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THE BOOK–SHORE TYPE LAW OF A GAUSSIAN PROCESS WITH STATIONARY INCREMENTS

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1. Introduction and results

Let $a_T\,(0 < T < \infty)$ be a nondecreasing function of T for which

(i) $0 < a_T \le T$,

(ii) T/a_T is nondecreasing.

For instance, we can choose a_T as $1, \log T, T^\theta, (0<\theta<1), T/(\log T)^r$, $(0 < r < \infty)$ and $cT\,(0 < c \le 1)$, etc.

Under these conditions on a_T, Csörgö and Révész [4] obtained the following theorem for a standard Wiener process $\{W(t); t \ge 0\}$:

THEOREM A. *If* $a_T\,(0 < T < \infty)$ *satisfies the conditions* (i) *and* (ii), *then*

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|W(t+a_T)-W(t)|}{\beta_T\sqrt{a_T}} = 1 \quad a.s. \tag{1.1}$$

and

$$\limsup_{T\to\infty} \sup_{0\le s\le a_T} \sup_{0\le t\le T-a_T} \frac{|W(t+s)-W(t)|}{\beta_T\sqrt{a_T}} = 1 \quad a.s.$$

where $\beta_T = \sqrt{2\{\log(T/a_T) + \log\log T\}}$. *If, in addition, we have also*

(iii) $\lim_{T\to\infty}\{\log T - \log a_T\}/\log\log T = \infty$,

then we have

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|W(t+a_T)-W(t)|}{\beta_T\sqrt{a_T}} = 1 \quad a.s. \tag{1.2}$$

and

$$\lim_{T\to\infty} \sup_{0\le s\le a_T} \sup_{0\le t\le T-a_T} \frac{|W(t+s)-W(t)|}{\beta_T\sqrt{a_T}} = 1 \quad a.s.$$

On the other hand, Book and Shore [1] extended the result (1.2) of the above Csörgö-Révész theorem as follows:

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THEOREM B. *If* $a_T\,(0 < T < \infty)$ *satisfies the above conditions* (i), (ii) *and further*

(iii)$'$ $\lim_{T\to\infty}(\log T - \log a_T)/\log\log T = r, \quad 0 \le r \le \infty,$

then

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|W(t+a_T)-W(t)|}{\beta_T\sqrt{a_T}} = \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

For the standard Wiener process $\{W(t); t \ge 0\}$, the Strassen's law of iterated logarithm in [6] implies that for any $0 < c < 1$

$$(1.3) \qquad \limsup_{T\to\infty} \sup_{0\le t\le T-cT} \frac{|W(t+cT)-W(t)|}{\sqrt{2c\log\log T}} = 1 \qquad \text{a.s.}$$

For $0 < c < 1$ if we set $a_T = cT$ in (1.1), we get the result (1.3). Clearly, $a_T = cT\,(0 < c < 1)$ fails to satisfy the condition (iii) of Theorem A, but it satisfies the condition (iii)$'$ of Theorem B. Thus the Strassen's law of iterated logarithm is complemented as follows:

$$\liminf_{T\to\infty} \sup_{0\le t\le T-cT} \frac{|W(t+cT)-W(t)|}{\sqrt{2c\log\log T}} = 0 \qquad \text{a.s.}$$

We are going to extend Theorems A and B to a Gaussian process with stationary increments. Let $\{X(t) : 0 \le t < \infty\}$ be a almost surely continuous Gaussian process with $X(0) = 0$, $E\{X(t)\} = 0$ and stationary increments: $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$, where $\sigma(y)$ is a function of $y \ge 0$ (for example, if $\{X(t) : 0 \le t < \infty\}$ is a standard Wiener process, then $\sigma(t) = \sqrt{t}$). Further assume that $\sigma(t)$, $t > 0$, is a nondecreasing continuous, regularly varying function with exponent $\gamma\,(0 < \gamma < 1)$ at infinity (or zero). A positive function $q(t)$, $t > 0$, is said to be *regularly varying with exponent* $\gamma > 0$ *at* $a\,(a = \infty$ or $0)$ if, for all $x > 0$, one has

$$\lim_{t\to a} \frac{q(xt)}{q(t)} = x^\gamma.$$

Let us define continuous parameter processes $X_1(T), X_2(T), \cdots,$

$X_6(T)$ by

$$X_1(T) = \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-s} \frac{|X(t+s)-X(t)|}{\beta_T\sigma(a_T)},$$

