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On the Almost Sure Convergence of Weighted Sums

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가중합에 대한 거의 확실한 수렴성 연구

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Let $\{X, X_n, n\geq 1\}$ be i.i.d. random variables with mean zero and $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ a triangular array of constants. In this paper we give sufficient conditions on X and $\{a_{ni}\}$ such that $\sum_{i=1}^{n} a_{ni}X_i$ converges to zero almostly surely.

$\{X, X_n, n\geq 1\}$은 독립이고 평균이 영으로 같은 확률분포를 갖는 확률변수 열이고, $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$은 수열일 때 가중합 $\sum_{i=1}^{n} a_{ni}X_i$가 0에 확률 1로 수렴할 충분조건을 제시한다.

Key words : almost sure convergence, weighted sums, i.i.d. random variables

1. Introduction

Let $\{X, X_n, n\geq 1\}$ be independent and identically distributed(i.i.d.) random variables with mean zero and $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ a triangular array of constants. Chow[2], Chow and Lai,[3] and Thrum[4] have obtained the following result on almost sure convergence for the cases p=2, $1\leq p<2$ and $p>2$, respectively.

Theorem 1.1.

Let $\{X, X_n, n\geq 1\}$ be i.i.d. random variables with $EX=0$ and $E|X|^p<\infty$ for some $p\geq 1$. Let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be a triangular array of constants satisfying

$$\sum_{i=1}^{n} a_{ni}^2 = O\left(\frac{1}{n^{2/p}}\right). \qquad (1)$$

Then $\sum_{i=1}^{n} a_{ni}X_i \to 0$ almost surely(a.s.).

In this paper, we prove Theorem 1.1 for the case $1<p\leq 2$ under the weaker condition than (1).

Throughout this paper $C>0$ stands for a constant which may be different in various places.

2. Main Result

To prove our main result, we need the following lemma which is a generalization of Lemma 6 in Choi and Sung.[1)]

Lemma 2.1.

Let $\{X_n, n\geq 1\}$ be independent random variables with $EX_n=0$ and $\sup_{n\geq 1}|X_n|\leq C$ for some constant C>0. Let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be a triangular array of constants such that

$$\sum_{i=1}^{n} a_{ni}^2 = O(\frac{1}{n^{\alpha}}) \qquad (2)$$

for some $\alpha>0$. Then $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s.

Proof. From an inequality $e^x\leq 1+x+\frac{1}{2}x^2e^{|x|}$ for all $x\in R$, we have

$$\begin{aligned} Eexp(ta_{ni}X_i) &\leq 1+E[\frac{1}{2}t^2a_{ni}^2X_i^2\exp(t|a_{ni}||X_i|)] \\ &\leq 1+Ct^2a_{ni}^2\exp(C\frac{t}{n^{\alpha/2}}) \\ &\leq \exp[Ct^2a_{ni}^2\exp(C\frac{t}{n^{\alpha/2}})] \end{aligned}$$

for any t>0. Using the independence of X_n, we have

$$\begin{aligned} &E\exp(t\sum_{i=1}^{n}a_{ni}X_i)=\prod_{i=1}^{n}E\exp(ta_{ni}X_i) \\ &\leq \exp[\frac{Ct^2}{n^{\alpha}}\exp(C\frac{t}{n^{\alpha/2}})]. \end{aligned}$$

Let $\varepsilon>0$ be given. By putting $t=2\log n/\varepsilon$, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty}P(\sum_{i=1}^{n}a_{ni}X_i>\varepsilon)\leq\sum_{n=1}^{\infty}e^{-t\varepsilon}Eexp(t\sum_{i=1}^{n}a_{ni}X_i) \\ &\leq\sum_{n=1}^{\infty}\exp[-t\varepsilon+C\frac{t^2}{n^{\alpha}}\exp(C\frac{t}{n^{\alpha/2}})] \\ &\leq C\sum_{n=1}^{\infty}\exp(-2\log n)<\infty. \end{aligned}$$

By the Borel-Cantelli lemma, we have

$$\limsup_{n\to\infty}\sum_{i=1}^{n}a_{ni}X_i\leq 0 \ a.s.$$

By replacing X_i by $-X_i$ from the above statement we obtain

$$\liminf_{n\to\infty}\sum_{i=1}^{n}a_{ni}X_i\geq 0 \ a.s.$$

Thus the conclusion follows.

Now we state and prove our main result.

Theorem 2.2.

Let $\{X, X_n, n\geq 1\}$ be i.i.d. random variables with $EX=0$ and $E|X|^p<\infty$ for some $1<p\leq 2$. Let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be a triangular array of constants such that

$$\sum_{i=1}^{n}|a_{ni}|^q = O(\frac{1}{n^{q/p}}), \qquad (3)$$

where $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\sum_{i=1}^{n}a_{ni}X_i\to 0 \quad \text{a.s.}$$

Proof. Let $\varepsilon>0$ be given and choose a constant M such that $E|X|^pI(|X|>M)<\varepsilon$. Define

$$\begin{aligned} X_i' &= X_iI(|X_i|\leq M)-EXI(|X|\leq M), \\ X_i'' &= X_iI(|X_i|>M)-EXI(|X|>M). \end{aligned}$$

Then $EX_i'=0$ and X_i' are uniformly bounded by 2M. Also we have by (3) that

$$\sum_{i=1}^{n} a_{ni}^2 = \sum_{i=1}^{n}(|a_{ni}|^q)^{2/q} \le \sum_{i=1}^{n}(\sum_{j=1}^{n}|a_{nj}|^q)^{2/q} \le C\frac{1}{n^{2/p-1}}.$$

Hence, by Lemma 2.1 with $\alpha=\frac{2}{p}-1$ if $1<p<2$, $\alpha=1$ if p=2, we have

$$\sum_{i=1}^{n} a_{ni}X_i' \to 0 \ a.s. \qquad (4)$$

On the other hand, by the Holder inequality, the strong law of large numbers and C_p-inequality,

$$|\sum_{i=1}^{n} a_{ni}X_i''| \le (\sum_{i=1}^{n}|a_{ni}|^q)^{1/q}(\sum_{i=1}^{n}|X_i''|^p)^{1/p}$$
$$\le C\frac{1}{n^{1/p}}(\sum_{i=1}^{n}|X_i''|^p)^{1/p} = C\left(\frac{\sum_{i=1}^{n}|X_i''|^p}{n}\right)^{1/p}$$
$$\to C(E|X_1''|^p)^{1/p} \le C2\varepsilon^{1/p} \text{ a.s.}$$

This result and (4) imply $\limsup_{n\to\infty}|\sum_{i=1}^{n} a_{ni}X_i| \le C2\varepsilon^{1/p}$ a.s. The conclu- sion now follows, since $\varepsilon>0$ is arbitrary.

Remark.

Theorem 2.2 is a generalization of Theorem 1.1 for the case $1<p\le 2$, since the condition (1) of Theorem 1.1 implies the condition (3) of Theorem 2.2.

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