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A CHARACTERIZATION OF MCSHANE INTEGRABILITY

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Abstract. In this paper we prove that for functions from [0,1] into a totally ordered AL-space, Mcshane integrability and absolute Mcshane integrability are equivalent.

1. Introduction

In 1990 Gordon[4] introduced the Mcshane integral of Banach-valued functions. This integral is a generalized Riemann integral of functions which have values in a Banach space. For real-valued functions the Mcshane integral and the Lebesgue integral are equivalent. Gordon[4] and Fremlin and Mendoza[2] have developed the properties of this integral. A Bochner integrable function is Mcshane integrable [4], and a Mcshane integrable function is Pettis integrable [2]. Many authors have studied the Bochner integral and the Pettis integral ([1],[3],[5],[6]).

In this paper we prove that for functions from [0,1] into a totally ordered AL-space, Mcshane integrability and absolute Mcshane integrability are equivalent.

2. Preliminaries

Unless otherwise stated, we always assume that X is a real Banach space with dual X^* and $([0,1],\Sigma,\mu)$ is the Lebesgue measure space.

Gorden [4] introduced the Mcshane integral of Banach-valued functions.

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Definition 2.1. A Mcshane partition of [0,1] is a finite collection $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ such that $\{[a_i, b_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of [0,1] covering [0,1] and $t_i \in [0,1]$ for each $i \leq n$. A gauge on [0,1] is a function $\delta : [0,1] \to (0, \infty)$. A Mcshane partition $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ is subordinate to a gauge δ if $[a_i, b_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i \leq n$. If $f : [0,1] \to X$ and if $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ is a Mcshane partition of $[0,1]$, we will denote $f(\mathcal{P})$ for $\sum_{i=1}^{n} f(t_i)(b_i - a_i)$. A function $f : [0,1] \to X$ is Mcshane integrable on $[0,1]$, with Mcshane integral ω, if for every $\varepsilon > 0$ there exists a gauge $\delta : [0,1] \to (0, \infty)$ such that $\|\omega - f(\mathcal{P})\| < \varepsilon$ for every Mcshane partition $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ of $[0,1]$ subordinate to δ.

Gorden [4] obtained the following theorem which is useful to prove our result.

Theorem 2.2 [4]. *The function $f : [0,1] \to X$ is Mcshane integrable on $[0,1]$ if and only if for each $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are Mcshane partitions of $[0,1]$ subordinate to δ.*

Definition 2.3. A function $f : [0,1] \to X$ is absolutely Mcshane integrable on $[0,1]$ if for each $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t_i') - f(t_j'')\| \mu([a_i', b_i'] \cap [a_j'', b_j'']) < \varepsilon$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \leq j \leq h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ.

3. Main Result

In this section, we give a characterization of Mcshane integrablility in terms of absolute Mcshane integrability.

Lemma 3.1. *$f : [0,1] \to X$ is Mcshane integrable if and only if for each $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that*

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} [f(t_i') - f(t_j'')] \mu([a_i', b_i'] \cap [a_j'', b_j'']) \| < \varepsilon$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ *and* $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ *are Mcshane partitions of* $[0, 1]$ *subordinate to* δ.

Proof. Let $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ be any Mcshane partitions of $[0, 1]$. Then

$$\begin{aligned} f(\mathcal{P}') &= \sum_{i=1}^{k} f(t_i')(b_i' - a_i') \\ &= \sum_{i=1}^{k} \sum_{j=1}^{h} f(t_i')\mu([a_i', b_i'] \cap [a_j'', b_j'']) \end{aligned}$$

and

$$\begin{aligned} f(\mathcal{P}'') &= \sum_{j=1}^{h} f(t_j'')(b_j'' - a_j'') \\ &= \sum_{i=1}^{k} \sum_{j=1}^{h} f(t_j'')\mu([a_i', b_i'] \cap [a_j'', b_j'']) \end{aligned}$$

Hence $\|f(\mathcal{P}') - f(\mathcal{P}'')\| = \|\sum_{i=1}^{k} \sum_{j=1}^{h} [f(t_i') - f(t_j'')]\mu([a_i', b_i'] \cap [a_j'', b_j''])\|$ Therefore by Theorem 2.2, $f : [0, 1] \to X$ is Mcshane integrable if and only if for each $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that $\|\sum_{i=1}^{k} \sum_{j=1}^{h} [f(t_i') - f(t_j'')]\mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon$ whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ are Mcshane partitions of $[0, 1]$ subordinate to δ. □

The following is a main result of this paper.

