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ON QUASI-FUZZY H-CLOSED SPACE AND CONVERGENCE

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Abstract. In this paper, we discuss quasi-fuzzy H-closed space and introduce θ-convergence of prefilter in fuzzy topological space. And we define θ-closed fuzzy set using by θ-convergence.

1. Preliminaries

Let X be a set and I be the closed unit interval. Then a function F from X into I is called a *fuzzy set* in X. For any fuzzy set F, $\{x \in X \,|\, F(x) > 0\}$ is called the *support* of F and denoted by $suppF$. i.e. $suppF = \{x \in X \,|\, F(x) > 0\}$. And for any $\alpha \in (0,1]$, a fuzzy set x_α in X is called a *fuzzy point* if its support is a singleton $\{x\}$ and its value is α on its support. That is,

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Definition 1.1. Let X be a nonempty set and I be the closed unit interval. A family δ of functions from X into I is called a *fuzzy topology* on X if

(1) $\phi, X \in \delta$
(2) for all $U_i \in \delta$, $\cup U_i \in \delta$
(3) if $U_1, U_2 \in \delta$, then $U_1 \cap U_2 \in \delta$.

The pair (X, δ) is called a *fuzzy topological space*. A member of δ is called an *open set*. And a fuzzy set F in X is said to be *closed* if $F^c = X - F$ is open in X, i.e. $F^c \in \delta$.

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DEFINITION 1.2. Let (X, δ) be a fuzzy topological space and $A \in I^X$. Let $\overline{A} = \cap\{F|F :$ closed and $A \subset F\}$, which is called the *closure* of A. And $\overset{\circ}{A} = \cup\{U|U :$ open and $U \subset A\}$, which is called the interior of A.

2. Quasi-fuzzy H-closed space

DEFINITION 2.1. Let (X, δ) be a fuzzy topological space. Then (X, δ) is said to be *quasi-fuzzy H-closed* if any open cover $\{U_\lambda | \lambda \in \Lambda\}$ of X has a finite subfamily $\{U_{\lambda_1}, U_{\lambda_2}, \cdots, U_{\lambda_n}\}$ such that $\cup_{i=1}^{n}\overline{U_{\lambda_i}} = X$.

PROPOSITION 2.2. *A fuzzy topological space (X, δ) is quasi-fuzzy H-closed if and only if for every collection of fuzzy open sets $\{U_j\}_{j\in J}$ of X having the finite intersection property we have $\cap_{j\in J}\overline{U}_j \neq \phi$.*

A fuzzy set F in (X, δ) is called *regular closed* if $F = \overline{\overset{\circ}{F}}$ and a fuzzy set U in (X, δ) is called *regular open* if $U = \overset{\circ}{\overline{U}}$.

The following theorem shows that in the definition of quasi-fuzzy H-closedness we may work with fuzzy regular closed or fuzzy regular open sets.

THEOREM 2.3. *In a fuzzy topological space (X, δ) the following conditions are equivalent.*

(1) *(X, δ) is quasi-fuzzy H-closed space.*
(2) *For every collection $\{F_j\}_{j\in J}$ of fuzzy regular closed sets such that $\cap_{j\in J}F_j = \phi$, there is a finite subfamily $\{F_1, F_2, \cdots, F_n\}$ such that $\cap_{j=1}^{n}\overset{\circ}{F}_j = \phi$.*
(3) *$\cap_{j=1}^{n}\overline{U_j} \neq \phi$ holds for every collection of fuzzy regular open sets $\{U_j\}_{j\in J}$ with the finite intersection property.*
(4) *Every fuzzy regular open cover of X contains a finite subcollection whose closures cover X.*

3. θ-convergence in fuzzy topological space

DEFINITION 3.1. A non-empty subset $\mathcal{F} \subset I^X$ is called a *prefilter* if

(i) for all $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
(ii) if $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$,
(iii) $\phi \notin \mathcal{F}$.

In a fuzzy topological space, we have various concepts of prefilter convergence. In this paper we take the following convergence concept. A prefilter $\mathcal{F}$ in (X, δ) *converges* to $x_\alpha(\mathcal{F} \longrightarrow x_\alpha)$ if for any open set V containing x_α there exists $F \in \mathcal{F}$ with $F \subset V$ and $\mathcal{F}$ *accumulates* to $x_\alpha(\mathcal{F} \propto x_\alpha)$ if for each $F \in \mathcal{F}$ and for any open set V containing x_α, $F \cap V \neq \phi$.

