탄소성 대변형 거동에서의 손상의 운동학

The Kinematics of Damage for Elasto-Plastic Large Deformation

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요 약 : 탄소성 대변형에서의 손상의 운동학을 연속체 역학적 구도 안에서 유효 응력의 개념을 통하여 4차 유효 손상 텐사를 이용하여 소개하였다. 손상 변형의 운동학적인 기술의 부재로 인하여 소변형 문제에서는 고체의 손상의 특성을 기술하기 위해서는 다음의 두 가지 가정 (변형률 등가의 가정 또는 에너지 등가의 가정)중의 하나가 일반적으로 채택되어진다. 본 연구에서 제안된 방법은 대변형에 적용될 수 있는 손상 거동의 운동학적인 일반화된 방법을 제공한다. 이 방법은 소 변형률에 국한되는 변형률 등가의 가정이나 에너지 등가의 가정 방식이 아닌 변형장의 운동학을 직접 고려하여 손상 거동의 운동학을 2차 손상 텐사의 함수인 4차 유효 손상 텐사를 이용하여 탄성 및 소성 영역에서 표현하였다.

ABSTRACT : In this paper the kinematics of damage for finite strain, elasto-plastic deformation is introduced using the fourth-order damage effect tensor through the concept of the effective stress within the framework of continuum damage mechanics. In the absence of the kinematic description of damage deformation leads one to adopt one of the following two different hypotheses for the small deformation problems. One uses either the hypothesis of strain equivalence or the hypotheses of energy equivalence in order to characterize the damage of the material. The proposed approach in this work provides a general description of kinematics of damage applicable to finite strains. This is accomplished by directly considering the kinematics of the deformation field and furthermore it is not confined to small strains as in the case of the strain equivalence or the strain equivalence approaches. In this work, the damage is described kinematically in both the elastic domain and plastic domain using the fourth order damage effect tensor which is a function of the second-order damage tensor. The damage effect tensor is

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본 논문에 대한 토의를 1997년 12월 30일까지 학회로 보내주시면 토의회답을 게재하겠습니다.

explicitly characterized in terms of a kinematic measurure of damage through a second-order damage tensor. Two kinds of second-order damage tensor representations are used in this work with respect to two reference configurations.

핵 심 용 어 : 탄소성 대변형, 연속체 역학, 손상유효텐사
KEYWORDS : elast-plastic deformation, finite strain, continuum mechanics, damage effect tensor

1. Introduction

In 1958, Kachanov [1] introduced the concept of effective stress in damaged materials. This pioneering work started the subject that is now known as continuum damage mechanics. Research in this area has steadily grown and reached a stage that warrants its use in today's engineering applications. Continuum damage mechanics is now widely used in different areas including brittle failure (Krajcinovic [2,3], Krajcinovic and Foneska [4], Lubarda et al. [5], Ju and Lee [6], ductile failure (Lemaitre [7,8], Chaboche [9,10,11,12], Chow and Wang [13]), composite materials (Allen et al. [14], Boyd et al. [15], Voyiadjis and Kattan [16], Voyiadjis and Park [17,18] and fatigue (Chow and Wei [19], Voyiadjis and Echle [20]). In this theory, a continuous damage variable is defined and used to represent degradation of the material which reflects various types of damage

at the micro-scale level like nucleation and growth of voids, cavities, micro-crack, and other microscopic defects.In continuum damage mechanics, the effective stress tensor is usually not symmetric. This leads to a complicated theory of damage mechanics involving micropolar media and the Cosserat continuum. Therefore, to avoid such a theory, symmetrization of the effective stress tensor is used to formulate a continuum damage theory in the classical sense (Lee et al. [21], Sidoroff [22], Cordebois and Sidoroff [23], Murakami and Ohno [24], Betten [25], and Lu and Chow [26]). Recently, Voyiadjis and Park [27] reviewed a linear transformation tensor, defined as a fourth-order damage effect tensor and focused on its geometric symmetrization method in order to describe the kinematics of damage using the second-order damage tensor. Park and Voyiadjis [28] introduced the kinematics of damage in the finite deformation field using the damage effect tensor which does not only symmetrize the effective stress tensor but can also be related to the deformation gradient of damage.

The kinematics of damage is described here using the second-order damage tensor. The deformation gradient of damage is defined using the second-order damage tensor. The Green deformation tensor of the damage elasto-plastic deformation is also derived.

2. Theoretically Preliminares

A continuous body in an initial undeformed configuration that consists of the material volume Ω^0 is denoted by C^0, while the elasto-plastic damage deformed configuration at time t after the body is subjected to a set of external agencies is denoted by C^t. The corresponding material volume at time, t is denoted by Ω^t. Upon elastic unloading from the configuration C^t an intermediate stress free configuration is denoted by C^{dp}. In the framework of continuum damage mechanics a number of fictitious configurations, based on the effective stress concept, are assumed that are obtained by fictitiously removing all the damage that the body has undergone. Thus the fictitious configuration of the body denoted by $\overline{C}^t$ is obtained from C^t by fictitiously removing all the damage that the body has undergone at C^t. Also the fictitious configuration denoted by $\tilde{C}^F$ is assumed which is obtained from C^{dp} by fictitiously removing all the damage that the body has undergone at C^{dp}. While the configuration $\overline{C}^F$ is the intermediate configuration upon unloading from the configuration $\overline{C}^t$. The initial undeformed body may have a pre-existing damage state. The initial fictitious effective configuration denoted by $\overline{C}^0$ is defined by removing the initial damage from the initial undeformed configuration of the body. In the case of no initial damage existing in the undeformed body, the initial fictitious effective configuration is identical to the initial undeformed configuration. Cartesian tensors are used in this work and the tensorial index notation is employed in all equations. The tensors used in the text are denoted by boldface letters. However, superscripts in the notation do not indicate tensorial index but merely stand for corresponding deformation configurations such as "e" for elastic, "p" for plastic, and "d" for damage etc. The barred and tilded notations refer to the fictitious effective configurations

