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FUZZY STOCHASTIC PROCESS UNDER PROBABILITY SPACES

Choon Il Park

1. Introduction

Fuzzy random variables are used to describe fuzzy stochastic phenomena mathematically for one-period time.

For phenomena that are similar to this kind of dynamic fuzzy stochastic phenomena, it is not enough to describe and observe their evolutional procedures. For these requirements, many authors have studied fuzzy stochastic processes [1, 2, 11, 12, 13, 15].

Fuzzy random variables and fuzzy random vectors generalize random variables and random vectors, respectively: they also generalized random sets [4, 6, 7, 9, 10].

Karlin [6] introduced the concept of fuzzy variables as a function $\Omega \to F(R)$, where (Ω, A, P) is a probability space, and $F(R)$ denotes all piecewise continuous functions $u : R \to [0,1]$ for the real line R.

Puri and Ralescu [10] defined the notion of a fuzzy random variable as a function $F : \Omega \to F(R^n)$, where $F(R^n)$ denotes all functions $u : R^n \to [0,1]$ such that $\{x \in R^n : u(x) \leq \alpha\}$ is nonempty and compact for each $\alpha \in (0,1]$.

In this paper, $X : \Omega \to F_0(R)$ denotes a measurable fuzzy set-valued function, where $F_0(R) = \{A : R \to [0,1]\}$ and $\{x \in R : A(x) \leq \alpha\}$ is a bounded closed interval for each $\alpha \in (0,1]$.

2. Dynamic fuzzy sets and fuzzy random variables

Let U be a nonempty usual set, $P(U)$ denote the set of all subsets of U, and $F(U)$ denote the set of all fuzzy subsets of U. For $A \in F(U)$ we define two subsets of U as follows :

$$A_\alpha = \{x \in U : A(x) \leq \alpha\} \quad \text{for any} \quad \alpha \in [0,1]$$

$$A_{\overline{\alpha}} = \{x \in U : A(x) \leq \alpha\}^c \quad \text{for any} \quad \alpha \in [0,1].$$

We have the following lemmas.

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LEMMA 2.1. *For $A \in F(U)$, let $\{B_\alpha : \alpha \in [0,1]\}$ be a class of subsets of U such that $A_{\overline{\alpha}} \subset B_\alpha \subset A_\alpha$ for any $\alpha \in [0,1]$. Then*

$$A = \bigcup_{\alpha \in [0,1]} \alpha B_\alpha.$$

Let (R, B) be the Borel measurable space and $F_0(R)$ denote the set of fuzzy subsets $A : R \to [0,1]$ with following properties :

(1) $\{x \in R : A(x) = 1\} \neq \phi$

(2) $A_\alpha = \{x \in R : A(x) \geq \alpha\}$ *is a bounded closed interval in R for each $\alpha \in (0,1]$, i.e., $A \in F_0(R)$ is a bounded closed fuzzy number (BCFN).*

LEMMA 2.2. *Let*

$$H : (0,1] \to I(R) = \{[x,y] : x, y \in R, x \leq y\},$$
$$\alpha \mapsto H(\alpha) = [m_{\alpha_1}, n_{\alpha_2}]$$

satisfy the following condition :

$$\alpha_1 < \alpha_2 \Rightarrow [m_{\alpha_1}, n_{\alpha_1}] \supset [m_{\alpha_2}, n_{\alpha_2}].$$

Then

$$(1) \qquad A = \bigcup_{\alpha \in (0,1]} \alpha H(\alpha) = \bigcup_{\alpha \in (0,1]} [m_{\alpha_2}, n_{\alpha_2}] \in F_0(R);$$

$$(2) \qquad A_\alpha = \bigcap_{n=1}^{\infty} H(\alpha_n) = \bigcap_{n=1}^{\infty} [m_{\alpha_n}, n_{\alpha_n}],$$

where $\alpha_n = (1 - \frac{1}{(n+1)})\alpha$ and $\alpha \in (0,1]$.

