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On the Restricted Fibred Mapping Projection in a Quasitopos

Young-Sun Kim
Department of Applied Mathematics, Pai Chai University

Quasitopos에서의 제한된 화이버 함수에 대한 연구

김영선
배재대학교 응용수학과

In this paper, we introduce various exponential laws and fibrations in a quasitopos over a base. And the properties of the restricted fibred mapping projection in a quasitopos are investigated.

본 논문에서는 근위의 quasitopos에서의 여러가지 지수법칙과 화이버화를 소개하고 제한된 화이버 함수의 성질들을 조사한다.

Key words : Quasitopos, Fibrations, Evaluation map

I. Preliminaries

The purpose of this section is to collect some basic definitions and known results about the fibrewise category and quasitopos which we shall need in later sections.

Given an object B of a category C, the category C_B of objects over B is defined as follows. An object over B is a pair (X, p) consisting of an object X of C and a morphism $p: X \to B$ of C called the projection ; in practice X alone is usually a sufficient notation. If X , Y are objects over B with projections p, q, then a morphism $f: X \to Y$ of C is a morphism over B if $q \circ f = p$. Composition in C_B is defined according to the composition in C. The isomorphisms (= equvalences) of the category C_B are called ismorphisms (equivalences) over B. Notice that B itself is regarded as an object over B with the projection 1_B. Let X be an object over B with the projection $p: X \to B$. The inverse image $p^{-1}(b)$ of each b in B is denoted by X_b and known as the fibre over b. The category of all sets and functions between them will be denoted by Set. In the category Set_B, we can consider a bifunctor $\times_B$. Specifically, let X, Y be sets over B with projections p , q respectively. A fibre product $X \times_B Y$ is the subset of $X \times Y$ consisting of pairs (x, y) such that $p(x) = q(y)$ with the projection r given by $r(x, y) = p(x) = q(y)$. In fact, $X \times_B Y$ is a product of X and Y in the category Set_B.

Definition 1.1 A concrete category (C, U) over *Set* is topological if every $U-$structured source in *Set* has a unique $U-$initial lift where, $U : C \rightarrow Set$ is the underlying functor.

Dually,we have

Definition 1.2 A concrete category (C, U) over *Set* is cotopological if every $U-$structured sink in *Set* has a unique $U-$final lift.

Remark. If C is a concrete category, then C is topological if and only if it is cotopological.

Definition 1.3 A concrete category C over *Set* is a c-category if every constant map is a morphism.

Definition 1.4 A topological c-category C is a quasitopos if final eqi-sinks in C are closed under the formation of pullbacks.

Defintion 1.5 A concrete category C is cartesian closed if it satisfies the follwing conditions :

(i) C has finite products.

(ii) For any object X of C, the funtor $X \times \quad : C \rightarrow C$ has a right adjoint.

Remark. A topological category C is a quasitopos if and only of for each object B in C, the category C_B is cartesian closed.

II. Fibrewise Exponential Laws in a Quasitopos

Let C be a quasitopos. In this section, we obtain the various exponential laws in the category C_B. From now on, C will denote a quasitopos otherwise specified throughout this section.

It is known that a topological category C is a quasitopos if and only if for each object B in C, the category C_B is cartesian closed. Hereafter we obtain the internal function space structure in C_B. For objects X and Y over B,

let $\hom_B(X, Y) = \coprod_{b \in B} \hom(X_b, Y_b)$ as a set, where $\hom(X_b, Y_b)$ is the set of all morphisms from X_b to Y_b. Since C is cotopological, we can endow $\hom_B(X, Y)$ with the final $C-$structure with respect to the sink $\{f \mid f : Z \rightarrow \hom_B(X, Y)$ set map over B such that $ev \circ (1_X \times_B f)$ is a morphsm in C where Z is in $C_B\}$. Then $\hom_B(X, Y)$ is an object over B with projection $(pq) : \hom_B(X, Y) \rightarrow B$ defined by $(pq)(g) = c$ for $g \in \hom(X_c, Y_c)$. Also, the evaluation map $ev : X \times_B \hom_B(X, Y) \rightarrow Y$ defined by $ev\ (x, f) = f(x)$ is a morphism over B. Furthermore, the funtor $X \times_B \quad : C_B \rightarrow C_B$ is a left adjoint of the functor $\hom_B(X, \quad) : C_B \rightarrow C_B$ via ev.

Theorem 2.1 (The First Exponential Law)

For objects X, Y, Z over B, there is a natural isomorphsm

$\theta : \hom_B(X \times_B Y, Z) \rightarrow \hom_B(X, \hom_B(Y, Z$

given by $\theta(f)(x)(y) = f(x, y)$

Proof. see [4].

For objects X, Y over B, let $Hom_B(X, Y)$ be the set of all morphisms over B from X to Y. Give this the subspace structure of

hom (X, Y), where hom (X, Y) is the set of all morphisms from X to Y. Then the evaluation map
$ev: X\times Hom_B(X, Y) \to Y$ is a morphism in C.

Theorem 2.2 For objects X and Y in C_B, we have an isomorphism
$\sigma: Hom_B(X, Y) \to Hom_B(B, \mathrm{hom}_B(X, Y))$
given by $\sigma(\phi)(b) = \phi_b$ where
$\phi: X \to Y$ is a morphism over B and $\phi_b: X_b \to Y_b$.
Proof. see [4].

Theorem 2.3 (The Second Exponential Law) For objects X, Y, Z over B, there is a natural isomorphism
$\theta: Hom_B(X \times_B Y, Z) \to Hom_B(X, \mathrm{hom}_B(Y, Z))$ where $\theta(f)(x)(y) = f(x, y)$.
Proof. see [4].