3. Description of Damage State

The damage state can be described using an even order tensor (Leckie [29], Onat [30] and Betten [31]). Ju [32] pointed out that even for isotropic damage one should employ a damage tensor(not a scalar damage variable) to characterize the state of damage in materials. However, the damage generally is anisotropic due to the external agency condition or the material nature itself. Although the fourth-order damage tensor can be used directly as a linear transformation tensor to define the effective stress tensor, it is not easy to characterize physically the fourth-order damage tensor compared to the second-order damage tensor. In this work, the damage is considered as a symmetric second-order tensor. However, damage tensor for the finite elasto-plastic deformation can be defined in two reference systems [33]. The first one is the damage tensor denoted by Φ representing the damage state with respect to the current damaged configuration, $\overline{C}^{t}$. Another one is denoted by ϕ and is representing the damage state with respect to the elastically unloaded damage configuration, $\overline{C}^{dp}$. Both are given by Murakami [34] as follows

$$\phi_{ij} = \sum_{k=1}^{n} \hat{\phi}_k \, \hat{n}_i^k \, \hat{n}_j^k \text{ (no sum in k)} \tag{1}$$

and

$$\psi_{ij} = \sum_{k=1}^{3} \hat{\psi}_k \, \widehat{m}_i^k \, \widehat{m}_j^k \text{ (no sum in k)} \tag{2}$$

where $\hat{n}^k$ and $\widehat{m}^k$ are eigenvectors corresponding to the eigenvalues, $\hat{\psi}_k$ and $\hat{\phi}_k$ of the damage tensors, ϕ and ψ, respectively. Equations (1) and (2) can be written alternatively as follows

$$\phi_{ij} = b_{ir} b_{js} \hat{\phi}_{rs} \tag{3}$$

and

$$\psi_{ij} = c_{ir} c_{js} \hat{\psi}_{rs} \tag{4}$$

The damage tensors in the coordinate system that coincides with the three orthogonal principal directions of the damage tensors, $\hat{\Phi}_{rs}$ and $\hat{\phi}_{rs}$, in equations (3) and (4) are obviously of diagonal form and are given by

$$\hat{\phi}_{ij} = \begin{bmatrix} \hat{\phi}_1 & 0 & 0 \\ 0 & \hat{\phi}_2 & 0 \\ 0 & 0 & \hat{\phi}_3 \end{bmatrix} \tag{5}$$

$$\hat{\psi}_{ij} = \begin{bmatrix} \hat{\psi}_1 & 0 & 0 \\ 0 & \hat{\psi}_2 & 0 \\ 0 & 0 & \hat{\psi}_3 \end{bmatrix} \tag{6}$$

and the second order transformation tensors, b and c are given by

$$b_{ir} = \begin{bmatrix} n_1^1 & n_2^{\,1} & n_3^{\,1} \\ n_1^{\,2} & n_2^{\,2} & n_3^{\,2} \\ n_1^{\,3} & n_2^{\,3} & n_3^{\,3} \end{bmatrix} \tag{7}$$

$$b_{ir} = \begin{bmatrix} m_1^1 & m_2^{\,1} & m_3^{\,1} \\ m_1^{\,2} & m_2^{\,2} & m_3^{\,2} \\ m_1^{\,3} & m_2^{\,3} & m_3^{\,3} \end{bmatrix} \tag{8}$$

The relation between the damage tensors ϕ and ψ is shown in section 5.

4. Fourth-Order Anisotropic Damage Effect Tensor

In a general state of deformation and

damage, the effective stress tensor $\bar{\sigma}$ is related to the Cauchy stress tensor σ by the following linear transformation (Murakani and Ohno [24])

$$\bar{\sigma}_{ij} = M_{ikjl}\sigma_{kl} \tag{9}$$

where M is a fourth-order linear transformation operator called the damage effect tensor. Depending on the form used for M, it is very clear from equation (9) that the effective stress tensor $\bar{\sigma}$ is generally nonsymmetric. Using a non-symmetric effective stress tensor as given by equation (9) to formulate a constitutive model will result in the introduction of the Cosserat and a micropolar continua. However, the use of such complicated mechanics can be easily avoided if the proper fourth-order linear transformation tensor is formulated in order to symmetrize the effective stress tensor. Such a linear transformation tensor called the damage effect tensor is obtained in the literature [8,21] using symmetrization methods.

One of the symmetrization methods given by Cordebois and Sidoroff [35] and Lee et al. [21] is expressed as follows

$$\bar{\sigma}_{ij} = (\delta_{ik} - \phi_{ik})^{-1/2}\sigma_{kl}(\delta_{jl} - \phi_{jl})^{-1/2} \tag{10}$$

The fourth-order damage effect tensors corresponding to equations (10) is defined such that

$$M_{ikjl} = (\delta_{ik} - \phi_{ik})^{-1/2}(\delta_{jl} - \phi_{jl})^{-1/2} \tag{11}$$

In order to describe the kinematics of damage, the physical meaning of the fourth-order damage effect tensor should be interpreted and not merely given as the symmertrization of the effective stress. In this work, the fourth-order damage effect tensor given by equation (11) will be used because of its geometrical symmetrization of the effective stress [35]. However, it is very difficult to obtain the explicit representation of $(\delta_{ik} - \phi_{ik})^{-1/2}$. The explicit representation of the fourth-order damage effect tensor M using the second-order damage tensor ϕ is of particular importance in the implementation of the constitutive modeling of damage mechanics. Therefore, the damage effect tensor M equation (11) should be obtained using the coordinate transformation of the principal damage direction coordinate system. Thus the fourth-order damage effect tensor given by equation (11) can be written as follows (Voyiadjis and Park [27])

$$M_{ikjl} = b_{mi}b_{nj}b_{pk}b_{ql}\hat{M}_{mnpq} \tag{12}$$

where $\hat{M}$ is a fourth-order damage effect tensor with reference to the principal damage direction coordinate system. The fourth-order damage effect tensor $\hat{M}$ can be written as follows (Voyiadjis and Park [27])

$$\hat{M}_{mnpq} = \hat{a}_{mp}\ \hat{a}_{np} \tag{13}$$

where the second-order tensor a in the principal damage direction coordinate system is given by

$$\hat{a}_{mp} = [\delta_{mp} - \hat{\phi}_{mp}]^{-\frac{1}{2}} = \begin{bmatrix} \frac{1}{\sqrt{1-\phi_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\phi_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-\phi_3}} \end{bmatrix} \tag{14}$$

