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ON FUZZY S-OPEN MAPS

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Abstract. We introduce the concepts of fuzzy s-open functions, and s-closed functions. And we investigate several properties of such functions. In particular, we study the relation between fuzzy s-continuous maps and fuzzy s-open maps(s-closed maps).

1. Introduction

Fuzzy topological spaces were first introduced in the literature by Chang [1] who studied a number of the basic concepts including fuzzy continuous maps and compactness. And fuzzy topological spaces are a very natural generalization of topological spaces. In 1983, A.S. Mashhour. et al.[3] introduced supra topological spaces and studied s-continuous functions and s^*-continuous functions. In 1987, M.E. Abd El-Monsef .et al.[2] introduced the fuzzy-supra topological spaces and studied fuzzy supra-continuous functions and characterized a number of basic concepts. Also fuzzy-supra topological spaces are a generalization of supra topological spaces. In [4], the author introduced the fuzzy s-continuous function and established a number of properties. In this paper, we introduce the fuzzy s-open map and the fuzzy s-closed map, and we establish a number of characterizations. Let X be a set and let $I = [0, 1]$. Let I^X denote the set of all mapping $a\colon X \to I$.

A member of I^X is called a fuzzy subset of X. And unions and intersections of fuzzy sets are denoted by $\vee$ and $\wedge$ respectively and defined by

$$\vee\, a_i = \sup\{a_i(x) \mid i \in J \textit{ and } x \in X\},$$
$$\wedge\, a_i = \inf\{a_i(x) \mid i \in J \textit{ and } x \in X\}.$$

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DEFINITION 1.1[1]. A fuzzy topology T on X is a collection of subsets of I^X such that

(1) $0, 1 \in T$,
(2) if $a, b \in T$, then $a \wedge b \in T$,
(3) if $a_i \in T$ for all $i \in J$, then $\vee a_i \in T$.

(X, T) is called a fuzzy topological space. Members of T are called fuzzy open sets in (X, T) and complement of a fuzzy open set is called a fuzzy closed set. And $cl(a)$ and $int(a)$ denote the closure, interior of fuzzy set a respectively.

DEFINITION 1.2[5]. Let f be a mapping from a set X into a set Y. Let a and b be the fuzzy sets of X and Y, respectively. Then $f(a)$ is a fuzzy set in Y, defined by

$$f(a)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} a(z), & \text{if } f^{-1}(y) \neq \emptyset,\ y \in Y \\ 0, & \text{otherwise,} \end{cases}$$

and $f^{-1}(b)$ is a fuzzy set in X, defined by $f^{-1}(b)(x) = b(f(x))$, $x \in X$.

DEFINITION 1.3[2]. A subfamily T^* of I^X is said to be a fuzzy supra-topology on X if

(1) $1 \in T^*$,
(2) if $a_i \in T^*$ for all $i \in J$, then $\vee a_i \in T^*$.

(X, T^*) is called a fuzzy supra-topological space. The elements of T^* are called fuzzy supra-open sets in (X, T^*). And a fuzzy set a is supra-closed iff $co(a) = 1 - a$ is a fuzzy supra-open set. And the fuzzy supra-topological spaces T^* is denoted by fsts.

DEFINITION 1.4[2]. The supra closure of a fuzzy set a is denoted by $scl(a)$, and given by

$$scl(a) = \wedge\{s \mid s \text{ is a fuzzy supra-closed set and } a \leq s\}.$$

The supra interior of a fuzzy set a is denoted by $si(a)$ and given by

$$si(a) = \vee\{t \mid t \text{ is a fuzzy supra-open set and } t \leq a\}.$$

Definition 1.5[2]. Let (X, T) be a fuzzy topological space and T^* be a fuzzy supra-topology on X. We call T^* a fuzzy supra-topology associated with T if $T \subset T^*$.

Definition 1.6. Let $f\colon (X, T^*) \to (Y, S^*)$ be a mapping between two fuzzy supra-topological spaces.

(1) f is a fuzzy supra-continuous function if $f^{-1}(S^*) \subseteq T^*$ [2],
(2) f is a fuzzy s-continuous function if the inverse image of each fuzzy open set in (Y, S) is T^*-fuzzy supra-open in X [4],
(3) f is a fuzzy supra open map if the image of each fuzzy supra-open in T^* is S^*-fuzzy supra-open in X [2].

2. Fuzzy s-open maps and fuzzy s-closed maps

Definition 2.1. A fuzzy mapping $f\colon (X, T) \to (Y, S)$ is called fuzzy s-open (respectively, fuzzy s-closed) if the image of each fuzzy open (respectively, fuzzy closed) set in (X, T), is S^*-fuzzy supra-open (respectively, fuzzy supra-closed) in (Y, S^*).

Clearly, every fuzzy open (fuzzy closed) map is a fuzzy s-open map (fuzzy s-closed map). And every fuzzy supraopen map is a fuzzy s-open map. But the converses of these implications are not true, which are clear from the following examples.

