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APPROXIMATION AND CONVERGENCE OF ACCRETIVE OPERATORS

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Abstract. We show that if X is a reflexive Banach space with a uniformly Gâteaux differentiable norm, then the convergence of semigroups acting on Banach spaces X_n implies the convergence of resolvents of generators of semigroups.

In this paper we show that if X is a reflexive Banach space with a uniformly Gâteaux differentiable norm, then the convergence of a sequence of semigroups acting on different Banach spaces X_n implies the convergence of the resolvents of the generators defined on X_n. This improves Theorem 5.3 in [5]. Combining our result with Theorem 3.1 in [5], we can derive a nonlinear version of Trotter-Kato Theorem. This version is useful for studying convergence of numerical approximations of solutions to partial differential equations (see [5]).

Let X be a Banach space. We denote the identity operator by I and the closure of a subset D of X by $cl(D)$. An operator $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ is said to be accretive if

$$|x_1 - x_2| \leq |x_1 - x_2 + r(y_1 - y_2)|$$

for $[x_i,\ y_i] \in A$, $i = 1,\ 2$, and $r > 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = X$ for all $r > 0$. Let $J_r^A = (I + rA)^{-1}$, $r > 0$, be the resolvent of A.

A semigroup on a subset C of X is a function $S : [0,\ \infty) \times C \to C$ satisfying $S(t_1 + t_2)x = S(t_1)S(t_2)x$ for $t_1,\ t_2 \geq 0$ and $x \in C$, $|S(t)x - S(t)y| \leq |x - y|$ for $x,\ y \in C$, $S(0)x = x$ for $x \in C$, and $S(t)x$ is continuous in $t \geq 0$ for each $x \in C$.

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If A is accretive and $R(I + rA) \supset cl(D(A))$ for $r > 0$, then there exists a semigroup S on $cl(D(A))$ such that for each $x \in cl(D(A))$ and $t \geq 0$,

$$S(t)x = \lim_{r \to 0} (I + rA)^{-[t/r]}x,$$

uniformly on bounded $t-$intervals (see [2, 6]), where $[\cdot]$ denotes the Gaussian bracket. We shall say that the semigroup $S(t)$ is generated by $-A$.

We first introduce an approximating sequence of Banach spaces. Let X and X_n be Banach spaces with norms $|\cdot|$ and $|\cdot|_n$, respectively. For every $n \geq 1$ there exist bounded linear operators $P_n : X \to X_n$ and $E_n : X_n \to X$ satisfying

(1) $||P_n|| \leq 1$ and $||E_n|| \leq 1$ for all n,
(2) $|E_n P_n x - x| \to 0$ as $n \to \infty$ for all $x \in X$,
(3) $P_n E_n = I_n$, where I_n is the identity on X_n.

The introduction of X_n, P_n and E_n is motivated by the approximation of differential equations via difference equations, since the difference operators act on spaces different from the one on which the differential operator acts. For examples of $\{P_n\}$ and $\{E_n\}$, see [5].

Recall that the norm of a Banach space X is said to be uniformly Gâteaux differentiable if for each $y \in U = \{x \in X : |x| = 1\}$, $\lim_{t \to 0}(|x + ty| - |x|)/t$ exists uniformly for $x \in U$. Every Banach space with a uniformly convex dual is a reflexive Banach space with a uniformly Gâteaux differentiable norm. If the norm of X is uniformly Gâteaux differentiable, the duality mapping $J : X \to X^*$ defined by $Jx = \{x^* \in X^* : (x,\ x^*) = |x|^2 = |x^*|^2\}$ is single-valued and uniformly continuous on bounded subsets of X from the strong topology of X to the weak star topology of X^*.

THEOREM 1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let A be an accretive operator in X such that $R(I + rA) \supset cl(D(A))$ for $r > 0$, and let S be the semigroup generated by $-A$. For each n, let A_n be an accretive operator in X_n such that $R(I + rA_n) \supset cl(D(A_n))$ for $r > 0$, and let S_n be the semigroup generated by $-A_n$. Suppose that $P_n(cl(D(A))) \subset cl(D(A_n))$ for each n and $cl(D(A))$ is convex. If*

$$\lim_{n \to \infty} E_n S_n(t) P_n x = S(t)x \tag{I}$$

for $x \in cl(D(A))$ and the convergence is uniform on bounded $t-$ intervals, then

$$\lim_{n\to\infty} E_n J_r^{A_n} P_n x = J_r^A x \tag{II}$$

for $r > 0$ and $x \in cl(D(A))$.