DEFINITION 2.1. $\{A(t) : t \in T \subset R\}$ is a dynamic fuzzy set (DFS) in U with respect to T if $A(t) \in F(U)$ for every $t \in T$. In particular, $\{A(t) : t \in T\}$ is a normal dynamic fuzzy set (NDFS) if $A(t) \in F_0(R)$ for every $t \in T$.

DEFINITION 2.2. Let $A(t)$ be a NDFS with respect to T and

$$\begin{aligned} A_\alpha &: T \to I(R) = \{[x,y] : x, y \in R, \quad x \le y\} \\ t &\mapsto A_\alpha(t) = (A(t))_\alpha = [A_\alpha^-(t), A_\alpha^+(t)]. \end{aligned}$$

DEFINITION 2.3. Let (Ω, A, P) be a probability space and ζ is a set-valued mapping

$$\begin{aligned} \zeta &: \Omega \to I(R) = \{[x,y] : x, y \in R, \quad x \le y\} \\ \omega &\mapsto \zeta(\omega) = [\zeta^-(\omega), \zeta^+(\omega)]. \end{aligned}$$

DEFINITION 2.4. Let (Ω, A, P) be a probability space. A fuzzy set-valued mapping $X : \Omega \to F_0(R)$ is a fuzzy random variable (FRV) if for every $B \in R$ and every $\alpha \in (0,1]$.

$$X_\alpha^-(B) = \{\omega \in \Omega : X_\alpha(\omega) \cap B \neq \phi\} \in A.$$

A fuzzy set-valued mapping $X : \Omega \to F_0^m(R) = F_0(R) \times \cdots \times F_0(R)$ is represented by $X(\omega) = (X(1,\omega), \cdots, X(m,\omega))$.

THEOREM 2.1. *$X(\omega)$ is a FRV if and only if $X_\alpha(\omega) = [X_\alpha^-(\omega), X_\alpha^+ (\omega)]$ is a random interval for every $\alpha \in (0,1]$ and*

$$\begin{aligned} X(\omega) &= \bigcup_{\alpha \in (0,1]} \alpha X_\alpha(\omega) \\ &= \bigcup_{\alpha \in (0,1]} \alpha [X_\alpha^-(\omega), X_\alpha^+(\omega)] \end{aligned}$$

$$(X(\omega))_\alpha = X_\alpha(\omega) = \bigcap_{n=1}^{\infty} [X_\alpha^-(\omega_n), X_{\alpha_n}^+(\omega)]$$

where for any $\alpha \in (0,1]$, $\alpha_n = (1 - \frac{1}{(n+1)})\alpha$,

$$\begin{aligned} X_\alpha^-(\omega) &= \inf X_\alpha(\omega) \\ &= \inf\{x \in R : X(\omega)(x) \ge \alpha\} \end{aligned} \tag{2.1}$$

$$(2.2) \qquad \begin{aligned} X_\alpha^+(\omega) &= \sup X_\alpha(\omega) \\ &= \sup\{x \in R : X(\omega)(x) \geq \alpha\} \end{aligned}$$

with $X(\omega)(x)$ is the membership function of $X(\omega)$.

Proof. It follows from (2.1), (2.2) and $X(\omega) \in F_0(R)$ that for any $\alpha \in (0,1]$,

$$(2.3) \qquad X_\alpha(\omega) = [X_\alpha^-(\omega), X_\alpha^+(\omega)], \quad X_\alpha^-(\omega), X_\alpha^+(\omega) \in X_\alpha(\omega).$$