Substituting equation (13) into equation (12), one obtains the following relation

$$\hat{M}_{ikjl} = \hat{b}_{mi}\ \hat{b}_{nj}\ \hat{b}_{pk}\ \hat{b}_{ql}\ \hat{a}_{mp}\ \hat{a}_{nq} = a_{ik}\, a_{jl} \tag{15}$$

Using equation (15), a second-order tensor a is defined as follows

$$a_{ik} = b_{mi} b_{pk}\ \hat{a}_{mp} \tag{16}$$

The matrix form of equation (16) is as follows (Voyiadjis and Park [27])

$$[a] = [b]^T [\hat{a}]\ [b] \tag{17}$$

$$= \begin{bmatrix} \frac{b_{11}b_{11}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{21}b_{21}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{31}b_{31}}{\sqrt{1-\hat{\phi}_3}} & \frac{b_{11}b_{12}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{21}b_{22}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{31}b_{32}}{\sqrt{1-\hat{\phi}_3}} & \frac{b_{11}b_{13}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{21}b_{23}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{31}b_{33}}{\sqrt{1-\hat{\phi}_3}} \\ \frac{b_{12}b_{11}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{22}b_{21}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{32}b_{31}}{\sqrt{1-\hat{\phi}_3}} & \frac{b_{12}b_{12}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{22}b_{22}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{32}b_{32}}{\sqrt{1-\hat{\phi}_3}} & \frac{b_{12}b_{13}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{22}b_{23}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{32}b_{33}}{\sqrt{1-\hat{\phi}_3}} \\ \frac{b_{13}b_{11}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{23}b_{21}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{33}b_{31}}{\sqrt{1-\hat{\phi}_3}} & \frac{b_{13}b_{12}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{23}b_{22}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{33}b_{32}}{\sqrt{1-\hat{\phi}_3}} & \frac{b_{13}b_{13}}{\sqrt{1-\hat{\phi}_1}} + \frac{b_{23}b_{23}}{\sqrt{1-\hat{\phi}_2}} + \frac{b_{33}b_{33}}{\sqrt{1-\hat{\phi}_3}} \end{bmatrix}$$

5. The Kinematics of Large Deformation with Damage

A position of a particle in C^0 at t^o is denoted by X and can be defined at its corresponding position in C^t at t, denoted by χ. Futhermore, assuming that the deformation is smooth regardless of damage, one can assume a one-to-one mapping such that

$$\chi_k = \chi_k(X, t) \tag{18}$$

or

$$X_k = X_k(\chi, t) \tag{19}$$

The corresponding deformation gradient is expressed as follows

$$F_{ij} = \frac{\partial \chi_i}{\partial X_j} \tag{20}$$

and the change in the squared length of a material filament dX is used as a measure of deformation such that

$$(ds)^2 - (dS)^2 = d\chi_i d\chi_i - dX_i dX_i = 2E_{ij}\, dX_i dX_j \tag{21}$$

or

$$(ds)^2 - (dS)^2 = 2\varepsilon_{ij}\, d\chi_i\, d\chi_j \tag{22}$$

where $(ds)^2$ and $(dS)^2$ the squared lengths of the material filaments in the deformed with damage configuration C^t, and the initial undeformed configuration C^0, respectively. E and ε are the Lagrngian and Eulerian strain tensors respectively and are given by

$$E_{ij} = \frac{1}{2}[F_{ki}F_{kj} - \delta_{ij}] = \frac{1}{2}(C_{ij} - \delta_{ij}) \tag{23}$$

$$\varepsilon_{ij} = \frac{1}{2}[\delta_{ij} - F_{ki}^{-1}F_{kj}^{-1}] = \frac{1}{2}(\delta_{ij} - B_{ij}^{-1}) \tag{24}$$

where C and B are the right Cauchy-Green and the left Cauchy-Green tensors, respectively.

The velocity vector field in the current configuration at time t is given by

$$\nu_i = \left(\frac{d\chi_i}{dt}\right) \tag{25}$$

The velocity gradient in the current configuration at time t is given by

$$\begin{aligned} L_{ij} &= \frac{\partial \nu_i}{\partial \chi_l} \\ &= \dot{F}_{ir}F_{ik}^{-1} \\ &= D_{ij} + W_{ij} \end{aligned} \tag{26}$$

where the dot designates the material time derivative and where D and W are the rate of deformation (stretching) and the vorticity, respectively. The rate of deformation, D is equal to the symmetric part of the velocity gradient L while the vorcity, W is the antisymmetric part of the velocity gradient L such that

$$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) \tag{27}$$

$$W_{ij} = \frac{1}{2}(L_{ij} - L_{ji}) \tag{28}$$

Strain rate measures are obtained by differentiating equations (21) and (22) such that

$$\begin{aligned} \frac{d}{dt}[(ds)^2 - (dS)^2] &= 2dX_i\dot{E}_{ij}dX_j \\ &= 2d\chi_i D_{ij} d\chi_j \\ &= 2dX_i F_{ik} D_{ij} F_{jm} dX_m \\ &= 2d\chi_i[\dot{\varepsilon}_{ij} + \varepsilon_{ik}L_{kj} + L_{ik}\varepsilon_{kj}]d\chi_j \end{aligned} \tag{29}$$

By comparing the first equation and the third of equation (29) one obtains the rate of the Lagrangian strain that is the projection of D onto the reference frame as follows

$$\dot{E}_{ij} = F_{ki}D_{ki}F_{ij} \tag{30}$$

while the deformation rate D is equal to the Cotter-Rivin convected rate of the Eulerian strain as follows

$$D_{ij} = \dot{\varepsilon}_{ij} + \varepsilon_{ik}L_{kj} + L_{ik}\varepsilon_{kj} \tag{31}$$

The convected derivative shown in

equation (31) can also be interpreted as the Lie derivative of the Eulerian strain ([36]).