It is known [5] that (II) implies (I) in any Banach space. That is, (I) and (II) are equivalent. Thus we have a complete nonlinear analog of Trotter-Kato Theorem [3] and our result is a generalization of Theorem 5.3 in [5]. In contrast with the linear case, (I) does not imply (II) in all Banach spaces, even in the one space case, that is, $X = X_n$ and $E_n = P_n = I$ for all n (see [2]). In the one space case, our result includes Theorem 1 in [7]. To prove Theorem 1, we start with the following lemma.

LEMMA 2. *For each fixed n, let A_n be an accretive operator in X_n satisfying $R(I + rA_n) \supset cl(D(A_n))$ for $r > 0$ and let S_n be the semigroup generated by $-A_n$. Suppose that $P_n x \in cl(D(A_n))$ for $x \in X$ and $[x_n,\ z_n] \in A_n$. Then*

$$\begin{aligned}|E_n x_n - E_n S_n(T) P_n x|^2 &- |E_n x_n - E_n P_n x|^2 \\ &\le 2 \int_0^T < E_n z_n,\ E_n x_n - E_n S_n(t) P_n x >_s dt,\end{aligned}$$

where for $x,\ y \in X$, $< x,\ y >_s = \sup\{(x,\ y^*) : y^* \in J(y)\}$.

Proof. Since $[x_n,\ z_n] \in A_n$, $\frac{1}{r}((J_r^{A_n})^{k-1} P_n x - (J_r^{A_n})^k P_n x) \in A_n (J_r^{A_n})^k P_n x$ and A_n is accretive,

$$\begin{aligned}&|x_n - (J_r^{A_n})^k P_n x|_n \\ &\quad \le |x_n - (J_r^{A_n})^k P_n x + \alpha(z_n - \frac{1}{r}((J_r^{A_n})^{k-1} P_n x - (J_r^{A_n})^k P_n x))|_n\end{aligned}$$

for $\alpha > 0$. So we have

$$\begin{aligned}|E_n x_n - &E_n (J_r^{A_n})^k P_n x| \\ &\le |E_n x_n - E_n (J_r^{A_n})^k P_n x \\ &\quad + \alpha(E_n z_n - \frac{1}{r}(E_n (J_r^{A_n})^{k-1} P_n x - E_n (J_r^{A_n})^k P_n x))|.\end{aligned}$$

By Lemma 1.1 in [4], there exists $\eta^* \in J(E_n x_n - E_n(J_r^{A_n})^k P_n x)$ such that

$$(E_n z_n - \frac{1}{r}(E_n(J_r^{A_n})^{k-1}P_n x - E_n(J_r^{A_n})^k P_n x),\ \eta^*) \geq 0.$$

Hence we have

$$\begin{aligned}
&(E_n z_n,\ \eta^*) \\
&\quad \geq \frac{1}{r}(E_n(J_r^{A_n})^{k-1}P_n x - E_n(J_r^{A_n})^k P_n x,\ \eta^*) \\
&\quad = \frac{1}{r}|E_n x_n - E_n(J_r^{A_n})^k P_n x|^2 - \frac{1}{r}(E_n x_n - E_n(J_r^{A_n})^{k-1}P_n x,\ \eta^*) \\
&\quad \geq \frac{1}{2r}(|E_n x_n - E_n(J_r^{A_n})^k P_n x|^2 - |E_n x_n - E_n(J_r^{A_n})^{k-1}P_n x|^2).
\end{aligned}$$

For $kr \leq t < (k+1)r$,

$$\begin{aligned}
&|E_n x_n - E_n(J_r^{A_n})^k P_n x|^2 - |E_n x_n - E_n(J_r^{A_n})^{k-1}P_n x|^2 \\
&\qquad \leq 2r(E_n z_n,\ \eta^*)\ \leq\ 2r < E_n z_n,\ E_n x_n - E_n(J_r^{A_n})^{[t/r]}P_n x >_s \\
&\qquad \leq 2\int_{kr}^{(k+1)r} < E_n z_n,\ E_n x_n - E_n(J_r^{A_n})^{[t/r]}P_n x >_s dt.
\end{aligned}$$

Add these inequalities for $k = 1,\ 2,\ \cdots,\ [T/r]$. Then

$$\begin{aligned}
&|E_n x_n - E_n(J_r^{A_n})^{[T/r]}P_n x|^2 - |E_n x_n - E_n P_n x|^2 \\
&\qquad \leq 2\int_r^{([T/r]+1)r} < E_n z_n,\ E_n x_n - E_n(J_r^{A_n})^{[t/r]}P_n x >_s dt.
\end{aligned}$$