$$X_2(T) = \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-s} \frac{X(t+s)-X(t)}{\beta_T\sigma(a_T)},$$

$$X_3(T) = \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-a_T} \frac{|X(t+s)-X(t)|}{\beta_T\sigma(a_T)},$$

$$X_4(T) = \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-a_T} \frac{X(t+s)-X(t)}{\beta_T\sigma(a_T)},$$

$$X_5(T) = \sup_{0\le t\le T-a_T} \frac{|X(t+a_T)-X(t)|}{\beta_T\sigma(a_T)},$$

$$X_6(T) = \sup_{0\le t\le T-a_T} \frac{X(t+a_T)-X(t)}{\beta_T\sigma(a_T)},$$

respectively. Clearly, $X_1(T)$ is the largest process and $X_6(T)$ is the smallest one of all $X_i(T)$, $i = 1,\dots,6$.

In this paper we shall investigate almost sure limiting values of $X_i(T)$, $i = 1, 2, \cdots, 6$, under varying conditions on a_T. Thus we are concerning only with behavior of functions near at infinity. We often use the letter c for a positive absolute constant which may be different from line to line if necessary.

The following theorem is an extension of Theoren A to a Gaussian process, which is proved in Csáki et al. [3] and Choi [2].

THEOREM C. *Let a_T be a nondecreasing function of T such that*

(i) $0 < a_T \le T$,

(ii) *T/a_T is nondecreasing.*

Let the Gaussian process $\{X(t); 0 \le t < \infty\}$ in the above statements satisfy the condition which for any $a \le b \le c \le d$

(iii) $E\{(X(b)-X(a))(X(d)-X(c))\} \le 0$.

Then

$$\limsup_{T\to\infty} X_i(T) = 1, \qquad a.s.,$$

where $i = 1, 2, \cdots, 6$. Moreover, if we have also

(iv) $\lim_{T\to\infty}(\log T - \log a_T)/\log\log T = \infty$

and if either the condition (iii) *it holds or*

(v) $\sigma^2(t)$ *is twice continuously differentiable which satisfies*

$$|(\sigma^2(t))''| \le c\sigma^2(t)/t^2, \quad t > 0,$$

where c *is a positive constant, then*

$$\lim_{T\to\infty} X_i(T) = 1, \qquad \text{a.s.,} \tag{1.4}$$

where $i = 1, 2, \cdots, 6$.

Note that the condition (iv) of Theorem C is satisfied in cases of $a_T = 1$, $(\log\log T)^\beta\,(0 < \beta < \infty)$, $(\log T)^\beta\,(0 < \beta < \infty)$ and $T^\theta(\log T)^\alpha\,(0 < \theta < 1, -\infty < \alpha < \infty)$, etc. But in case of $a_T = T/(\log T)^r\,(0 < r < \infty)$, it is not satisfied. Thus we investigate this case:

THEOREM 1. *Let* a_T *be a nondecreasing function of* T *such that*

(i) $0 < a_T \le T/(\log T)^r$ *for all* $0 < r < \infty$,

(ii) T/a_T *is nondecreasing.*

Assume that the above Gaussian process $\{X(t); 0 \le t < \infty\}$ *satisfies the condition* (iii) *of Theorem* C. *Then*

$$\liminf_{T\to\infty} X_i(T) \ge \sqrt{\frac{r}{1+r}}, \qquad \text{a.s},$$

where $i = 1, 2, \cdots, 6$.

The following theorem complements its lack for gaps in $T/(\log T)^r < a_T \le T$, $0 < r < \infty$ and exactly yields the "liminf" value. Theorem 2 is an extension of Theorem B for Wiener processes, and it gives the same value as the result (1.4) of Theorem C only when $r = \infty$ in Theorem 2. Theorem 1 also needs to prove Theorem 2.

THEOREM 2. *Let* a_T *be a nondecreasing function of* T *for which*

(i) $0 < a_T \le T$,

(ii) T/a_T *is nondecreasing,*

(iii) $\lim_{T\to\infty}(\log T - \log a_T)/\log\log T = r, \quad 0 \le r \le \infty$.

