

CONFORMAL CHANGE OF THE TENSOR $S_{\lambda\mu}{}^{\nu}$ FOR THE SECOND CATEGORY IN 6-DIMENSIONAL g -UFT

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ABSTRACT. We investigate change of the torsion tensor induced by the conformal change in 6-dimensional g -unified field theory. These topics will be studied for the second class with the second category in 6-dimensional case.

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([8],1957). CHUNG([6],1968) also investigated the same topic in 4-dimensional $*g$ -unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional case, and for the second class with the first category in 6-dimensional case were investigated by CHO([1],1992, [2],1994, [3],1995).

In the present paper, we investigate change of the torsion tensor $S_{\omega\mu}{}^{\nu}$ induced by the conformal change in 6-dimensional g -unified field theory. These topics will be studied for the second class with the second category in 6-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be

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referred to CHUNG([4],1982;[3],1988), CHO([1],1992;[2],1994;[3],1995).

2.1. n -dimensional g -unified field theory

The n -dimensional g -unified field theory (n - g -UFT hereafter) was originally suggested by HLAVATÝ([8],1957) and systematically introduced by CHUNG([7],1963).

Let X_n ¹ be an n -dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \rightarrow x^{\nu'}$, for which

$$(2.1) \quad \text{Det} \left(\left(\frac{\partial x}{\partial x'} \right) \right) \neq 0.$$

In the usual Einstein's n -dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ ² :

$$(2.2) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

where

$$(2.3) \quad \text{Det}((g_{\lambda\mu})) \neq 0 \quad \text{Det}((h_{\lambda\mu})) \neq 0.$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$(2.4) \quad h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu.$$

In our n - g -UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma_{\omega\mu}^\nu$ with the following transformation rule :

$$(2.5) \quad \Gamma_{\omega'\mu'}^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma_{\beta\gamma}^\alpha + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}} \right)$$

¹Throughout the present paper, we assumed that $n \geq 2$.

²Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \dots, n$ and follow the summation convention.

and satisfies the system of Einstein's equations

$$(2.6) \quad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}$$

where D_ω denotes the covariant derivative with respect to $\Gamma_{\lambda\mu}^\nu$ and

$$(2.7) \quad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda\mu]}^\nu$$

is the *torsion tensor* of $\Gamma_{\lambda\mu}^\nu$. The connection $\Gamma_{\lambda\mu}^\nu$ satisfying (2.6) is called the *Einstein's connection*.

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \dots$ are frequently used :

$$(2.8)a \quad \mathfrak{g} = \text{Det}((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = \text{Det}((h_{\lambda\mu})) \neq 0, \\ \mathfrak{t} = \text{Det}((k_{\lambda\mu})),$$

$$(2.8)b \quad g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{t}}{\mathfrak{h}},$$

$$(2.8)c \quad K_p = k_{[\alpha_1}{}^{\alpha_1} \dots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \dots)$$

$$(2.8)d \quad {}^{(0)}k_{\lambda}{}^\nu = \delta_{\lambda}^\nu, \quad {}^{(1)}k_{\lambda}{}^\nu = k_{\lambda}{}^\nu, \quad {}^{(p)}k_{\lambda}{}^\nu = {}^{(p-1)}k_{\lambda}{}^\alpha k_{\alpha}{}^\nu,$$

$$(2.8)e \quad K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu},$$

$$(2.8)f \quad \sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\{\Gamma_{\lambda\mu}^\nu\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

$$(2.9)a \quad K_0 = 1; K_n = k \text{ if } n \text{ is even; } \quad K_p = 0 \text{ if } p \text{ is odd,}$$

$$(2.9)b \quad g = 1 + K_2 + \cdots + K_{n-\sigma},$$

$$(2.9)c \quad {}^{(p)}k_{\lambda\mu} = (-1)^{p(p)} k_{\mu\lambda}, \quad {}^{(p)}k^{\lambda\nu} = (-1)^{p(p)} k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T :