Then for any $x \in R$ and any $\alpha \in (0,1]$,

$$\begin{aligned} &\{\omega \in \Omega \ ; \ X_\alpha^-(\omega) \leq x \ \} \\ =&\{\omega \in \Omega \ ; \ X_\alpha^-(\omega) > x \ \}^c \\ =&\{\omega \in \Omega \ ; \ X_\alpha^-(\omega) \in (x, +\infty)\}^c \\ =&\{\omega \in \Omega \ ; \ [X_\alpha^-(\omega), X_\alpha^+(\omega)] \subset (x, +\infty)\}^c \\ =&\{\omega \in \Omega \ ; \ X_\alpha(\omega) \cap (x, +\infty) \neq \emptyset\}^c \in A, \end{aligned}$$

$$\begin{aligned} \{\omega \in \Omega \ ; \ X_\alpha^+(\omega) \leq x \ \} &= \{\omega \in \Omega \ ; \ X_\alpha^+(\omega) \in (-\infty, x]\} \\ &= \{\omega \in \Omega \ ; \ [X_\alpha^-(\omega), X_\alpha^+(\omega)] \subset (-\infty, x]\} \\ &= \{\omega \in \Omega \ ; \ X_\alpha(\omega) \cap (-\infty, x] \neq \emptyset \ \} \in A. \end{aligned}$$

Let

$$\begin{aligned} H : (0,1] \to I(R) = \{[x,y]; x, y \in R, x \leq y\}, \\ \alpha \mapsto H(\omega) = [X_\alpha^-(\omega), X_\alpha^+(\omega)]. \end{aligned}$$

When $\alpha_1 < \alpha_2$, we have $X_{\alpha_1}(\omega) \supset X_{\alpha_2}(\omega)$. Thus

$$[X_{\alpha_1}^-(\omega), X_{\alpha_1}^+(\omega)] \supset [X_{\alpha_2}^-(\omega), X_{\alpha_2}^+(\omega)]$$

by (2.3). It follows from Lemma 2.2 that

$$\begin{aligned} X(\omega) &= \bigcup_{\alpha \in (0,1]} \alpha H(\alpha) \\ &= \bigcup_{\alpha \in (0,1]} \alpha [X_\alpha^-(\omega), X_\alpha^+(\omega)], \end{aligned}$$

$$(X(\omega))_\alpha = X_\alpha(\omega) = \bigcap_{n=1}^{\infty} [X_{\alpha_n}^-(\omega), X_{\alpha_n}^+(\omega)],$$

where $\alpha_n = (1 - \frac{1}{(n+1)})\alpha)\alpha, \alpha \in (0,1]$. Conversely, let

$$B_1 = \{b \in B\,; I(b) \text{ is an isolated set}\}.$$

Then

$$B \setminus B_1 = \bigcap_{n=1}^{\infty} < a_k, b_k > .$$

where $< a_k, b_k >$ could be any one of the following : $(a_k, b_k), [a_k, b_k], [a_k, b_k)$ and $(a_k, b_k]$. Then for any $\alpha \in (0,1]$,

$$\begin{aligned}
& \{\omega \in \Omega\,; X_\alpha(\omega) \subset B\} \\
= & \left\{\omega \in \Omega\,; [X_\alpha^-(\omega), X_\alpha^+(\omega)] \subset B_1 \cup \left(\cup_{n=1}^{\infty} < a_k, b_k >\right)\right\} \\
= & \left(\bigcup_{c \in B} \{\omega \in \Omega\,; X_\alpha^-(\omega) = X_\alpha^+(\omega) = c \in B_1\}\right) \\
& \qquad \bigcup \left(\bigcup_{n=1}^{\infty} \{\omega \in \Omega; [X_\alpha^-(\omega), X_\alpha^+(\omega)] \subset < a_k, b_k >\}\right) \\
= & \left(\{\omega \in \Omega\,; X_\alpha(\omega) \in B_1\} \cap (X_\alpha^- - X_\alpha^+)^{-1}\{0\}\right) \\
& \qquad \bigcup \left(\bigcup_{n=1}^{\infty} ((X_\alpha^-)^{-1} < a_k, +\infty) \bigcap (X_\alpha^+)^{-1}(-\infty, b_k >)\right).
\end{aligned}$$

