynamics theory is the study of the existence of such flow ϕ. In [5], Kim, Kye and Lee studied the embedding problem of the orbital Lipschitz stability and showed that if a orbitally Lipschitz stable diffeomorphism f on M is C^1 embedded in a flow ϕ on M then ϕ is also orbitally Lipschitz stable ; but they claimed that if a homeomorphism f on M is C^0 embedded in a flow ϕ on M and f is orbitally Lipschitz stable under ϕ, then ϕ need not be orbitally Lipschitz stable.

Here we will show that if a homeomorphism f on a compact space M is C^0 embedded in a flow ϕ on M and f is orbitally Lyapunov stable under ϕ then ϕ is also orbitally Lyapunov stable.

DEFINITION 5. Let f be a homeomorphism on M, and suppose f is C^0 embedded in a flow ϕ on M. We say that f is orbitally Lyapunov stable at $p \in M$ under ϕ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(f^n(x), f^n(y)) < \epsilon$$

for any $n \in \mathbb{Z}$ and any $x, y \in O(p)$ with $d(x, y) < \delta$, where $O(p)$ is the orbit of ϕ through p.

The followimg example shows that even if a diffeomorphism f on M is C^1 embedded in a flow ϕ on M and f is Lyapunov stable under ϕ , ϕ need not be Lyapunov stable.

EXAMPLE 6. Let us conder the C^1 flow ψ on $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ generated by the differential system ;

$$\begin{cases} \dot{x} = -y + \frac{x^2+y^2+1}{2y} \\ \dot{y} = x \end{cases}$$

Then the orbit of ψ is periodic with the period 2π, and the orbit $O(0, b)$ passing through a point $(0, b), b > 0$, in M is the circle

$$\{(x, y) \in M : x^2 + (y - \frac{1+b^2}{2b})^2 = (\frac{1-b^2}{2b})^2\}.$$

Let $\phi : M \times \mathbb{R} \to M$ be the flow on M given by

$$\phi(p, t) = \psi(p, 2\pi t),$$

and let $\phi^1 = f$. Then it is clear that f is Lyapunov stable at every point of M. However we can see that ϕ is not Lyapunov stable.

THEOREM 7. *Let M be a compact metric space, and suppose a homeomorphism f on M is C^0 embedded in a flow ϕ on M. Then ϕ is orbitally Lyapunov stable if and only if f is orbitally Lyapunov stable under ϕ.*

Proof. Since f is embedded in a flow ϕ on M, there exists $u \in \mathbb{R}^+$ with $\phi^u = f$. Suppose ϕ is not orbitally Lyapunov stable at $p \in M$, and let

$$L_{xy} = sup\{d(\phi^t(x), \phi^t(y)) : 0 \leq t \leq u\},$$

for any $x, y \in M$. Then we can choose $\epsilon > 0$ such that for any $\delta > 0$ there exist $x, y \in O(p)$ satisfying

$$L_{xy} < \delta \quad \text{and} \quad d(\phi^t(x), \phi^t(y)) > \epsilon.$$

for some $t \in \mathbb{R}$. Since f is orbitally Lyapunov stable at $p \in M$ under ϕ, given $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$d(f^n(x), f^n(y)) < \epsilon,$$

for any $n \in \mathbb{Z}$ and $x, y \in O(p)$ with $d(x, y) < \delta_1$. For the $\delta_1 > 0$, we choose $z, w \in O(p)$ satisfying

$$L_{zw} < \delta_1 \quad \text{and} \quad d(\phi^s(z), \phi^s(w)) > \epsilon$$

for some $s \in \mathbb{R}$. Select $n \in \mathbb{Z}$ and $0 \leq \alpha < u$ such that $s = nu + \alpha$. Let $\phi^\alpha(z) = z'$ and $\phi^\alpha(w) = w'$. Then we have

$$d(z', w') < L_{zw} < \delta_1, \quad \text{and}$$

$$d(f^n(z'), f^n(w') = d((\phi^{nu}(z'), \phi^{nu}(w') = d(\phi^s(z), \phi^s(w)) > \epsilon$$

The contradiction proves the theorem.

Acknowledgments I would like to express my gratitude to Ewha Women's University for the great help during my study in the last two semesters(MAR '96 - FEB '97).

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