$$(2.10)a \quad {}^{pqr}T = {}^{pqr}T_{\omega\mu\nu} = T_{\alpha\beta\gamma} {}^{(p)}k_{\omega}{}^{\alpha(q)} k_{\mu}{}^{\beta(r)} k_{\nu}{}^{\gamma},$$

$$(2.10)b \quad T = T_{\omega\mu\nu} = {}^{000}T,$$

$$(2.10)c \quad 2 {}^{pqr}T_{\omega[\lambda\mu]} = {}^{pqr}T_{\omega\lambda\mu} - {}^{pqr}T_{\omega\mu\lambda},$$

$$(2.10)d \quad 2 {}^{(pq)r}T_{\omega\lambda\mu} = {}^{pqr}T_{\omega\lambda\mu} + {}^{qpr}T_{\omega\lambda\mu}.$$

We then have

$$(2.11) \quad {}^{pqr}T_{\omega\lambda\mu} = - {}^{qpr}T_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma_{\lambda\mu}^{\nu}$, using the above abbreviations it was shown that the connection is of the form

$$(2.12) \quad \Gamma_{\omega\mu}^{\nu} = \{ \omega_{\mu}^{\nu} \} + S_{\omega\mu}{}^{\nu} + U^{\nu}{}_{\omega\mu}$$

where

$$(2.13) \quad U_{\nu\omega\mu} = S_{(\omega\mu)\nu}{}^{100} + S_{\nu(\omega\mu)}{}^{(10)0}.$$

The above two relations show that *our problem of determining $\Gamma_{\omega\mu}^{\nu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}{}^{\nu}$* . On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

$$(2.14) \quad S = B - 3 {}^{(110)}S$$

where

$$(2.15) \quad 2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta} k_{\omega]}{}^{\alpha} k_{\nu}{}^{\beta}.$$

2.2. Some results for the second class with the second category in 6- g -UFT

In this section, we introduce some results of 6- g -UFT without proof, which are needed in our subsequent considerations.

They may be referred to CHO([4],1993).

DEFINITION 2.1. In 6- g -UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class with the second category, if $K_4 \neq 0, K_6 = 0$.

THEOREM 2.2. (Main recurrence relations) For the second class with the second category in 6-UFT, the following recurrence relation hold

$$(2.16) \quad {}^{(p+4)}k_\lambda{}^\nu = -K_2{}^{(p+2)}k_\lambda{}^\nu - K_4{}^{(p)}k_\lambda{}^\nu, \quad (p = 0, 1, 2, \dots).$$

THEOREM 2.3. (For the second class with the second category in 6- g -UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is

$$(2.17) \quad (1 + K_2 + K_4)(1 - K_2 + K_4)(1 - K_4)(1 - 3K_2 + 9K_4) \times \\ \times [(1 - K_2 - 3K_4)^2 - 4K_4((K_2)^2 - 4K_4)] \neq 0.$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$(2.18) \quad (S - B)(1 + K_2 + K_4)[(1 - K_2 + 5K_4)^2 - 4K_4(2 - K_2)^2] \\ = 4B_1(K_4 - 1) + B_2(1 - K_2 + 5K_4) + 2B_3(1 - 2K_2 + (K_2)^2 - 5K_4)$$

where

$$B_1 = (K_4)^2 B + 2 \binom{(12)3}{1} B + K_2 K_4 \binom{002}{1} B + (2K_4 - (K_2)^2) \binom{112}{1} B - \\ - 2K_4 \binom{(12)1}{1} B + K_4(2 + 2K_2 + (K_2)^2) \binom{110}{1} B + K_2 \binom{222}{1} B + \\ + 2K_4 \binom{(20)2}{1} B - 2K_4(1 + K_2) \binom{(10)3}{1} B - K_4(1 + K_2) \binom{220}{1} B - \\ - 2K_4(1 + K_2)^2 \binom{(10)1}{1} B$$