5.1 A Multiplicative Decomposition

A schematic drawing representing the kinematics of elasto-plastic damage deformation is shown in Figure 1. C^0 is the initial undeformed configuration of the body which may have an initial damage in the material. C^t represents the current elasto-plastically deformed and damaged configuration of the body. The configuration $\overline{C}^0$ represents the initial configuration of the body that is obtained by fictitiously removing the initial damage from the C^0 configuration. If the initial configuration is undamaged consequently there is no difference between configurations C^0 and $\overline{C}^0$. Configuration $\overline{C}^t$ is obtained by fictitiously removing the damage from configuration C^t. Configuration C^{dp} is an intermediate configuration upon elastic unloading. In the most general case of large deformation processes, damage may be involved due to void and microcrack development because of external agencies. Although damage in the microlevel is a material discontinuity, damage can be considered as an irreversible deformation process in the framework of Continuum Damage Mechanics. Furthermore, one assumes that upon unloading from the elasto-palstic damage state, the elastic part of the deformation can be completely recovered while no additional plastic deformation and damage takes place. Thus upon unloading the elasto-plastic

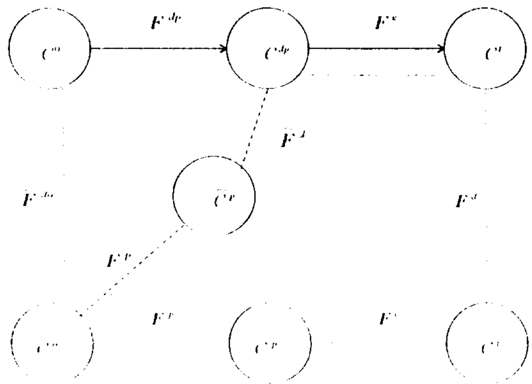

Fig 5.1 Schematic representation of elasto-plastic damage deformation configurations

damage deformed body from the current configuration C^t will elastically unload to an intermediate stress free configuration denoted by C^{dp} as shown in Fig 1. Although the damage process is an irreversible deformation hermodynamically, however, deformation due to damage itself can be partially or completely recovered upon unloading due to closure of micro-cracks or contraction of micro-voids. Nevertheless, recovery of damage deformation does not mean the healing of damage. No materials are brittle or ductile. The deformation gradient tensor and the Green deformation tensor of the elasto-plastic damage deformation can be obtained through Path I, Path II or Path III as shown in Figure 1. Considering Path I the deformation gradient referred to the undeformed configuration, C^0 is denoted by F and is polarly decomposed into the elastic deformation gradient denoted by F^e and the damage-plastic deformation gradient denoted by F^{dp} such that

$$F_{ij} = F^e_{ik} F^{dp}_{kj} \tag{32}$$

The elastic deformation gradient is given by

$$F^e_{ij} = \frac{\partial \chi_i}{\partial \chi^{dp}_j} \tag{33}$$

The corresponding damage-plastic deformation gradient is given by

$$F^{dp}_{ij} = \frac{\partial \chi^{dp}_i}{\partial X_j} \tag{34}$$

The Right Cauchy G._en deformation tensor, C, is given by

$$C_{ij} = F^{dp}_{nk} F^e_{ki} F^e_{nm} F^{dp}_{mj} \tag{35}$$

The finite deformation damage models by Ju [37] and Zbib [38] emphasize that "added flexibility" due to the existence of microcracks or microvoids is already embedded in the deformation gradient implicity. Murakami [33] presented the kinematics of damage deformation using the second-order damage tensor. However, the lack of an explicit formulation for the kinematics of finite deformation with damage leads to the failure in obtaining an explicit derivation of the kinematics that directly consider the damage deformation. Although most finite strain elasto-plastic deformation processes involve damage such as micro-voids, nucleations and micro-crack development due to external agencies, however, only the elastic and plastic deformation processes are cosidered kinematically due to the complexity in the involvement of damage deformation. In this work, the kinematics of damage will be explicitely characterized based on continuum damage mechanics. The elastic deformation gradient corresponds to elastic stretching and rigid body rotations due to both internal and external constraints. The plastic deformation gradient is arising from purely irreversible processes due to dislocations in the material. Damage may be initiated and evolves in both the elastic and plastic deformation processes. Particularly, damage in the elastic deformation state is termed elastic damage which is the case for most brittle materials while damage in the plastic deformation state is termed plastic damage which is mainly for ductile materials. Additional deformation due to damage consists of damage itself with additional deformation due to elastic and plastic deformation. This causes loss of elastic and plastic stiffness. In this work, kinematics of damage deformation is completely described for both damage and the coupling of damage with elasto-plastic deformation. The total Lagrangian strain tensor is expressed as follows

$$
\begin{aligned}
E_{ij} &= \frac{1}{2}(F^{dp}_{ki}F^{dp}_{kj} - \delta_{ij}) \\
&\quad + \frac{1}{2}F^{dp}_{mi}(F^{e}_{km}F^{e}_{kn} - \delta_{mn})F^{dp}_{nj} \\
&= E^{dp}_{ij} + F^{dp}_{mi}\varepsilon^{e}_{mn}F^{dp}_{nj} \\
&= E^{dp}_{ij} + E^{e}_{ij}
\end{aligned}
\tag{36}
$$

where E^{dp} and E^{e} are the Lagrangian damage-plastic strain tensor and the Lagrangian elastic strain tensor measured with respect to the reference configuration C^0, respectively. While ε^{e} is the Lagrangian elastic strain tensor measured with respect to the intermediate configuration C^{dp}. Similarly, the Eulerian strains corresponding to deformation gradients F^{e} and F^{dp} are given by

$$
e^{dp}_{ij} = \frac{1}{2}(\delta_{ij} - F^{dp-1}_{ki}F^{dp-1}_{kj}) \tag{37}
$$

$$
\varepsilon^{e}_{ij} = \frac{1}{2}(\delta_{ij} - F^{e-1}_{ki}F^{e-1}_{kj}) \tag{38}
$$