Since $B_1 = B \setminus \bigcup_{n=1}^{\infty} < a_k, b_k >$ and $\{0\}$ are Borel measurable and X_α^-, X_α^+, $X_\alpha^- - X_\alpha^+$ are A-measurable, we obtain

$$\{\omega \in \Omega; X_\alpha(\omega) \subset B\} = \{\omega \in \Omega; X_\alpha(\omega) \cap B \neq \phi\} \in A.$$

3. Fuzzy random functions

A family of fuzzy random variables $X(t) = \{X(t,\omega); t \in T\}$ is a fuzzy random function. Here the parameter set T could be any one of the following ;

$R, R^+ = [0,\infty), [a,b] \subset R$, $Z = \{0, \pm 1, \pm 2, \cdots\}$, $Z^+ = \{0,1,2,\cdots\}$, $\{0,1,2,\cdots\}$. If $T = \{1,2,\cdots,m\}, X(t)$ is a fuzzy random vector. If $T = Z$ or Z^+, one sometimes speaks of a fuzzy random sequence. If $T = R$ or R^+ or $[a,b], X(t)$ is a fuzzy stochastic process. In all cases we give a general definition as follows.

DEFINITION 3.1. A fuzzy random function $X(t) = \{X(t,\omega); t \in T\}$ is a fuzzy set-valued function from the space $T \times \Omega$ to $F_0(R), X(t,\cdot)$ is FRV on (Ω, A, P) for every fixed $t \in T$; and $X(\cdot,\omega)$ a NDFS with respect to the parameter set T for every fixed $\omega \in \Omega, X(\cdot,\omega)$ is a fuzzy sample function or a fuzzy trajectory.

DEFINITION 3.2. Let (Ω, A, P) be a probability space, $T \subset R$ and ζ a set-valued mapping $\zeta : T \times R \to I(R) = \{[x,y] : x,y \in R, x \leq y\}$.

$$(t,\omega) \mapsto \zeta(t,\omega) = [\zeta^-(t,\omega), \zeta^+(t,\omega)].$$

Then $\zeta(t,\omega) = [\zeta^-(t,\omega), \zeta^+(t,\omega)]$ is an interval random function if $\zeta^-(t,\omega)$ and $\zeta^+(t,\omega)$ are both random functions, i,e., when t is fixed,

$$\zeta(t,\cdot) = [\zeta^-(t,\cdot), \zeta^+(t,\cdot)]$$

is a random interval on (Ω, A, P), and when ω is fixed,

$$\zeta(\cdot,\omega) = [\zeta^-(\cdot,\omega), \zeta^+(\cdot,\omega)]$$

is an interval-valued function on T.

THEOREM 3.1. $X(t) = \{X(t,\omega); t \in T\}$ *is a fuzzy random function if and only if for every* $\alpha \in [0,1]$,

$$\begin{aligned} X_\alpha(t) &= \{X_\alpha(t,\omega) : t \in T\} \\ &= \{[X_\alpha^-, X_\alpha^+(t,\omega)] : t \in T\} \end{aligned}$$

is an interval random function for each $t \in T$ *and every* $\omega \in \Omega$ *, and*

$$X(t,\omega) = \bigcup_{\alpha\in(0,1]} \alpha X_\alpha(t,\omega) = \bigcup_{\alpha\in(0,1]} \alpha[X_\alpha^-, X_\alpha^+(t,\omega)]$$

$$X(t,\omega) = \bigcap_{n=1}^{\infty} X_{\alpha_n}(t,\omega) = \bigcap_{n=1}^{\infty} [X_{\alpha_n}^-, X_{\alpha_n}^+(t,\omega)]$$

where for every $\alpha \in (0,1]$.

$$X_\alpha^-(t,\omega) = \inf X_\alpha(t,\omega) = \inf\{x \in R : X(t,\omega)(x) \geq \alpha\},$$

$$X_\alpha^+(t,\omega) = \sup X_\alpha(t,\omega) = \sup\{x \in R : X(t,\omega)(x) \geq \alpha\}$$

$$\alpha_n = (1 - \frac{1}{n+1})\alpha.$$

Proof. Omit (See Theorem 2.1)