$$\begin{aligned}
B_2 &= -(K_4)^2 B + 2((K_2)^2 - 1 + K_4 + 2K_2K_4) \overset{(10)1}{B} + (2 + K_2) \overset{112}{B} - \\
&\quad - \overset{222}{B} - K_4 \overset{002}{B} + 2 \overset{(20)2}{B} + 2(K_2 + 2K_4) \overset{(10)3}{B} + 2K_4 \overset{(20)0}{B} - \\
&\quad - ((K_2)^2 - 1 + K_4 + 2K_2K_4) \overset{110}{B} + (K_2 - 1 + 2K_4) \overset{220}{B} \\
B_3 &= 2(K_4)^2 B + 2 \overset{(12)3}{B} - K_4 \overset{002}{B} + K_2 \overset{112}{B} + 2(1 + K_2) \overset{(21)1}{B} - \\
&\quad - \overset{222}{B} + 2K_4 \overset{(10)3}{B} - (1 + K_4)(1 + K_2) \overset{110}{B} + (1 + K_4) \overset{220}{B} + \\
&\quad + 2K_4(1 + K_2) \overset{(10)1}{B} - 2K_4 \overset{(20)0}{B} .
\end{aligned}$$

3. Conformal change of the 6-dimensional torsion tensor for the second class with the second category

In this final chapter we investigate the change $S_{\lambda\mu}{}^\nu \rightarrow \bar{S}_{\lambda\mu}{}^\nu$ of the torsion tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and \bar{X}_n are conformal if and only if

$$(3.1) \quad \bar{g}_{\lambda\mu}(x) = e^\Omega g_{\lambda\mu}(x)$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the torsion tensor $S_{\lambda\mu}{}^\nu$. An explicit representation of the change of 6-dimensional torsion tensor $S_{\lambda\mu}{}^\nu$ for the second class with the second category will be exhibited in this chapter.

AGREEMENT 3.1. Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote \bar{T} the same function of $\bar{g}_{\lambda\mu}$. In particular, if T is a tensor, so is \bar{T} . Furthermore, the indices of T (\bar{T}) will be raised and/or lowered by means of $h^{\lambda\nu}(\bar{h}^{\lambda\nu})$ and/or $h_{\lambda\mu}(\bar{h}_{\lambda\mu})$.

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1],1992, [2],1994, [3],1995).

THEOREM 3.2. *In n - g -UFT, the conformal change (3.1) induces the following changes :*

$$(3.2)a \quad \begin{aligned} {}^{(p)}\bar{k}_{\lambda\mu} &= e^{\Omega^{(p)}} k_{\lambda\mu}, & {}^{(p)}\bar{k}_\lambda{}^\nu &= {}^{(p)}k_\lambda{}^\nu, \\ {}^{(p)}\bar{k}^{\lambda\nu} &= e^{-\Omega^{(p)}} k^{\lambda\nu} \end{aligned}$$

$$(3.2)b \quad \bar{g} = g, \quad \bar{K}_p = K_p, \quad (p = 1, 2, \dots).$$

THEOREM 3.3. *(For all classes in 6- g -UFT). The change of the tensor $B_{\omega\mu\nu}$ induced by the conformal change (3.1) may be given by*

$$(3.3) \quad \begin{aligned} \bar{B}_{\omega\mu\nu} &= e^\Omega (B_{\omega\mu\nu} + k_{\nu[\omega} \Omega_{\mu]} - k_{\omega\mu} \Omega_\nu \\ &\quad - h_{\nu[\omega} k_{\mu]}{}^\delta \Omega_\delta + 2{}^{(2)}k_{\nu[\omega} k_{\mu]}{}^\delta \Omega_\delta + k_{\omega\mu} {}^{(2)}k_\nu{}^\delta \Omega_\delta). \end{aligned}$$

Now, we are ready to derive representations of the changes $S_{\omega\mu}{}^\nu \rightarrow \bar{S}_{\omega\mu}{}^\nu$ in 6- g -UFT for the second class with the second category induced by the conformal change (3.1).

