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An Application of the Multigrid Method to Eigenvalue Problems

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복합마디방법의 고유치문제에 응용

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We apply a full multigrid scheme to computing eigenvalues of the Laplace eigenvalue problem with Dirichlet boundary condition. We use finite difference method to get an algebraic equation and apply inverse power method to estimating the smallest eigenvalue. Our result shows that the combined method of inverse power method and full multigrid scheme is very effective in calculating eigenvalue of the eigenvalue problem.

Dirichlet 경계조건을 갖는 Laplace 고유치방정식의 고유치를 구하는 데 복합마디방법을 이용하였다. 유한차분법을 적용하여 행렬 고유치방정식을 만들고 이 방정식의 고유치를 구하기 위하여 역거듭제곱방법과 전체복합마디법을 사용하였다. 그 결과 고유치를 기존의 방법보다 더욱 빠르게 구할 수 있었다.

Key words : multigrid method, eigenvalue, inverse power method, full multigrid scheme, finite difference method

I. Introduction

For the computation of eigenvalues of a self-adjoint operator, we need to solve generalized matrix eigenvalue problems whose matrices are usually very sparse with large scale. The generalized matrix problem $Ax=\lambda Bx$ may be transformed to

$$(A-\sigma B)^{-1}Bx=\mu x,$$

which is self-adjoint with respect to the inner product induced by the matrix B. If we apply the inverse power method to this problem, we need to solve algebraic equation

$$(A-\sigma B)y=Bx$$

for given σ and x. It could be solved by direct method with some sparse technique. If we can however consider the algebraic equation as a disrete problem of the given equation, we can apply the idea of multigrid method to solving this problem whenever need. Thus we may get an approximation to an eigenvalue very fast comparing to the direct method. In this paper we compute the smallest eigenvalue of Laplace operator with Dirischlet boundary condition on a square domain and compare the results of the mixed method with multigrid idea to those of block tridiagonal Gauss elimination method. It shows that the former is very effective.

II. Eigenvalue Problems

We consider the solution of the second order partial differential eigenvalue problem

$$-\Delta u = \lambda u$$

in a square

$$\Omega = \{(x, y): 0 < x < 1, 0 < y < 1\}$$

with the boundary conditions $u=0$ on the boundary of Ω. For a given k, let

$$n = 2^k - 1 \text{ and } \quad h = \frac{1}{2^k}.$$

Let

$$\Omega^h = \{(x_i, y_j) | x_i = ih, y_j = jh, \ i, j = 1, \ldots, n\}$$

be the interior grid points at level k. If we apply finite differnce scheme with 5 stencils, we get the following generalized algebraic eigenvalue equation

$$A^h u^h = \lambda^h B^h u^h$$

where

$$A^h = \begin{cases} \text{Trid}(-I, D, -I) & \text{if rexicographic ordering} \\ \begin{pmatrix} D_b & H \\ H^t & D_r \end{pmatrix} & \text{if red}-\text{black ordering} \end{cases}$$

and $B^h = h^2 I$ with rank of n^2.

In a case of discrete problem like $Ax=b$, the multigrid method may be applied if we consider the equation $Ax=b$ as a disdrete problem of a continuous equation. Thus we can apply the idea of multigrid method in the inverse power method to approximate the smallest eigenvalue.

Mixed Algorithm of Inverse Power Method with Multigrid:

- Take an initial unit vector x_0 with $x_0 B^h x_0 = 1$.
- For $\nu = 1, 2, \cdots$, maxit
  1. $x = \text{MGM}(A^h, B^h, x_{\nu-1}, h)$
  2. $\beta_\nu = x^T B^h x_{\nu-1}$
  3. $r = \sqrt{x^T B^h x}$
  4. $x_\nu = \frac{x}{r}$
  5. $\lambda^{(h,\nu)} = \frac{1}{\beta_\nu}$

Note that $x = \text{MGM}(A, B, y, h)$ means solving the equation $Ax = By$ in x using the multigrid method.

III. Numerical Results

We use a finite difference discretization with 5 stencils so that we get

$$A^h u^h = \lambda^h B^h u^h.$$

Since we apply inverse power method to the equation, we need to compute solutions of

$$A^h x = B^h x_{\nu-1}$$

for the given right hand side. For the solutions of the equation, we employ both a block tridiagonal Gauss elimination and full multigrid scheme. For the full multigrid scheme we use a red-block Gauss-Seidel method. The results are shown in Tables 1 and 2. This shows that the mixed method produces the result as mush as 81 times fast at level 6 and 884 times fast at level 8. It is clear that the higher the level is, the faster it produces. The reason is that the complexity of the multigrid method is $O(n)$ while that of Gauss Elimination method is $O(n^3)$.

Table 1. With Gauss method

level	λ_1	cpu(sec.)	maxit
2	18.7451	0.00	16
3	19.4868	0.00	14
4	19.6758	0.08	14
5	19.7233	1.21	14
6	19.7352	12.97	14
7	19.7382	176.48	14
8	19.7389	2652.62	14

Table 2. With multigrid method

level	λ_1	cpu(sec.)	maxit
2	18.7451	0.00	16
3	19.4868	0.00	14
4	19.6758	0.00	14
5	19.7233	0.03	14
6	19.7352	0.16	14
7	19.7382	0.77	14
8	19.7389	3.00	14
exact	19.7392		

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