Definition 3.2. A non-empty subset $\mathfrak{S} \subset I^X$ is called a *base* for a prefilter if

(i) for all $A, B \in \mathfrak{S}$ there is a $C \in \mathfrak{S}$ such that $C \subset A \cap B$,
(ii) $\phi \notin \mathfrak{S}$.

The prefilter $\mathcal{F}$ generated by $\mathfrak{S}$ is defined as

$$\mathcal{F} = \{F \in I^X \,|\, \exists B \in \mathfrak{S}\, s.t. B \subset F\}$$

and is denoted by $[\mathfrak{S}]$. A subset $\mathfrak{S}$ of $\mathcal{F}$ is a base for $\mathcal{F}$ if and only if for all $F \in \mathcal{F}$ there is a $B \in \mathfrak{S}$ such that $B \subset F$.

Definition 3.3. A prefilter $\mathcal{F}$ in a fuzzy topological space (X, δ) *θ-converges* to $x_\alpha(\mathcal{F} \underset{\theta}{\longrightarrow} x_\alpha)$ if for any open set V containing x_α there exists $F \in \mathcal{F}$ such that $F \subset \overline{V}$.

Definition 3.4. The prefilter $\mathcal{F}$ in a fuzzy topological space (X, δ) *θ-accumulates* to x_α $(\mathcal{F} \underset{\theta}{\propto} x_\alpha)$ if for each $F \in \mathcal{F}$ and any open set V containing x_α, $F \cap \overline{V} \neq \phi$.

Let $\mathcal{F}$ be a prefilter in (X, δ). If $\mathcal{F} \longrightarrow x_\alpha$ (or $\mathcal{F} \propto x_\alpha$), then $\mathcal{F} \underset{\theta}{\longrightarrow} x_\alpha$ (or $\mathcal{F} \underset{\theta}{\propto} x_\alpha$). However the converse does not hold.

Proposition 3.5. *Let $\mathcal{F}$ be a prefilter in (X, δ). If $\mathcal{F} \underset{\theta}{\longrightarrow} x_\alpha$, then $\mathcal{F} \underset{\theta}{\propto} x_\alpha$.*

A prefilter $\mathcal{M}$ on X is called a *maximal prefilter* if it is not properly contained in any other prefilter on X.

LEMMA 3.6. *If X is a set and $\mathcal{F}$ a prefilter on X, then $\mathcal{F}$ is a maximal prefilter if and only if $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ for any fuzzy set in X.*

PROPOSITION 3.7. *Let $\mathcal{M}$ be a maximal prefilter in (X, δ). Then $\mathcal{M} \underset{\theta}{\propto} x_\alpha$ if and only if $\mathcal{M} \underset{\theta}{\longrightarrow} x_\alpha$.*

Proof. ($\Leftarrow$) Obvious.
($\Rightarrow$) For any open set V containing x_α, since $\mathcal{M}$ is a maximal prefilter, $\overline{V} \in \mathcal{M}$ or $X - \overline{V} \in \mathcal{M}$. Thus there is $M \in \mathcal{M}$ such that $M \subset \overline{V}$ or $M \subset X - \overline{V}$. Since $\mathcal{M} \underset{\theta}{\propto} x_\alpha$, for any $M \in \mathcal{M}$, $M \cap \overline{V} \neq \phi$. Thus $M \subset X - \overline{V}$ is impossible. Therefore $M \subset \overline{V}$ for all open set V containing x_α. Hence $\mathcal{M} \underset{\theta}{\longrightarrow} x_\alpha$. □ □

DEFINITION 3.8. A fuzzy point x_α in a fuzzy topological space (X, δ) is in the *θ-closure* of a fuzzy set K in X (θ-$\overline{K}$) if for any open set V containing x_α, $\overline{V} \cap K \neq \phi$.

DEFINITION 3.9. A fuzzy set K in (X, δ) is *θ-closed* if it contains its θ-closure (i.e. θ-$\overline{K} \subset K$).