THEOREM 3.4. *The conformal change (3.1) induces the following change :*

$$(3.4) \quad \begin{aligned} \overline{{}^{(10)1}B}_{\omega\mu\nu} &= e^\Omega [2 {}^{(10)1}B_{\omega\mu\nu} + (-2{}^{(4)}k_{\nu[\omega} k_{\mu]}{}^\delta \\ &\quad + 2{}^{(2)}k_{\nu[\omega} k_{\mu]}{}^\delta - k_{\nu[\omega} {}^{(2)}k_{\mu]}{}^\delta) \Omega_\delta - {}^{(3)}k_{\nu[\omega} \Omega_{\mu]}]. \end{aligned}$$

THEOREM 3.5. *The conformal change (3.1) induces the following change :*

$$(3.5) \quad \begin{aligned} \overline{{}^{ppq}B}_{\omega\mu\nu} &= e^\Omega [{}^{ppq}B_{\omega\mu\nu} + (-1)^p \{ 2^{(p+q+2)} k_{\nu[\omega} {}^{(p+1)}k_{\mu]}{}^\delta \\ &\quad + {}^{(2p+1)}k_{\omega\mu} {}^{(2+q)}k_\nu{}^\delta - {}^{(2p+1)}k_{\omega\mu} {}^{(q)}k_\nu{}^\delta \\ &\quad + {}^{(p+q+1)}k_{\nu[\omega} {}^{(p)}k_{\mu]}{}^\delta - {}^{(p+q)}k_{\nu[\omega} {}^{(p+1)}k_{\mu]}{}^\delta \} \Omega_\delta]. \end{aligned}$$

$$\left(\begin{array}{l} p = 0, 1, 2, 3, 4, \dots \\ q = 0, 1, 2, 3, 4, \dots \end{array} \right)$$

THEOREM 3.6. *The change $S_{\omega\mu}{}^\nu \rightarrow \bar{S}_{\omega\mu}{}^\nu$ induced by conformal change (3.1) may be represented by*