Example. Let $X = I$. Consider the fuzzy sets;

$$a(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1/2 \\ 1/2, & \text{if } 1/2 < x \le 1, \end{cases}$$

$$b(x) = \begin{cases} 1/2, & \text{if } 0 \le x < 1/4 \\ 2x, & \text{if } 1/4 \le x \le 1/2 \\ 0, & \text{if } 1/2 < x \le 1, \end{cases}$$

$$c(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ 0, & \text{if } 1/2 < x \le 1. \end{cases}$$

(1) Let $T_1 = \{0, a, 1\}$ be a fuzzy topology on X and let the collection $T_1^* = \{0, a, b, c, a \vee c, a \vee b, 1\}$ be an associated fuzzy supra-topology with T_1. Let $f\colon (X, T_1) \to (X, T_1)$ be a fuzzy mapping defined by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1/2 \\ 1 - x, & \text{if } 1/2 < x \le 1. \end{cases}$$

Clearly, we have $f(a) = b$ and $f(1) = c$. Since b and c are fuzzy supra-open in T_1^*, f is a fuzzy s-open mapping. But since b and c are not fuzzy open in T_1, f is not a fuzzy open mapping.

(2) Let $T = \{0, b, 1\}$ be a fuzzy topology on X. Let $T^* = \{0, a, b, a \vee b, 1\}$ and $S^* = \{0, b, c, 1\}$ are associated fuzzy supra-topologies with T. Consider a fuzzy mapping $f\colon (X, T^*) \to (X, S^*)$ defined by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1/2 \\ 1/2, & \text{if } 1/2 < x \le 1. \end{cases}$$

We obtain $f(b) = b$ and $f(1) = c$, thus f is a fuzzy s-open map. But for a fuzzy supra-open set a in T^*, $f(a)$ is not fuzzy supra-open in S^*. Consequently, f is not a fuzzy supraopen map.

THEOREM 2.2. *Let $f\colon (X, T_1) \to (Y, T_2)$ be a fuzzy function. Then the followings are equivalent :*

(1) *f is a fuzzy s-open map.*
(2) *$f(int(a)) \le si(f(a))$ for each fuzzy set a in X.*

Proof. (1) $\Rightarrow$ (2). Since $int(a) \le a$, we have $f(int(a)) \le f(a)$. By hypothesis, $f(int(a))$ is fuzzy supra-open, and because $si(f(a))$ is the largest fuzzy supra-open set in $f(a)$, thus $f(int(a)) \le si(f(a))$.

(2) $\Rightarrow$ (1). Let a be a fuzzy open in X. We have $si(f(a)) \le f(a)$. By hypothesis, $f(a) \le si(f(a))$. Thus $f(a)$ is a fuzzy supra-open in Y. □ □

THEOREM 2.3. *A fuzzy mapping $f\colon (X, T_1) \to (Y, T_2)$ is fuzzy s-closed iff $scl(f(a)) \le f(cl(a))$ for each fuzzy set a in X.*

Proof. If f is fuzzy s-closed map, then $f(cl(a))$ is a fuzzy supra-closed set in Y. And we have $f(a) \le f(cl(a))$, thus $scl(f(a)) \le f(cl(a))$.

Conversely, let a be a fuzzy closed set. Then $f(a) \le scl(f(a)) \le f(cl(a)) = f(a)$, thus $f(a)$ is a fuzzy supra-closed set in Y. □ □

THEOREM 2.4. *Let $f\colon (X, T_1) \to (Y, T_2)$ and $g\colon (Y, T_2) \to (Z, T_3)$ be fuzzy mappings.*

(1) *If $(g \circ f)$ is fuzzy s-open and f is fuzzy continuous surjective, then g is also fuzzy s-open .*

(2) *If $(g \circ f)$ is a fuzzy open map and g is fuzzy s-continuous injective, then f is fuzzy s-open.*

Proof. (1) Let a be any fuzzy open set in Y. Then $f^{-1}(a)$ is fuzzy open in X. Since $(g \circ f)$ is fuzzy s-open, $(g \circ f)(f^{-1}(a))$ is a fuzzy supra-open set in Z. And $(g \circ f)(a \circ f) = g(a)$, since f is surjective. Therefore the map g is fuzzy s-open.

(2) Let a be fuzzy open in X. Then $(g \circ f)(a) = g(f(a))$ is fuzzy open in Z. Since g is fuzzy s-continuous and injective, $g^{-1}(g(f(a))) = g(f(a)) \circ g = f(a)$ is a fuzzy supra-open set. Hence, f is fuzzy s-open. □ □

THEOREM 2.5. *Let (X, T_1) and (Y, T_2) be fts. If $f\colon (X, T_1) \to (Y, T_2)$ is a fuzzy bijective mapping, then following statements are equivalent:*

(1) *f is a fuzzy s-open map.*
(2) *f is a fuzzy s-closed map.*
(3) *f^{-1} is fuzzy s-continuous.*

Proof. (1) $\Rightarrow$ (2). Let a be a fuzzy closed set in X. Then $f(1-a) = 1-f(a)$ is fuzzy supra-open in Y, since f is a fuzzy s-open map. Hence $f(a)$ is fuzzy supra-closed in Y.

(2) $\Rightarrow$ (3). Let a be a fuzzy closed set in X. We have $(f^{-1})^{-1}(a) = f(a)$. Since f is a fuzzy s-closed map, $f(a)$ is fuzzy supra-closed in Y. Therefore, f is fuzzy s-continuous.

(3) $\Rightarrow$ (1). Let a be a fuzzy open set in X. Since f^{-1} is fuzzy s-continuous, $(f^{-1})^{-1}(a) = f(a)$ is fuzzy supra-open in Y. Hence f is a fuzzy s-open map. □ □

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