The total Eulerian strain tensor can be expressed as follows

$$
\begin{aligned}
\varepsilon_{ij} &= \varepsilon^{e}_{ij} + F^{e-1}_{ki}e^{dp}_{km}F^{e-1}_{mj} \\
&= \varepsilon^{e}_{ij} + \varepsilon^{dp}_{ij}
\end{aligned}
\tag{39}
$$

The strain e^{dp} is referred to the intermediate configuration C^{dp}, while the strains, ε, ε^{e} and ε^{dp} are defined relative to the current configuration as a reference. The relationship between the Lagrangian and Eulerian strains is obtained directly in the form

$$
E_{ij} = F_{ki}\varepsilon_{kl}F_{lj} \tag{40}
$$

The change in the squared length of a materrial filament deformed elastically from C^{t} to C^{dp} is given by

$$
\begin{aligned}
(ds)^2 - (dS^{dp})^2 &= d\chi_i d\chi_i - d\chi^{dp}_i d\chi^{dp}_i \\
&= 2dX_i E^{e}_{ij} dX_i
\end{aligned}
\tag{41}
$$

However, the change in the squared length of a material filament deformed due to damage and plastic deformation from C^{dp} to C^0 is given by

$$
(ds^{dp})^2 - (dS)^2 = 2dX_i E^{dp}_{ij} dX, \tag{42}
$$

The kinematics of finite strain elasto-plastic deformation including damage is completely described in Path I. In order to describe the kinematics of damage and plastic deformation, the deformation gradient given by equation (32) may be further decomposed into

$$
F_{ij} = F^{e}_{ik}F^{d}_{km}F^{p}_{mj} \tag{43}
$$

However, it is very difficult to characterize physically only the kinematics of deformation due to damage inspite of its obvious physical phenomena. The damage, however, may be defined through the effective stress concept. Similarly the kinematics of damage can be described using the effective kinematic configuration. Considering Path II the deformation gradient can be alternatively expressed as follows

$$F_{ij} = \overline{F}^{d}_{ik} \overline{F}^{e}_{km} \overline{F}^{p}_{mn} \overline{F}^{do}_{nj} \tag{44}$$

where $\overline{F}^{d}$ is the fictitious damage deformation gradient from configuration $\overline{C}^{t}$ to C^{t} and is given by

$$F^{d}_{ij} = \frac{\partial \chi_i}{\partial \overline{\chi}_j} \tag{45}$$

The elastic deformation gradient in the effective configuration is given by

$$\overline{F}_{ij}{}^{e} = \frac{\partial \overline{\chi}_i}{\partial \overline{\chi}_j{}^{p}} \tag{46}$$

The corresponding plastic deformation gradient in the effective configuration is given by

$$F^{p}_{ij} = \frac{\partial \chi^{p}_{i}}{\partial \overline{X}_j} \tag{47}$$

while the fictitious initial damage deformation gradient from configuration $\overline{C}^{0}$ to C^{0} is given by

$$\overline{F}^{do}_{ij} = \frac{\partial \overline{X}_i}{\partial \overline{X}_j} \tag{48}$$

Similar to Path I, the Right Cauchy Green deformation tensor, C^{t}, is given by

$$C_{ij} = \overline{F}^{do}_{mk} \overline{F}^{p}_{kp} \overline{F}^{q}_{kp} \overline{F}^{d}_{qi} \overline{F}^{d}_{mn} \overline{F}^{p}_{nr} \overline{F}^{p}_{rs} \overline{F}^{do}_{sj} \tag{49}$$

The Lagrangian damage strain tensor measured with respect to the fictious configuration $\overline{C}^{t}$ is given by

$$\overline{\varepsilon}^{d}_{ij} = \frac{1}{2} (\overline{F}^{d}_{ki} \overline{F}^{d}_{kj} - \delta_{ij}) \tag{50}$$

and the corresponding Lagrangian effective elastic strain tensor measured with respect to the fictious configuration $\overline{C}^{p}$ is given by

$$\overline{\varepsilon}^{e}_{ij} = \frac{1}{2} (\overline{F}^{e}_{ki} \overline{F}^{e}_{kj} - \delta_{ij}) \tag{51}$$

The Lagrangian effective plastic strain tensor measured with respect to the fictious undamaged initial configuration $\overline{C}^{0}$ is given by

$$\overline{\varepsilon}^{p}_{ij} = \frac{1}{2} (\overline{F}^{p}_{ki} \overline{F}^{P}_{kj} - \delta_{ij}) \tag{52}$$

The total Lagrangian strain tensor is therefore expressed as follows

$$\begin{aligned} \overline{E}_{ij} = & \frac{1}{2} (\overline{F}^{do}_{ki} \overline{F}^{do}_{kj} - \delta_{ij}) \\ & + \frac{1}{2} \overline{F}^{do}_{mi} (\overline{F}^{p}_{km} \overline{F}^{p}_{kn} - \delta_{mn}) \overline{F}^{do}_{nj} \\ & + \frac{1}{2} \overline{F}^{do}_{ni} \overline{F}^{p}_{rn} (\overline{F}^{e}_{qr} \overline{F}^{e}_{qs} - \delta_{rs}) \overline{F}^{p}_{sm} \overline{F}^{do}_{mj} \\ & + \frac{1}{2} \overline{F}^{do}_{wi} \overline{F}^{p}_{nw} \overline{F}^{e}_{rn} (\overline{F}^{d}_{qr} \overline{F}^{d}_{qs} - \delta_{rs}) \overline{F}^{e}_{sm} \overline{F}^{p}_{mk} \overline{F}^{do}_{kj} \end{aligned} \tag{53}$$