$$\begin{aligned}
(3.6) \quad \bar{S}_{\omega\mu}{}^\nu = & S_{\omega\mu}{}^\nu + \frac{1}{C} [a_1 k_{\omega\mu} \Omega^\nu + a_2 k^\nu{}_{[\omega} \Omega_{\mu]} \\
& + a_3 h^\nu{}_{[\omega} k_{\mu]}{}^\delta \Omega_\delta + a_4 \delta^\nu{}_{[\omega} k_{\mu]} \\
& + a_5 k^\nu{}_{[\omega} {}^{(2)} k_{\mu]}{}^\delta \Omega_\delta + a_6 {}^{(2)} k^\nu{}_{[\omega} k_{\mu]}{}^\delta \Omega_\delta \\
& + a_7 k_{\omega\mu} {}^{(2)} k^{\nu\delta} \Omega_\delta + a_8 {}^{(3)} k_{\omega\mu} \Omega^\nu \\
& + a_9 {}^{(3)} k^\nu{}_{[\omega} \Omega_{\mu]} + a_{10} \delta^\nu{}_{[\omega} {}^{(3)} k_{\mu]}{}^\delta \Omega_\delta \\
& + 2a_{11} {}^{(3)} k^\nu{}_{[\omega} {}^{(2)} k_{\mu]}{}^\delta \Omega_\delta + 2a_{12} {}^{(2)} k^\nu{}_{[\omega} {}^{(3)} k_{\mu]}{}^\delta \Omega_\delta \\
& + a_{13} {}^{(3)} k_{\omega\mu} {}^{(2)} k^{\nu\delta} \Omega_\delta],
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \alpha^2 \beta (1 + 4\beta) - 2\alpha \beta (1 + \beta + 2\beta^2) + \beta (1 - 13\beta^2) - C, \\
a_2 &= 2\alpha^3 \beta - \alpha^2 \beta (\alpha - 2\beta) + 2\alpha \beta^2 (1 - 2\beta) + \beta^2 (3\beta - 4) + C, \\
a_3 &= \beta^2 (2\alpha^2 - 5\alpha - 9\beta + 7) - C, \\
a_4 &= -2\alpha^3 \beta + \alpha^2 \beta (1 + 12\beta) - 9\alpha \beta^2 - \beta (3 + 5\beta + 18\beta^2), \\
a_5 &= 2\alpha^4 - \alpha^3 (2\beta + 3) - \alpha^2 (1 + 9\beta + 4\beta^2) \\
& \quad + \alpha (2 - 10\beta - \beta^2 + 8\beta^3) + \beta (6 + 13\beta + 19\beta^2), \\
a_6 &= -2\alpha^4 + \alpha^3 (1 + 18\beta) + 2\alpha^2 \beta (1 - 8\beta) - \alpha (2 + 16\beta \\
& \quad + 59\beta^2 + 8\beta^3) + \beta (27\beta^2 - 58\beta - 10) - 1 + 2C, \\
a_7 &= -\alpha^2 \beta (1 + 4\beta) + 2\alpha \beta (1 + \beta) + \beta (13\beta^2 + 4\alpha \beta^2 - 1) + C, \\
a_8 &= 3\alpha^3 + \alpha^2 (5\beta + 8\beta^2 - 4) - \alpha (2 + 36\beta + 5\beta^2) \\
& \quad + 7\beta (2 - 6\beta - 3\beta^2) + 3, \\
a_9 &= \alpha^2 (1 - 8\beta) - 2\alpha (1 - 6\beta^2) + \beta (8\beta^2 + 35\beta - 12) + 1, \\
a_{10} &= 2\alpha^2 \beta (-5 + 2\beta) + 2\alpha \beta (3 - 6\beta + 4\beta^2) + 4\beta (1 + 2\beta - 2\beta^2), \\
a_{11} &= 2\alpha^4 - \alpha^3 (1 + 3\beta) - 4\alpha^2 \beta^2 + \alpha (1 + 7\beta + 4\beta^2) \\
& \quad - \beta (3 - 7\alpha - 4\alpha\beta) - 2, \\
a_{12} &= 2\alpha^4 + \alpha^3 (2\beta - 15) + \alpha^2 (22 - 19\beta + 4\beta^2) \\
& \quad + \alpha (-8 + 35\beta - 6\beta^2) - 3\beta + 1,
\end{aligned}$$

$$a_{13} = -4\alpha^4 - \alpha^3(1 - 8\beta) + 11\alpha^2\beta - \alpha(8 - 16\beta + 21\beta^2) + \beta(5\beta^2 + 2\beta - 10) - 3,$$

where $\alpha = K_2$, $\beta = K_4$,

$$(3.7) \quad C = (1 + \alpha + \beta)[(1 - \alpha + 5\beta)^2 - 4\beta(2 - \alpha)^2].$$

Proof. In virtue of (2.18) and Agreement (3.1), we have
(3.8)

$$\begin{aligned} & (\bar{S} - \bar{B})(1 + \bar{K}_2 + \bar{K}_4) \times [(1 - \bar{K}_2 + 5\bar{K}_4)^2 - 4\bar{K}_4(2 - \bar{K}_2)^2] \\ &= 4\bar{B}_1(\bar{K}_4 - 1) + \bar{B}_2(1 - \bar{K}_2 + 5\bar{K}_4) + 2\bar{B}_3(1 - 2\bar{K}_2 + (\bar{K}_2)^2 - 5\bar{K}_4). \end{aligned}$$

The relation (3.6) follows by substituting (3.2), (3.3), (3.4), (3.5), (2.16), (3.7) into (3.8). \square \square

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