Assume that the above Gaussian process $\{X(t); 0 \le t < \infty\}$ *satisfies the condition which, for* $t > 0$,

(iv) $\sigma(t) = t^{\gamma}, \quad 0 < \gamma \le 1/2.$

Then we have

$$\liminf_{T\to\infty} X_i(T) = \sqrt{\frac{r}{1+r}} \qquad \text{a.s.},$$

where $i = 1, 2, \ldots, 6$ *if* $r > 0$, *and* $i = 1, 3, 5$ *if* $r = 0$.

We note that the condition (iii) of Theorem C is weaker than that (iv) of Theorem 2, but the condition (iii) of Theorem 2 contains that (iv) of Theorem C.

2. Proofs

For proving our Theorem 1, we shall make use of the following lemma:

LEMMA 1 (Slepian [5]). *Suppose that* $\{X_i : i = 1, 2, \ldots, n\}$ *and* $\{Y_i : i = 1, 2, \ldots, n\}$ *are jointly standardized normal random variables with*

$$\text{covariance } (X_i, X_j) \le \text{covariance } (Y_i, Y_j), \quad i \ne j.$$

Then for any real number u_n,

$$P\{X_i \le u_n; i = 1, 2, \ldots, n\} \le P\{Y_j \le u_n; j = 1, 2, \ldots, n\}.$$

Proof of Theorem 1. Considering the order of magnitude of $X_i(T)$, $i = 1, 2, \ldots, 6$, it suffices to prove

$$\liminf_{T\to\infty} X_6(T) \ge \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

For given $T > 0$ large enough, let us define a positive integer n_T by $n_T = [T/a_T]$ where $[y]$ denotes the greatest integer not exceeding y. By the assumption (i) of a_T, the integers n_T are increasing and $n_T \to \infty$ as $T \to \infty$. For $j = 1, 2, \ldots, n_T$, define incremental random variables

$$Z_T(j) = X(ja_T) - X((j-1)a_T).$$

From the condition (iii) it follows that for $i \neq j$

$$\text{covariance}\,(Z_T(i), Z_T(j)) \leq 0.$$

Applying Lemma 1 for $X_j = Z_T(j)/\sigma(a_T)$, $j = 1, 2, \ldots, n_T$, we have for any $0 < \epsilon < 1$

$$\begin{aligned} P\{X_6(T) &< \sqrt{(1-\epsilon)r/(1+r)}\} \\ &= P\Big\{ \sup_{0\leq t\leq T-a_T} \frac{X(t+a_T)-X(t)}{\sigma(a_T)} < u_T \Big\} \\ &\leq P\Big\{ \sup_{1\leq j\leq n_T} \frac{Z_T(j)}{\sigma(a_T)} < u_T \Big\} \\ &\leq \{\Phi(u_T)\}^{n_T} \end{aligned}$$

where $u_T = \sqrt{(1-\epsilon)r/(1+r)}\sqrt{2\{\log(T/a_T) + \log\log T\}}$ and $\Phi(\cdot)$ denotes the standard normal distribution function. Since, for large T

$$\{\Phi(u_T)\}^{n_T} \leq \exp(-c\{(T/a_T)\log T\}^{-(1-\epsilon)r/(1+r)} n_T),$$

we have

$$P\{X_6(T) < \sqrt{(1-\epsilon)r/(1+r)}\} \leq \exp(-c(\log T)^{\epsilon r}).$$

Let $0 < \alpha < 1$ and set $T_k = \exp(k^\alpha), k \in N$, where N is a set of positive integers. Then the above inequality yields

$$P\{X_6(T_k) < \sqrt{(1-\epsilon)r/(1+r)}\} \leq \exp(-ck^{\alpha r\epsilon}).$$

Using the Borel-Cantelli lemma, we obtain

$$\liminf_{k\to\infty} X_6(T_k) \geq \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

For given T_k, let T be in $T_k \leq T \leq T_{k+1}$, $k \in N$. Then by the similar techniques as in the proof of Lemma 4.6 of Choi [2], we have

$$\liminf_{T\to\infty} X_6(T) \geq \liminf_{k\to\infty} X_6(T_k) \qquad \text{a.s.}$$

This proves Theorem 1.