The Lagrangian initial damage strain tensor measured with respect to the reference configuration $\overline{C}^0$ is denoted by

$$\overline{E}_{ij}^{do} = \frac{1}{2}\left(\overline{F}_{ki}^{do}\overline{F}_{kj}^{do} - \delta_{ij}\right) \tag{54}$$

The Lagrangian plastic strain tensor measured with respect to the reference configuration C^o is denoted by

$$\overline{E}_{ij}^{p} = \overline{F}_{ki}^{do}\overline{\varepsilon}_{km}^{p}\overline{F}_{mj}^{do} \tag{55}$$

One now defines the Lagrangian elastic strain tensor measured with respect to the reference configuration C^o as follows:

$$\overline{E}_{ij}^{e} = \overline{F}_{ni}^{do}\overline{F}_{nk}^{p}\overline{\varepsilon}_{km}^{e}\overline{F}_{nr}^{p}\overline{F}_{rj}^{do} \tag{56}$$

and the corresponding Lagrangian damage strain tensor measured with respect to the reference configuration C^0 is given by

$$\overline{E}_{ij}^{d} = \overline{F}_{wi}^{do}\overline{F}_{wn}^{p}\overline{F}_{nk}^{e}\overline{\varepsilon}_{km}^{d}\overline{F}_{mr}^{p}\overline{F}_{sj}^{do} \tag{57}$$

The total Lagrangian strain is now given as follows through the additive decomposition of the corresponding strains

$$E_{ij} = \overline{E}_{ij}^{do} + \overline{E}_{ij}^{p} + \overline{E}_{ij}^{e} + \overline{E}_{Ij}^{d} \tag{58}$$

The change in the squared length of a material filament deformed due to fictitiously removing of damage from C^t to $\overline{C}^0$ is given by

$$\begin{aligned}(ds)^2 - (\overline{d}s)^2 &= d\chi_i d\chi_i - d\overline{\chi}_i d\overline{\chi}_i \\ &= 2dX_i\overline{E}_{ij}^{d}dX_j\end{aligned} \tag{59}$$

The change in the squared length of a material filament deformed elastically from $\overline{C}^0$ to $\overline{C}^p$ is given by

$$\begin{aligned}(d\overline{s})^2 - (d\overline{s}^p)^2 &= d\overline{\chi}_i d\overline{\chi}_i - d\overline{\chi}_i^{\,p} d\overline{\chi}_i^{\,p} \\ &= 2dX_i\overline{E}_{ij}^{e}dX_j\end{aligned} \tag{60}$$

The change in the squared length of a material filament deformed plastically from

$$\begin{aligned}(d\overline{s}^p)^2 - (d\overline{S})^2 &= d\overline{\chi}_i^{p}d\overline{\chi}_i^{p} - dX_i dX_i \\ &= 2\overline{E}_{ij}^{p}dX_i dX_j\end{aligned} \tag{61}$$

while the change in the squared length of a material filament deformed due to fictitious removing of the initial damage from $\overline{C}^0$ to C^0

$$\begin{aligned}(d\overline{S})^2 - (dS)^2 &= d\overline{X}_i d\overline{X}_i - dX_i dX_i \\ &= 2dX_i\overline{E}_{ij}^{do}dX_i\end{aligned} \tag{62}$$

Finally Path III gives the deformation gradient as follows

$$F_{ij} = F_{il}^{e}\hat{F}_{im}^{d}\hat{F}_{mn}^{p}\overline{F}_{mn}^{do} \tag{63}$$

where $\hat{F}^a$ is the fictious damage deformation gradient from configuration $\hat{C}^p$ to $\overline{C}^{dp}$ and is given by

$$\tilde{F}^{d}_{ij}=\frac{\partial \chi^{dp}_{i}}{\partial \chi^{p}_{j}} \tag{64}$$

and the corresponding plastic deformation gradient in the effective configuration is given by

$$\tilde{F}^{p}_{ij}=\frac{\partial \chi^{p}_{i}}{\partial \chi^{p}_{j}} \tag{65}$$

Similar to Path II, the Right Cauchy Green deformation tensor C is given by

$$C_{ij}=\overline{F}^{do}_{mk}\tilde{F}^{p}_{kp}\tilde{F}^{d}_{pq}F^{e}_{qi}F^{e}_{mn}\overline{F}^{d}_{nr}\tilde{F}^{p}_{rs}\overline{F}^{do}_{sj} \tag{66}$$

The Lagrangian damage strain tensor measured with respect to the fictitious intermediate configuration $\tilde{C}^{p}$ is given by

$$\varepsilon^{d}_{ij}=\frac{1}{2}(\tilde{F}^{d}_{ki}\tilde{F}^{d}_{kj}-\delta_{ij}) \tag{67}$$

The total Lagrangian strain tensor is expressed as follows

$$\begin{aligned}\overline{E}_{ij}= & \frac{1}{2}(\overline{F}^{do}_{ki}\overline{F}^{do}_{kj}-\delta_{ij})\\ &+\frac{1}{2}\overline{F}^{do}_{mi}(\tilde{F}^{p}_{km}\tilde{F}^{p}_{kn}-\delta mn)\overline{F}^{do}_{nj}\\ &+\frac{1}{2}\overline{F}^{do}_{ni}\tilde{F}^{p}_{rn}(\tilde{F}^{do}_{ni}\tilde{F}^{d}_{as}-\delta_{rs})\tilde{F}^{p}_{sm}\overline{F}^{do}_{mj}\\ &+\frac{1}{2}\overline{F}^{do}_{ui}\tilde{F}^{p}_{nu}\tilde{F}^{d}_{rn}(F^{e}_{ar}F^{e}_{as}-\delta_{rs})\tilde{F}^{d}_{sm}\tilde{F}^{p}_{mk}\overline{F}^{do}_{kj}\end{aligned} \tag{68}$$