Letting $r \to 0$, we obtain

$$\begin{aligned}
&|E_n x_n - E_n S_n(T)P_n x|^2 - |E_n x_n - E_n P_n x|^2 \\
&\qquad \leq 2\int_0^T < E_n z_n,\ E_n x_n - E_n S_n(t)P_n x >_s dt,
\end{aligned}$$

by the upper semicontinuity of $< \cdot, \cdot >_s$ and dominated convergence theorem (see [6]). □ □

Proof of Theorem 1. Let $x \in cl(D(A))$ and let $y_n = E_n J_r^{A_n} P_n x$ for $r > 0$. Then (see [1])

$$\begin{aligned} |y_n - E_n P_n x| &\le |J_r^{A_n} P_n x - P_n x|_n \\ &\le \frac{4}{r}\int_0^r |P_n x - S_n(t) P_n x|_n dt \\ &= \frac{4}{r}\int_0^r |E_n P_n x - E_n S_n(t) P_n x| dt. \end{aligned}$$

So $\{y_n\}$ is bounded. Define $f : cl(D(A)) \to R$ by

$$f(z) = \mathrm{LIM}\{|y_n - z|^2\},$$

where LIM is a Banach limit, $z \in cl(D(A))$ and $\{y_n\}$ is any subsequence of the original sequence which we continue to denote by $\{y_n\}$. Then f is continuous, convex and $f(z) \to \infty$ as $|z| \to \infty$. Since X is reflexive, f has its minimum $f(u)$ over $cl(D(A))$ for some $u \in cl(D(A))$.

For $0 < \eta \le 1$, we have

$$(z - u,\ J(y_n - u - \eta(z - u))) \le \frac{1}{2\eta}(|y_n - u|^2 - |y_n - u - \eta(z - u)|^2).$$

By taking LIM to both sides, we have

$$\begin{aligned} &\mathrm{LIM}\{(z - u,\ J(y_n - u - \eta(z - u)))\} \\ &\qquad \le \frac{1}{2\eta}(f(u) - f(u + \eta(z - u))) \le 0. \end{aligned}$$

By the uniform continuity of J, for each $\varepsilon > 0$ there exists η_0 such that

$$\mathrm{LIM}\{(z - u,\ J(y_n - u)\} \le \mathrm{LIM}\{(z - u,\ J(y_n - u - \eta(z - u)))\} + \varepsilon$$

for $\eta \le \eta_0$. Since ε is arbitrary,

$$\mathrm{LIM}\{(z - u,\ J(y_n - u))\} \le 0.$$

By Lemma 2, we have

$$\begin{aligned} &\frac{2}{r}\int_0^T (y_n - E_n P_n x,\ J(y_n - E_n S_n(t) P_n u)) dt \\ &\qquad \le |y_n - E_n P_n u|^2 - |y_n - E_n S_n(T) P_n u|^2, \end{aligned}$$

since $y_n = E_nP_ny_n$. Note that

$$\begin{aligned}
&(y_n - x,\ J(y_n - u)) - (y_n - E_nP_nx,\ J(y_n - E_nS_n(t)P_nu)) \\
&\ = (y_n - u + u - x,\ J(y_n - u)) \\
&\ \ - (y_n - E_nS_n(t)P_nu + E_nS_n(t)P_nu - E_nP_nx,\ J(y_n - E_nS_n(t)P_nu)) \\
&\ = |y_n - u|^2 - |y_n - E_nS_n(t)P_nu|^2 + (u - x,\ J(y_n - u)) \\
&\ \quad - (E_nS_n(t)P_nu - E_nP_nx,\ J(y_n - E_nS_n(t)P_nu)) \\
&\ = |y_n - u|^2 - |y_n - E_nS_n(t)P_nu|^2 + (u - x,\ J(y_n - u)) \\
&\ \quad - (E_nS_n(t)P_nu - u + u - x + x - E_nP_nx,\ J(y_n - E_nS_n(t)P_nu)) \\
&\ = |y_n - u|^2 - |y_n - E_nS_n(t)P_nu|^2 \\
&\ \quad + (u - x,\ J(y_n - u) - J(y_n - E_nS_n(t)P_nu)) \\
&\ \quad - (E_nS_n(t)P_nu - u,\ J(y_n - E_nS_n(t)P_nu)) \\
&\ \quad - (x - E_nP_nx,\ J(y_n - E_nS_n(t)P_nu)).
\end{aligned}$$