In proving Theorem 2 we shall use a form of the modulus of continuity for Gaussian processes (cf. Lemma 2), which is an extension of Lévy's modulus of continuity for Wiener processes.

LEMMA 2 [3]. (Moduli of continuity for a Gaussian process) *Assume that the condition* (iii) *of Theorem C holds. Then*

$$\lim_{h\downarrow 0} \sup_{0\le u\le 1-h} \frac{X(u+h)-X(u)}{\sqrt{2\log(1/h)}\sigma(h)} = 1, \tag{2.1}$$

$$\lim_{h\downarrow 0} \sup_{0\le u\le 1-h} \frac{|X(u+h)-X(u)|}{\sqrt{2\log(1/h)}\sigma(h)} = 1, \tag{2.2}$$

$$\lim_{h\downarrow 0} \sup_{0\le v\le h} \sup_{0\le u\le 1-h} \frac{X(u+v)-X(u)}{\sqrt{2\log(1/h)}\sigma(h)} = 1, \tag{2.3}$$

$$\lim_{h\downarrow 0} \sup_{0\le v\le h} \sup_{0\le u\le 1-h} \frac{|X(u+v)-X(u)|}{\sqrt{2\log(1/h)}\sigma(h)} = 1, \tag{2.4}$$

$$\lim_{h\downarrow 0} \sup_{0\le v\le h} \sup_{0\le u\le 1-h'} \frac{X(u+v)-X(u)}{\sqrt{2\log(1/h)}\sigma(h)} = 1, \tag{2.5}$$

$$\lim_{h\downarrow 0} \sup_{0\le v\le h} \sup_{0\le u\le 1-h'} \frac{|X(u+v)-X(u)|}{\sqrt{2\log(1/h)}\sigma(h)} = 1 \tag{2.6}$$

hold almost surely, where $0 < h' < h < 1$.

The proof of Theorem 2 applies the similar techniques as the proof of Book-Shore [1].

Proof of Theorem 2. When $r = \infty$, we have already proved in Theorem C. The only part of the proof is the "liminf" part when $0 \le r < \infty$. Since a_T/T is nonincreasing, either $a_T/T \to 0$ or $a_T/T \to \delta\,(0 < \delta \le 1)$ as $T \to \infty$. First suppose the case $a_T/T \to \delta\,(0 < \delta \le 1)$. Then $a_T \ge \delta T$ for all large T, and we must be in a case when $r = 0$ because in the condition (iii)

$$0 \le \lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} \le \lim_{T\to\infty} \frac{\log(1/\delta)}{\log\log T} = 0.$$

Let us denote $U(t) \overset{\mathrm{d}}{=} V(t)$ if $U(t)$ has the same distribution as $V(t)$. By the condition (iv),

$$\sigma(a_T)X(t/a_T) \overset{\mathrm{d}}{=} X(t).$$

Thus, in case $X_1(T)$, we have

$$\begin{aligned}
0 &\le X_1(T) \\
&= \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-s} \frac{|X(t+s)-X(t)|}{\sqrt{2(\log(T/a_T)+\log\log T)}\sigma(a_T)} \\
&\overset{\mathrm{d}}{=} \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-s} \frac{\sigma(a_T)|X((t+s)/a_T)-X(t/a_T)|}{\sqrt{2(\log(T/a_T)+\log\log T)}\sigma(a_T)} \\
&= \sup_{0\le p\le 1}\ \sup_{0\le q\le (T/a_T)-p} \frac{|X(q+p)-X(q)|}{\sqrt{2(\log(T/a_T)+\log\log T)}} \\
&\le \sup_{0\le p\le 1}\ \sup_{0\le q\le (T/a_T)-p} \frac{|X(q+p)-X(q)|}{\sqrt{2\log\log T}} \\
&\le \sup_{0\le p\le 1}\ \sup_{0\le q\le (1/\delta)-p} \frac{|X(q+p)-X(q)|}{\sqrt{2\log\log T}} \to 0 \qquad \text{a.s.,}
\end{aligned}$$

as $T \to \infty$ by the a.s. continuity of Gaussian process. So $X_1(T) \to 0$ in probability as $T \to \infty$ and hence there exists a subsequence $\{T_k : 1 \le k < \infty\}$ such that $X_1(T_k)$ converges almost surely to zero as $k \to \infty$. It follows that