PROPOSITION 3.1. *X and Y are two equivalent fuzzy random functions on (Ω, A, P) and T if and only if X_α and Y_α are two equivalent fuzzy random functions on (Ω, A, P) and T for every $\alpha \in (0,1]$. In addition*

$$X = \bigcup_{\alpha\in(0,1]} \alpha X_\alpha = Y = \bigcup_{\alpha\in(0,1]} \alpha Y_\alpha.$$

DEFINITION 3.3. Given a fuzzy stochastic process X on a probability space (Ω, A, P) and $T \subset R$, let $D = \{t_1, t_2, \cdots, t_n\}$ be a finite sequence of distinct elements of T. Consider the transformation $X_D = X_{t_1,t_2,\cdots,t_n} = (X(t_1,\cdot), X(t_2,\cdot), \cdots, X(t_n,\cdot))$ of Ω into $F_0^n(R)$

and let $F_{X_D} = F_{X_{t_1,t_2,\cdots,t_n}}$ be the n-dimensional fall-shadow distribution function of X_D. i.e.,

$$\begin{aligned} F_{X_D} &= F_{X_{t_1,t_2,\cdots,t_n}}(x_1, \cdots, x_n) \\ &= P\{\omega : x_i \in X_\alpha(t_i, \omega) : i = 1, 2, \cdots, n\}, \\ x &= (x_1, x_2, \cdots, x_n) \in R^n, \alpha \in (0, 1]. \end{aligned}$$

Let $\widetilde{D}$ be the collection of all the D. Then $\{F_{X_D} : D \in \widetilde{D}\}$ is the system of finite dimensional fall-shadow distribution functions determined by X.

PROPOSITION 3.2. *For two equivalent fuzzy stochastic processes X and Y on a probability space (Ω, A, P) and $T \subset R$ we have $F_{X_D} = F_{Y_D}$ for every $D \in \widetilde{D}$.*

4. Measurable fuzzy random functions

A fuzzy random function X on a probability space (Ω, A, P) and T is a measurable fuzzy set-valued function of $\omega \in \Omega$ for every $t \in T$. However X may not be a measurable fuzzy-set valued function for $(t, \omega) \in T \times \Omega$ with respect to the product of the probability measure and the Lebesgue measure. If it is fuzzy measurable and fuzzy integrable with respect to the product measure then we can apply the Fubini theorem.

Let L_T be the σ-field of Lebesgue measurable sets contained in $T \subset R$ when T is Lebesgue measurable and A_T be the σ-field of Borel sets contained in T when T is Borel set. Let $\sigma(L_T \times A)$ be the σ-field generated by the Cartesian product of L_T and A. We define $\sigma(A_T \times A)$ likewise. We write m_L for the Lebesgue measure on L_T or A_T.

Definition 4.1 A fuzzy random function $X(t, \omega)$ is Lebesgue (Borel) measurable if for every $B \in R$ and every $\alpha \in (0, 1]$,

$$\{(t, \omega) : X_\alpha(t, \omega) \cap B \neq \phi\} \in \sigma(L_T \times A),$$

$$\{(t, \omega) : X_\alpha(t, \omega) \cap B \neq \phi\} \in \sigma(A_T \times A),$$

respectively.

THEOREM 4.1. *A fuzzy random function $X(t,\omega)$ is Lebesgue (Borel) measurable if and only if for every $\alpha \in (0,1]$,*

$$X_\alpha(t,\omega) = [X_\alpha^-(t,\omega), X_\alpha^+(t,\omega)]$$

is a Lebesgue (or Borel) measurable interval random function and

$$X(t,\omega) = \bigcup_{\alpha\in(0,1]} \alpha[X_\alpha^-(t,\omega), X_\alpha^+(t,\omega)]$$

$$X(t,\omega) = \bigcap_{n=1}^{\infty} \alpha[X_{\alpha_n}^-(t,\omega), X_{\alpha_n}^+(t,\omega)]$$

where $\alpha_n = (1 - \frac{1}{(n+1)})\alpha$ and $\alpha > 0$.