By the uniform continuity of J, it follows that for each $\varepsilon > 0$ there exist T and n_0 such that

$$(y_n - x,\ J(y_n - u)) \le (y_n - E_nP_nx,\ J(y_n - E_nS_n(t)P_nu)) + \varepsilon$$

for all $0 \le t \le T$ and $n \ge n_0$. Hence we have

$$\begin{aligned}
&\frac{2}{r}\int_0^T (y_n - x,\ J(y_n - u))dt \\
&\qquad \le \frac{2}{r}\int_0^T (y_n - E_nP_nx,\ J(y_n - E_nS_n(t)P_nu))dt + \frac{2T}{r}\varepsilon \\
&\qquad \le |y_n - E_nP_nu|^2 - |y_n - E_nS_n(T)P_nu|^2 + \frac{2T}{r}\varepsilon \\
&\qquad \le (|y_n - u| + |u - E_nP_nu|)^2 \\
&\qquad\quad - (|y_n - S(T)u| - |S(T)u - E_nS_n(T)P_nu|)^2 + \frac{2T}{r}\varepsilon \\
&\qquad = |y_n - u|^2 - |y_n - S(T)u|^2 + K_n + \frac{2T}{r}\varepsilon,
\end{aligned}$$

where $K_n = 2|y_n - u|\ |u - E_nP_nu| + |u - E_nP_nu|^2 + 2|y_n - S(T)u||S(T)u - E_nS_n(T)P_nu| + |S(T)u - E_nS_n(T)P_nu|^2$. Applying LIM to both sides, we obtain

$$\frac{2T}{r}\mathrm{LIM}\{(y_n - x,\ J(y_n - u))\}$$
$$\le f(u) - f(S(T)u) + \frac{2T}{r}\varepsilon \le \frac{2T}{r}\varepsilon.$$

Since ε is arbitrary, $\mathrm{LIM}\{(y_n - x,\ J(y_n - u))\} \le 0$. Therefore $\mathrm{LIM}\{|y_n - u|^2\} = \mathrm{LIM}\{(y_n - x,\ J(y_n - u))\} + \mathrm{LIM}\{(x - u,\ J(y_n - u))\} \le 0$. So there exists a subsequence $\{y_{n_k}\}$ such that

$$\lim_{k\to\infty} |y_{n_k} - u| = 0.$$

For $s > 0$, let $z_s = (I + \frac{r}{s}(I - S(s)))^{-1}x$. It is known [8] that $\lim_{s\to 0} z_s = J_r^A x = v$. Suppose that $\lim_{m\to\infty} y_m = u$ for some subsequence $\{y_m\}$ of $\{y_n\}$. We complete the proof by showing that $u = v$.

For $T > 0$ we have

$$\frac{2}{r}\int_0^T (u - x,\ J(u - S(t)z_s))dt$$
$$\le |u - z_s|^2 - |u - S(T)z_s|^2.$$

By the uniform continuity of J, for given $\varepsilon > 0$

$$\frac{2T}{r}(u - x,\ J(u - v))$$
$$\le \frac{2}{r}\int_0^T (u - x,\ J(u - S(t)z_s))dt + \frac{2T}{r}\varepsilon$$
$$\le |u - z_s|^2 - |u - S(T)z_s|^2 + \frac{2T}{r}\varepsilon.$$

for all sufficiently small T and s. Let $s = T$. Then

$$\frac{2s}{r}(u - x,\ J(u - v))$$
$$\le |u - z_s|^2 - |u - z_s - \frac{s}{r}(z_s - x)|^2 + \frac{2s}{r}\varepsilon$$
$$\le \frac{2s}{r}(z_s - x,\ J(u - z_s)) + \frac{2s}{r}\varepsilon.$$

Therefore we have $(u - x,\ J(u - v)) \leq (v - x,\ J(u - v)) + \varepsilon$, that is, $|u - v|^2 \leq \varepsilon$. This completes the proof. □ □

Recall the notion of the limit inferior of a sequence of operators $\{B_n\}$. The operator $\liminf B_n$ is defined by $[x,\ y] \in \liminf B_n$ if and only if there exists a sequence $\{[x_n,\ y_n]\}$ such that $[x_n,\ y_n] \in B_n$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Combining Theorem 1 with Lemma 3.3 in [5], we establish the equivalency between convergence of semigroups and convergence of m-accretive operators.

Corollary 3. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let A be an m-accretive operator in X and let S be the semigroup generated by $-A$. For each n let A_n be an m-accretive operator in X_n and let S_n be the semigroup generated by $-A_n$. Suppose that $P_n(cl(D(A))) \subset cl(D(A_n))$ for each n and $cl(D(A))$ is convex. Then the following are equivalent.*
(I) $\lim_{n\to\infty} E_n S_n(t) P_n x = S(t)x$ for each $x \in cl(D(A))$ and the convergence is uniform on bounded t-intervals.
(II) $\lim_{n\to\infty} E_n J_r^{A_n} P_n x = J_r^A x$ for $r > 0$ and $x \in cl(D(A))$.
(III) $A \subset \liminf E_n A_n P_n$.

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