$$\liminf_{T\to\infty} X_1(T) = 0 \qquad \text{a.s.}$$

Also $X_3(T)$ and $X_5(T)$ are proved by the same way as $X_1(T)$. In the remainder of the proof, we shall consider only the case when $a_T/T \to 0$ as $T \to \infty$. Then there are two cases : $r > 0$ or $r = 0$. First consider the case $r > 0$. This does not imply $a_T/T \to \delta$ for some $\delta > 0$, and the a_T's in this case are contained in the set $\{a_T : 0 < a_T \le T/(\log T)^r,\ 0 < r < \infty\}$. Thus from Theorem 1

$$\liminf_{T\to\infty} X_i(T) \ge \sqrt{\frac{r}{1+r}}, \quad i = 1, 2, \cdots, 6, \qquad \text{a.s.} \tag{2.7}$$

Now let us prove

$$\liminf_{T\to\infty} X_i(T) \le \sqrt{\frac{r}{1+r}}, \quad i = 1, 2, \ldots, 6, \qquad \text{a.s.}$$

Set $B_T = \sqrt{1+\{\log\log T/\log(T/a_T)\}}$. Then $B_T \to \sqrt{(1+r)/r}$ as $T \to \infty$ by the condition (iii). Since $\sigma(T)X(t/T) \overset{\mathrm{d}}{=} X(t)$, we have, in case $X_2(T)$,

$$
\begin{aligned}
& X_2(T) \\
&= \sup_{0\le s\le a_T}\sup_{0\le t\le T-s} \frac{X(t+s)-X(t)}{\sqrt{2\log(T/a_T)}B_T\sigma(a_T)} \\
(2.8) \qquad &\overset{\mathrm{d}}{=} M_2(T) \\
&:= \sup_{0\le s\le a_T}\sup_{0\le t\le T-s} \frac{\sigma(T)\{X((t+s)/T)-X(t/T)\}}{\sqrt{2\log(T/a_T)}B_T\sigma(a_T)} \\
&= \sup_{0\le s/T\le a_T/T}\ \sup_{0\le t/T\le 1-s/T} \frac{X((t/T)+(s/T))-X(t/T)}{\sqrt{2\log(T/a_T)}B_T\sigma(a_T/T)}.
\end{aligned}
$$

Because we are in the case $h = a_T/T \to 0$ as $T \to \infty$, we have, from Lemma 2 ((2.3) or (2.5))

$$\lim_{h\downarrow 0}\sup_{0\le v\le h}\sup_{0\le u\le 1-v} \frac{X(u+v)-X(u)}{\sqrt{2\log(1/h)}\sigma(h)} = 1 \qquad \text{a.s.}$$

Thus in (2.8)

$$\lim_{T\to\infty} M_2(T) = \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

This implies that

$$\lim_{T\to\infty} X_2(T) = \sqrt{\frac{r}{1+r}} \qquad \text{in probability.}$$

Therefore we can find a subsequence $\{T_k : 1 \le k < \infty\}$ such that

$$\lim_{k\to\infty} X_2(T_k) = \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

Thus

$$\liminf_{T\to\infty} X_2(T) \le \sqrt{\frac{r}{1+r}} \qquad \text{a.s.} \tag{2.9}$$

By (2.7) and (2.9), we have

$$\liminf_{T\to\infty} X_2(T) = \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

Also, as for the others $X_i(T)$, $i = 1,3,4,5,6$, it is easily proved by the same method as $X_2(T)$. Consider the next case when $r = 0$. Clearly,

$$\text{(2.10)} \qquad \liminf_{T\to\infty} X_i(T) \geq 0, \quad i = 1,3,5, \qquad \text{a.s.}$$

If we define $\frac{1}{0} = \infty$, then by the same method as above, we can deduce

$$\lim_{T\to\infty} M_i(T) = 0, \quad i = 1,3,5, \qquad \text{a.s.}$$

and

$$\text{(2.11)} \qquad \liminf_{T\to\infty} X_i(T) \leq 0, \quad i = 1,3,5, \qquad \text{a.s.}$$

By (2.10) and (2.11) we have, for $r = 0$,

$$\liminf_{T\to\infty} X_i(T) = 0, \quad i = 1,3,5, \qquad \text{a.s.}$$

Thus the proof is complete.

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