Proof. The proof of the Theorem 4.1 is omitted (See Theorem 3.1)

DEFINITION 4.2. Let

$$F : T \times \Omega \to I(R) = \{[x,y] : x,y \in R, x \leq y\}$$
$$(t,\omega) \mapsto F(t,\omega) = [F^-(t,\omega), F^+(t,\omega)],$$

where F^- and F^+ are two-variable real-valued functions. The two-variable interval function $F(t,\omega)$ is integrable on $T \times \Omega$ if F^- and F^+ are integrable on $T \times \Omega$. The interval number

$$\left[\int_\Omega \int_T F^-(t,\omega)dtd\omega, \int_\Omega \int_T F^+(t,\omega)dtd\omega\right]$$

is the integral of $F(t,\omega)$ on $T \times \Omega$, and we denote

$$\int_\Omega \int_T F(t,\omega)dtd\omega = \left[\int_\Omega \int_T F^-(t,\omega)dtd\omega, \int_\Omega \int_T F^+(t,\omega)dtd\omega\right]$$

DEFINITION 4.3. A fuzzy random function $X(t,\omega)$ is integrable if every interval random function

$$X_\alpha(t,\omega) = [X_\alpha^-(t,\omega), X_\alpha^+(t,\omega)]$$

is integrable on $T \times \Omega$ for every $\alpha \in (0,1]$.

THEOREM 4.2. *Let $X(t,\omega)$ be a Lebesgue or Borel measurable fuzzy random function on $T \times \Omega$. If one of the two iterated integrals*

$$\int_T E(|X_\alpha(t,\cdot)|)m_L(dt)$$

and

$$E(\int_T |X_\alpha(t,\cdot)|m_L(dt))$$

is finite for every $\alpha \in (0,1]$. Then

(1) For every $\alpha \in (0,1]$,

$$\int_T E_\alpha(X(t,\cdot))m_L(dt)$$

is a closed interval.

(2) For every $\alpha \in (0,1]$,

$$\int_T E_\alpha(X(t,\cdot))m_L(dt) = E(\int_T X_\alpha(t,\cdot)m_L(dt)).$$

Proof. (1) It follows from Theorem 2.1 and Definition 4.2 that for every $\alpha \in (0,1]$.

$$\begin{aligned}\int_T E_\alpha(X(t,\cdot))m_L(dt) &= \int_T E(X_\alpha(t,\cdot))m_L(dt)\\ &= \int_T [E(X_\alpha^-(t,\cdot)), E(X_\alpha^+(t,\cdot))]\, m_L(dt)\\ &= \left[\int_T E(X_\alpha^-(t,\cdot)), \int_T E(X_\alpha^+(t,\cdot))m_L(dt)\right]\end{aligned}$$

i.e., $\int_T E_\alpha(X(t,\cdot))m_L(dt)$ is a closed interval for every $\alpha \in (0,1]$.
(2) It follows from Definition 4.2 and Fubini's theorem that for every

$\alpha \in (0,1]$,

$$
\begin{aligned}
E\Big(\int_T X_\alpha(t,\cdot)m_L(dt)\Big) &= E\Big(\int_T [X_\alpha^-(t,\cdot), X_\alpha^+(t,\cdot)]m_L(dt)\Big) \\
&= E\Big(\Big[\int_T X_\alpha^-(t,\cdot)m_L(dt), \int_T X_\alpha^+(t,\cdot)m_L(dt)\Big]\Big) \\
&= \Big[E\Big(\int_T X_\alpha^-(t,\cdot)m_L(dt)\Big), E\Big(\int_T X_\alpha^+(t,\cdot)m_L(dt)\Big)\Big] \\
&= \Big[\int_T E(X_\alpha^-(t,\cdot))m_L(dt), \int_T E(X_\alpha^+(t,\cdot))m_L(dt)\Big] \\
&= \int_T [E(X_\alpha^-(t,\cdot)), E(X_\alpha^+(t,\cdot))]m_L(dt) \\
&= \int_T E(X_\alpha(t,\cdot))m_L(dt) \\
&= \int_T E_\alpha(X(t,\cdot))m_L(dt).
\end{aligned}
$$