THEOREM 3.10. *A fuzzy point x_α in (X, δ) is in the θ-closure of a fuzzy set K if and only if there is a prefilter $\mathcal{F}$ in K which θ-converges to x_α.*

Proof. ($\Rightarrow$) Let $\mathcal{F} = \{\overline{V} \cap K |\, V$ is an open set containing $x_\alpha\}$. Since $\overline{V} \cap K \neq \phi$, $\mathcal{F}$ is a prefilter in K and obviously $\mathcal{F} \underset{\theta}{\longrightarrow} x_\alpha$.
($\Leftarrow$) Let $\mathcal{F}$ be a prefilter in K such that $\mathcal{F} \underset{\theta}{\longrightarrow} x_\alpha$. Then for any open set V containing x_α, there is an $F \in \mathcal{F}$ such that $F \subset \overline{V}$. So $F \cap \overline{V} \neq \phi$. Thus $K \cap \overline{V} \neq \phi$. Hence $x_\alpha \in \theta$-$\overline{K}$. □ □

THEOREM 3.11. *In any fuzzy topological space (X, δ),*

(1) *ϕ and X are θ-closed.*
(2) *Arbitrary intersections and finite unions of θ-closed sets are θ-closed.*

Proof. (1) Clear.
(2) Let $\{K_\lambda |\, \lambda \in \Lambda\}$ be the class of θ-closed sets. First, let $K = \underset{\lambda}{\cap} K_\lambda$. If $x_\alpha \in \theta\text{-}\overline{K}$, for any open set V containing x_α, $\overline{V} \cap K = \overline{V} \cap (\cap_\lambda K_\lambda) = \cap_\lambda(\overline{V} \cap K_\lambda) \neq \phi$. Thus $\overline{V} \cap K_\lambda \neq \phi$ for all $\lambda \in \Lambda$. Hence $x_\alpha \in \theta\text{-}\overline{K}_\lambda \subset K_\lambda$ for all $\lambda \in \Lambda$. Therefore $x_\alpha \in \underset{\lambda}{\cap} K_\lambda = K$. So $\theta\text{-}\overline{K} \subset K$. Hence K is a θ-closed set.

Secondly, let K_1 and K_2 be θ-closed, and $x_\alpha \notin K_1 \cup K_2$. then $x_\alpha \notin K_1, x_\alpha \notin K_2$. Since $x_\alpha \notin K_1$ and K_1 is θ-closed, there is an open set U containing x_α such that $\overline{U} \cap K_1 = \phi$. Similarly there is an open set V containing x_α such that $\overline{V} \cap K_2 = \phi$. Let $W = U \cap V$ then $x_\alpha \in W$ and W is open. And $\overline{W} = \overline{U \cap V} \subset \overline{U} \cap \overline{V}$. Hence $\overline{W} \cap (K_1 \cup K_2) = \phi$. Thus x_α is not a θ-closure point of $K_1 \cup K_2$. Therefore $\theta\text{-}\overline{K_1 \cup K_2} \subset K_1 \cup K_2$. Hence $K_1 \cup K_2$ is θ-closed. By induction, the claim holds. □ □

THEOREM 3.12. *Let (X, δ) be a fuzzy topological space. If (X, δ) is quasi-fuzzy H-closed, then every prefilter $\mathcal{F}$ is θ-accumulate to some fuzzy point x_α.*

Proof. Suppose that there is a prefilter $\mathcal{F}$ on X that does not θ-accumulate to every fuzzy point x_α. Then there are an open set V containing x_α and $F \in \mathcal{F}$ such that $F \cap \overline{V} = \phi$. Consider a fuzzy point x_1 for each $x \in X$. Then $\underset{x \in X}{\cup} \{V |\, x_1 \in V \in \delta\} = X$. Since X is quasi-fuzzy H-closed, there is a finite subfamily $\{V_i |\, i = 1, 2 \cdots n\}$ such that $\cup_{i=1}^n \overline{V_i} = X$. Let F_i be a member in $\mathcal{F}$ corresponding $V_i (1 \leqslant i \leqslant n)$. Then $F_i \cap \overline{V_i} = \phi$. Since $\mathcal{F}$ is a prefilter, there exists $F \in \mathcal{F}$ such that $F \subset \cup_{i=1}^n F_i$. Since $F \neq \phi$, there is $x \in X$ such that $F(x) + (\cup_{i=1}^n \overline{V_i})(x) > 1$. Hence $F \cap (\cup_{i=1}^n \overline{V_i}) \neq \phi$. Hence $F \cap \overline{V_i} \neq \phi$ for some i, which is contradiction. □ □

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