The Lagrangian damage strain tensor measured with respect to the reference configuration C^{o} is denoted by

$$\tilde{E}^{d}_{ij}=\overline{F}^{do}_{ki}\tilde{F}^{p}_{mk}\tilde{\varepsilon}^{d}_{mn}\tilde{F}^{p}_{nq}\overline{F}^{do}_{qj} \tag{69}$$

The Lagrangian elastic strain tensor measured with respect to the reference configuration C^{o} is denoted by

$$E^{e}_{ij}=\overline{F}^{do}_{li}\tilde{F}^{p}_{kl}\tilde{F}^{d}_{mk}\varepsilon^{e}_{mn}\tilde{F}^{d}_{nq}\tilde{F}^{p}_{qr}\overline{F}^{do}_{rj} \tag{70}$$

The corresponding total Lagrangian strain is now given by

$$E_{ij}=\overline{E}^{do}_{ij}+\tilde{E}^{p}_{ij}+\tilde{E}^{d}_{ij}\tilde{E}^{e}_{ij} \tag{71}$$

The change in the squared length of a material filament deformed due to fictitious removal of damage from C^{dp} to $\tilde{C}^{p}$ is given by

$$\begin{aligned}(ds^{dp})^{u}-(d\,\tilde{s}^{p})^{2} &= d\chi^{dp}_{i}d\chi^{dp}_{i}-d\tilde{\chi}^{p}_{i}d\tilde{\chi}^{p}_{i}\\ &=2dX_{i}\overline{E}^{d}_{ij}dX_{j}\end{aligned} \tag{72}$$

The change in the squared length of a material filament deformed plastically from $\overline{C}^{o}$ to $\tilde{C}^{p}$ is then given by

$$\begin{aligned}(d\,\tilde{s}^{p})^{2}-(d\,\overline{s}^{2}) &= d\tilde{\chi}^{p}_{i}d\tilde{\chi}^{d}_{i}-d\overline{X}_{i}\,d\overline{X}_{i}\\ &=2dX_{i}\tilde{E}^{d}_{ij}dX_{j}\end{aligned} \tag{73}$$

The total Lagrangian strain tensors obtained by considering the three paths are given by equations (36), (58) and (71). From the equivalency of these total strains, one obtains the explicit presentations of the kinematics of damage as follows. With the assumption of the equivalence between the elastic strain tensors given by equations (36) and (71), the damage-plastic deformation gradient given by (34) and the Lagrangian damage plastic strain tensor can be expressed as follows

$$F^{dp}_{ij} = \overline{F}^{do}_{ik} \widehat{F}^{p}_{kl} \widehat{F}^{d}_{lj} \tag{74}$$

and

$$E^{dp}_{ij} = \overline{E}^{do}_{ij} + \widehat{E}^{p}_{ij} + \widehat{E}^{d}_{ij} \tag{75}$$

Furthermore one obtains the following expression from equations (58) and (71) as follows

$$\overline{E}^{e}_{ij} + \overline{E}^{d}_{ij} = \widehat{E}^{d}_{ij} + E^{e}_{ij} \tag{76}$$

which concludes that $\widetilde{C}^{p}$ and $\overline{C}^{p}$ are the same. Substituting equations (57), (69) and (70) into equation (76), one obtains the effective Lagrangian elastic strain tensor as follows

$$\begin{aligned} \overline{E}^{e}_{ij} = {} & \overline{F}^{do}_{ki} \overline{F}^{p}_{mk} \\ & [\, \tilde{\varepsilon}^{d}_{mn} - \overline{F}^{e}_{qm} \tilde{\varepsilon}^{d}_{qr} \widehat{F}^{e}_{rn} \\ & + \widehat{F}^{d}_{qm} \varepsilon^{e}_{qr} \widehat{F}^{d}_{rn}] \overline{F}^{p}_{ns} \overline{F}^{do}_{sj} \end{aligned} \tag{77}$$

Using equations (56) and (77) one can now express $\bar{\varepsilon}$ as follows

$$\bar{\varepsilon}^{e}_{ij} = \tilde{\varepsilon}^{d}_{ij} - \overline{F}^{e}_{mi} \bar{\varepsilon}^{d}_{mn} + \widehat{F}^{d}_{mi} \varepsilon^{e}_{mn} \widehat{F}^{d}_{nj} \tag{78}$$

This expression gives a general relation of the effective elastic strain for finite strains of elasto-plasic damage deformation. For the special case when one assumes that

$$\tilde{\varepsilon}^{d}_{ij} - \overline{F}^{e}_{mi} \bar{\varepsilon}^{d}_{mn} \overline{F}^{e}_{nj} = 0 \tag{79}$$

equation (78) can be reduced to the following expression

$$\tilde{\varepsilon}^{e}_{ij} = \overline{F}^{d}_{ki} \varepsilon^{e}_{kl} \widehat{F}^{d}_{lj} \tag{80}$$

This relation is similar to that obtained without the consideration of the kinematics of damage and only utilizing the hypothesis of elastic energy equivalence. However, equation (80) for the case of finite strains is given by relation (78) which cannot be obtained through the hypothesis of elastic energy equivalence. Equation (79) maybe valid only for some special cases of the small strain theory.