5. Continuous fuzzy stochastic process

DEFINITION 5.1. An interval stochastic process

$$\zeta(t,\omega) = [\zeta^-(t,\omega), \zeta^+(t,\omega)] \tag{5.1}$$

on a probability space (Ω, A, P) and $T = [a,b] \subset R$ is continuous if $\zeta^\pm(\cdot,\omega)$ are continuous on T for every $\omega \in \Omega$. $\zeta(t,\omega)$ is almost surely continuous if there exists $N \in A$ with $P(N) = 0$ such that $\zeta^\pm(\cdot,\omega)$ are continuous on T for $\omega \in N^c$, the complement of N. It is uniformly continuous if $\zeta^\pm(\cdot,t)$ are uniformly continuous on T.

DEFINITION 5.2. A fuzzy stochastic process $X(t,\omega)$ on a probability space (Ω, A, P) and $T \subset R$ is continuous on T if and only if for every $\alpha \in (0,1]$, the level sample functions (5.1) are continuous on T for a.e. $\omega \in \Omega$. $X(t,\omega)$ is uniformly continuous if and only if for every $\alpha \in (0,1]$. (5.1) is uniformly continuous on T.

We get immediately the following proposition.

PROPOSITION 5.1. *Let $X(t,\omega)$ be a fuzzy stochastic process on a probability space (Ω, A, P) and $T \subset R$ which is separable relative to a countable dense subset $S \subset R$. If the fuzzy sample function $X(\cdot,\omega)$ is uniformly continuous on S, then $X(\cdot,\omega)$ must be uniformly continuous on T.*

THEOREM 5.1. *Let $X(t,\omega)$ be a separable fuzzy stochastic process. If there exist $\epsilon, \beta, k, h_0 > 0$ such that for every $\alpha \in (0,1]$,*

$$E(|X_\alpha^\pm(t+h,\cdot) - X_\alpha^\pm(t,\cdot)|^\epsilon) \leq k|h|^{1+\beta},$$

whenever $t, t+h \in T$, $|h| < h_0$, then $X(t,\omega)$ is uniformly continuous on T with probability 1.

Proof. For every $\alpha \in (0,1]$, the level sample function

$$X_\alpha(\cdot,\omega) = [X_\alpha^-(\cdot,\omega), X_\alpha^+(\cdot,\omega)]$$

is uniformly continuous on T with probability 1. Therefore $X(\cdot,\omega)$ is uniformly continuous on T with probability 1.

PROPOSITION 5.2. *Let $X(t,\omega)$ be a separable fuzzy stochastic process on a probability space (Ω, A, P) and a Lebesgue or Borel measurable set $T \subset R$. For each $t_0 \in T$ and positive integer n let*

$$\begin{aligned} I_{t_0,n} &= \{t \in R : [2^n t] = [2^n t_0]\}, \\ L_\infty(t_0,\omega) &= \lim_{n\to\infty} \inf_{T\cap I_{t_0}} X(t,\omega), \quad t_0 \in T, \omega \in \Omega, \\ U_\infty(t_0,\omega) &= \lim_{n\to\infty} \sup_{T\cap I_{t_0}} X(t,\omega), \quad t_0 \in T, \omega \in \Omega. \end{aligned}$$

If for $\omega \in \Omega$, $X(\cdot,\omega)$ is continuous at t_0 then

$$L_\infty(t_0,\omega) = X(t_0,\omega) = U_\infty(t_0,\omega).$$

The proof of the proposition is omitted.