Remark. The above two theorems are equivalent under the first exponential law.

III. The Restricted Fibred Mapping Projection.

In this section, we assume that the quasitopos C contains R equipped with a structure allowing addition and multiplication from $R\times R$ to R as morphisms. Further, every map $f: I\times I \to I$ is a morphism if and only if it is continuous in the usual sense where $I = [0, 1]$.

Definition 3.1
(1) A morphism $p: X \to B$ will be said to have covering homotopy property (CHP) with respect to a homotopy $H: A \times I \to B$ if for every morphism $h: A \to X$ such that $p \circ h(a) = H(a, 0)$ $(a \in A)$, there is a homotopy $G: A \times I \to X$ such that $p \circ G = H$ and $G(a, 0) = h(a)$ for all $a \in A$.

$$\begin{array}{ccc} A & \rightarrow & X \\ \downarrow & \nearrow G & \downarrow p \\ A \times I & \rightarrow & B \end{array}$$

(2) p will be called a Hurewiez fibration if it has CHP with respect to all homotopies $H: A\times I \to B$.
(3) p will be called a Dold fibration if it has CHP with respect to all homotopies $H: A\times I \to B$ such that $H(a, t) = H(a, 0)$ for all $0 \le t \le \frac{1}{2}$.

Theorem 3.2 Let $p: X \to B$, $q: Y \to B$ be morphisms in C.
(i) If p, q are Hurewiez fibrations, then so is $\mathrm{hom}_B(X, Y) \to B$.
(ii) If p, q are Dold fibrations, then so is $\mathrm{hom}_B(X, Y) \to B$.
Proof. see [4].

Definition 3.3 Let $j: X_b \to Y_b$ be a morphism for some $b \in B$. We define $(\mathrm{hom}_B(X, Y) : j)$ as the subspace of $\mathrm{hom}_B(X, Y)$ consisting of the path component of $\mathrm{hom}_B(X, Y)$ containing j, and $(pq : j): (\mathrm{hom}_B(X, Y) : j) \to B$ be the restriction of (pq) to $(\mathrm{hom}_B(X, Y) : j)$.

Proposition3.4 (The fibration proposition for $(pq : j)$)
Let $p: X \to B$, $q: Y \to B$ be morphisms in C. If p and q are Hurewiez fibrations, then so is $(pq : j)$.
Proof. Consider the diagram

$$\begin{array}{ccccc} A & \rightarrow & (\mathrm{hom}_B(X, Y) : j) & \rightarrow & \mathrm{hom}_B(X, Y) \\ \downarrow & & & & \downarrow \\ A \times I & & \rightarrow & & B \end{array}$$

Let $a \in A$ be fixed.
Then for any $t \in I$, there is a path from $h(a) = \hat{H}(a, 0)$ to $\hat{H}(a, t)$ $(I \to [0, t] \to I \to (\hom_B(X, Y) : j))$, where $h : A \to (\hom_B(X, Y) : j)$ and $\hat{H} : A \times I \to \hom_B(X, Y)$. Hence $h(a) = \hat{H}(a, 0)$, $\hat{H}(a, t) \in (\hom_B(X, Y) : j)$. Next for any $b \in A$, $h(b)$, $\hat{H}(b, t) \in (\hom_B(X, Y) : j)$. Since $h(a)$, $h(b) \in (\hom_B(X, Y) : j)$ (path-component), $\hat{H}(a, t) \in (\hom_B(X, Y) : j)$.

Definition 3.5 If $f : X \to Y$ is a morphism, then hom $(X, Y : f)$ will denote the subspace of $\hom(X, Y)$ consisting of morphisms homotopic to f.

Proposition3.6 (The fibre proposition for $(pq : j)$)
If $p : X \to B$ and $q : Y \to B$ are Hurewiez fibrations and B is simply connected, then the fibre of $(pq : j)$ over b is hom $(X_b, Y_b : j)$
i.e. $(\hom_B(X, Y) : j)_b$
$= \hom(X_b, Y_b : j)$.
Proof. The fibre consists of at least one path-component of the corresponding fibre of (pq) - for paths in B may be lifted into $\hom_B(X, Y)$ and every pair of points in B are the endpoints of a path $(\pi_0(B) = 0)$. The image in B of a path in $\hom_B(X, Y)$ that connects two points in the fibre over b is a loop at b and this loop may be shrunk to a point $(\pi_1(B) = 0)$. It follows from the fibration property for (pq) that the two points belong to the same path-component of the fibre of (pq) over b.

Proposition 3.7 (The section proposition for $(pq : j)$)

If $p: X \to B$ and $q: Y \to B$ are Hurewiez fibrations and B is simply connected space, then there is a morphism $f : X \to Y$ over B such that
$f \mid X_b \simeq j: X_b \to Y_b$ if and only if there is a section to $(pq : j)$; in fact there is a one-to-one corresponence between these morphisms $f : X \to Y$ over B and the sections $g : B \to (\hom_B(X, Y) : j)$ to $(pq : j)$ defined by $f(x) = g(b)(x)$ where $p(x) = b$.
Proof. f exists if and only if there is a section $h : B \to \hom_B(X, Y)$ to (pq) such that $h(b) \simeq j$. B is path-connected hence $h(B) \subset (\hom_B(X, Y) : j)$ and such maps h correspond to sections g to $(pq : j)$ under the relation $h(b) = g(b)$, $b \in B$ and the result is proved.

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