5.2 Fictitious Damage Deformation Gradients

The two fictitious deformation gradients given by equations (45) and (64) may be used to define the damage tensor in order to describe the damage behavior of solids. Since the fictitious effective deformed configuration denoted by $\overline{C}^{t}$ is obtained by removing the damages from the real deformed configuration denoted by C^{t}, therefore the differential volume of the fictitious effective deformed volumes denoted by $d\bar{\Omega}^t$ is obtained as follows [28].

$$d\bar{\Omega}^t = d\Omega^t - d\Omega^d = \sqrt{(1-\hat{\phi}_1)(1-\hat{\phi}_2)(1-\hat{\phi}_3)}\, d\Omega^t \tag{81}$$

or

$$d\Omega^t = \bar{J}^d\, d\bar{\Omega}^t \tag{82}$$

where Ω^a is the volume of damage in the configuration C^t and $\bar{J}^d$ is termed the Jacobian of the damage deformation which is the determinant of the fictitious damage deformation gradient. Thus the Jacobian of the damage deformation can be written as follows

$$\bar{J}^d = |\bar{F}^d_{ij}| = \frac{1}{\sqrt{(1-\hat{\phi}_1)(1-\hat{\phi}_2)(1-\hat{\phi}_3)}} \tag{83}$$

The determinant of the matrix [a] in equation (17) is given by

$$|[a]| = |[b]^T|\,|[\hat{a}]|\,|[b]| = |[a]| = \frac{1}{\sqrt{(1-\hat{\phi}_1)(1-\hat{\phi}_2)(1-\hat{\phi}_3)}} \tag{84}$$

Thus one assumes the following relation without loss of generality

$$\bar{F}_{ij}{}^d = [\delta_{ij} - \phi_{ij}]^{-\frac{1}{2}} \tag{85}$$

Although the identity is established between $\bar{J}^a$ and $|a|$, however, this is not sufficient to demonstrate the validity of equation (85). This relation is assumed here based on the physics of the geometrically symmetrized effective stress [28]. Similarly, the fictitious damage deformation gradient $\tilde{F}^a$ can be written as follows

$$\tilde{F}^d = [\delta_{ij} - \psi_{ij}]^{-\frac{1}{2}} \tag{86}$$

Finally, assuming that $\bar{\chi} = \tilde{\chi}$ based on equation (78) the relations between $\tilde{F}^a$, and $\bar{F}^a$, and ψ and ϕ are given by

$$\tilde{F}^d_{ij} = F^e_{ki}\, \bar{F}^d_{kl}\, F^{e-1}_{lj} \tag{87}$$

and

$$\psi_{ij} = F^e_{kl}\, \phi_{kl}\, F^{e-1}_{lj} \tag{88}$$

6. Constitutive Equation for Finite Elasto-Plastic Deformation with Damage Behavior

The kinematics discussed in the previous sections provide the basis for a finite deformation damage elasto-plasticity. In this section the basic structure of the constitutive equations are reviewed based on the generalized Hooke's law, originally obtained for small elastic strains such that the second Piola-Kirchoff stress tensor S is the gradient of free energy with respect to the Lagrangian elastic strain tensor. Considering three dimensional state of stress and strain, one obtains the following relation

$$
\begin{aligned}
S_{ij} &= \overline{Q}_{ijkl}(E_{kl}-E^{p}_{kl}-E^{d}_{kl}) \\
&= \overline{Q}_{ijkl}E^{e}_{kl} \\
&= \overline{Q}_{ijkl}(E^{e}_{kl}+E^{d}_{kl}) \qquad (89)\\
&= \overline{Q}_{ijkl}(E_{kl}-E^{d'}_{kl}-E^{p}_{kl})
\end{aligned}
$$

From the incremental analysis one obtains the following rate form of the constitutive equation by differentiating equation (89)

$$\dot{S}_{ij}=Q_{ijkl}(\dot{E}_{kl}-\dot{E}^{p}_{kl}-\dot{E}^{d}_{kl}) \qquad (90)$$

Consequently the constitutive equation of the elasto-pastic damage behavior can be written as follows

$$\dot{S}_{ij}=\overline{Q}_{ijkl^{DP}}\ \dot{E}_{kl} \qquad (91)$$

where Q^{DP} is the damage elasto-plastic stiffness and is expressed as follows

$$Q^{DP}_{ijkl}=\overline{Q}_{ikjl}-Q_{ijkl^{p}}-Q_{ijkl^{d}} \qquad (92)$$

where Q^{P} is the plastic stiffness and Q^{d} is the damage stiffness. Both Q^{P} and Q^{d} are the reduction in stiffness due to the plastic and damage deteriorations, respectively. Equation (92) shows the obvious reduction in stiffness due to both the plastic and damage deteriorations which is limited when two hypotheses as mentioned in abstract are used for coupling of the plastic and damage deteriorations. The plastic stiffness and the damage stiffness can be obtained by using the flow rule and damage evolution law, respectively. The details of the complete constitutive models using the proposed kinematics and the evolution laws of damage will be stated in the forthcoming paper.

7. Conclusion

The fourth-order anisotropic damage effect tensor, M, using the kinematic measure for damage expressed through the second-order damage tensor ϕ, is reviewed here in reference to the symmetrization of the effective stress tensor. This introduces a distinct kinematic measure of damage which is complimentary to the deformation kinematic measure of strain. A thermodynamically consistent evolution equation for the damage tensor, ϕ together with a generalized thermodynamic force conjugate to the damage tensor was presented in the paper by Voyiadjis and Park [17,18]. Voyiadjis and Venson [39] quantified the physical values of the eigenvalues, $\hat{\phi}_k(k=1,2,3)$, and the second-order damage tensor, ϕ, for the unidirectional fibrous composite by measuring the crack densities with the assumption that one of the eigendirections of the damage tensor coincides with the fiber direction.

The fourth-order anisotropic damage effect tensor used here is obtained through the geometrical symmetrization of the effective stress [23]. This tensor is used here for the kinematic description of damage. The explicit representation of the fourth-order damage effect tensor is obtained with reference to the principal damage direction coordinate system.

The damage elasto-plastic deformation for finite strain is also described here using the kinematics of damage. In this work the multiplicative decomposition of the deformation gradient and the additive decomposition of the Lagrangian strain tensor are used in order to describe the kinematics of damage. Both formulations are used to deduce seperately the strain due to damage and the coupled elasto-plastic, elastic-damage and plastic-damage strains.

The constitutive relation between the rate of the second Piola-Kirchhoff stress tensor and the Lagrangian strain rate, is established for the elasto-plastic model with damage. The resulting tangential elasto-plastic damage stiffness is obtained in the form of an additive

decomposition of the respective elastic, plastic and damage stiffnesses.

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(접수일자 : 1997. 9. 3)