THEOREM 5.2. *Let $X(t,\omega)$ be a fuzzy stochastic process on a probability space (Ω, A, P) and a Lebesgue or Borel measurable set $T \subset R$. If every fuzzy sample function of $X(t,\omega)$ is continuous on T, then*

$X(t,\omega)$ is separable relative to every countable dense subset $S \subset T$ and is Lebesgue or Borel measurable.

Proof. From the almost sure separability of $X(t,\omega)$ and Theorem 4.2 there exist a countable dense subset $S \subset T$ and $N_1 \in A$ with $P(N_1) = 0$ such that for every open interval $G \subset R$,

$$\text{(5.2a)} \qquad \inf_{T\cap G} X(t,\omega) = \inf_{S\cap G} X(s,\omega) \quad \text{when } \omega \in N_1^c$$

$$\text{(5.2b)} \qquad \sup_{T\cap G} X(t,\omega) = \sup_{S\cap G} X(s,\omega) \quad \text{when } \omega \in N_1^c.$$

Since $Y(t,\omega)$ is almost surely continuous there exists $N_2 \in A$ with $P(N_2) = 0$ such that $Y(\cdot,\omega)$ is continuous on T when

$$\text{(5.3)} \qquad \omega \in N_2^c.$$

From the equivalence of $X(t,\omega)$ and $Y(t,\omega)$ we have in particular for every $s \in S$ a set $N_s \in A$ with $P(N_s) = 0$ such that

$$\text{(5.4)} \qquad Y(s,\omega) = X(s,\omega) \qquad \text{when } \omega \in N_s^c.$$

Let $N = N_1 \cup N_2 \cup (\cup_{s\in S} N_s)$. Then $N \in A$ and $P(N) = 0$. For every $\omega \in N^c$, (5.2), (5.3) and (5.4) hold. We proceed to show that, for every $\omega \in N^c$, $X(\cdot,\omega)$ is continuous on T. Let $t_0 \in T$. From the continuity of $Y(\cdot,\omega)$ on T, for every $\epsilon > 0$ and every $\alpha \in (0,1]$ there exists $\delta > 0$ such that

$$|Y_\alpha^\pm(t,\omega) - Y_\alpha^\pm(t_0,\omega)| < \epsilon \qquad \text{when } t \in T, |t - t_0| < \delta,$$

With $G = (t_0 - \delta, t_0 + \delta)$ we have, for every $\alpha \in (0,1]$,

$$\begin{aligned} X_\alpha^\pm(t_0,\omega) &\geq \inf_{T\cap G} X_\alpha^\pm(t,\omega) \\ &= \inf_{S\cap G} X_\alpha^\pm(s,\omega) \\ &= \inf_{S\cap G} Y_\alpha^\pm(s,\omega) \\ &\geq Y_\alpha^\pm(t_0,\omega) - \epsilon, \end{aligned}$$

From the arbitriness of $\epsilon > 0$, we have $X_\alpha^\pm(t_0,\omega) \geq Y_\alpha^\pm(t_0,\omega)$ for every $\alpha \in (0,1]$. Similarly we have $X_\alpha^\pm(t_0,\omega) \leq Y_\alpha^\pm(t_0,\omega)$ for every $\alpha \in (0,1]$. So $[X_\alpha^-(t,\omega), X_\alpha^+(t_0,\omega)] = [Y_\alpha^-(t,\omega), Y_\alpha^+(t_0,\omega)]$ for every $\alpha \in (0,1]$. It follows from Proposition 3.1 that $X(t_0,\omega) = Y(t_0,\omega)$. From the arbitrariness of $t_0 \in T$ we have $X(t,\omega) = Y(t,\omega)$ for all $t \in T$ when $\omega \in N^c$. Thus $X(\cdot,\omega)$ is almost surely continuous .

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Department of Applied Mathematics
Korea Maritime University
Pusan 606-791, Korea