THEOREM 3.2. *Let X be a totally ordered AL-space. Then $f : [0, 1] \to X$ is Mcshane integrable if and only if f is absolutely Mcshane integrable.*

Proof. Suppose that $f : [0, 1] \to X$ is absolutely Mcshane integrable. Let $\varepsilon > 0$ be given. Then there exists a gauge δ on $[0, 1]$ such that $\sum_{i=1}^{k} \sum_{j=1}^{h} \|[f(t_i') - f(t_j'')\|\mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon$ whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ are Mcshane partitions of $[0, 1]$ subordinate to δ.
Therefore $\|\sum_{i=1}^{k} \sum_{j=1}^{h} [f(t_i') - f(t_j'')]\mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon$ whenever

$\mathcal{P}' = \{([a'_i, b'_i], t'_i); 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a''_j, b''_j], t''_j); 1 \le j \le h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ. By Lemma 3.1, f is Mcshane integrable.

For the converse, suppose that $f : [0,1] \to X$ is Mcshane integrable. Let $\varepsilon > 0$ be given. Then there exists a gauge δ on [0,1] such that $\| \sum_{i=1}^{k} f(t_i)(b_i - a_i) - \int_0^1 f d\mu \| < \frac{\varepsilon}{2}$ whenever $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \le i \le k\}$ is a Mcshane partition of $[0,1]$ subordinate to δ.
Let $\mathcal{P}' = \{([a'_i, b'_i], t'_i) : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a''_j, b''_j], t''_j) : 1 \le j \le h\}$ be any Mcshane partitions of $[0,1]$ subordinate to δ. Define $t'_{ij} = t'_i$ and $t''_{ij} = t''_j$ if $f(t'_i) \ge f(t''_j)$ and define $t'_{ij} = t''_j$ and $t''_{ij} = t'_i$ if $f(t'_i) < f(t''_j)$. Then $f(t'_{ij}) - f(t''_{ij}) \in X_+$ and $\|f(t'_{ij}) - f(t''_{ij})\| = \|f(t'_i) - f(t''_j)\|$ for $1 \le i \le k, 1 \le j \le h$. Moreover $\{([a'_i, b'_i] \cap [a''_j, b''_j], t'_{ij}) : 1 \le i \le k, 1 \le j \le h\}$ and $\{([a'_i, b'_i] \cap [a''_j, b''_j], t''_{ij}) : 1 \le i \le k, 1 \le j \le h\}$ are both Mcshane partitions of $[0,1]$ subordinate to δ. Hence

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} f(t'_{ij}) \mu([a'_i, b'_i] \cap [a''_j, b''_j]) - \int_0^1 f d\mu \| < \frac{\varepsilon}{2}$$

and

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} f(t''_{ij}) \mu([a'_i, b'_i] \cap [a''_j, b''_j]) - \int_0^1 f d\mu \| < \frac{\varepsilon}{2}.$$

Therefore

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} [f(t'_{ij}) - f(t''_{ij})] \mu([a'_i, b'_i] \cap [a''_j, b''_j]) \| < \varepsilon.$$

Since X is an AL-space and $f(t'_{ij}) - f(t''_{ij}) \in X_+$ for $1 \le i \le k, 1 \le j \le h$,

$$\begin{aligned}
&\sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t'_i) - f(t''_j)\| \mu([a'_i, b'_i] \cap [a''_j, b''_j]) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t'_{ij}) - f(t''_{ij})\| \mu([a'_i, b'_i] \cap [a''_j, b''_j]) \\
&= \| \sum_{i=1}^{k} \sum_{j=1}^{h} [f(t'_{ij}) - f(t''_{ij})] \mu([a'_i, b'_i] \cap [a''_j, b''_j]) \| < \varepsilon.
\end{aligned}$$

Therefore $f : [0,1] \to X$ is absolutely Mcshane integrable. □ □

COROLLARY 3.3. *Let X be a totally ordered AL-space and Y a Banach space. If $f : [0,1] \to X$ is Mcshane integrable and $g : X \to Y$ is Lipschitz continuous, then the composite function $g \circ f : [0,1] \to Y$ is Mcshane integrable.*

Proof. Let $\varepsilon > 0$ be given. Since g is Lipschitz continuous, there exists a $K > 0$ such that $\|g(x') - g(x)\| \leq K\|x' - x\|$ for all $x', x \in X$. Since f is Mcshane integrable, by Theorem 3.2, f is absolutely Mcshane integrable. Hence there exists a gauge δ on $[0,1]$ such that

$$\sum_{i=1}^{k}\sum_{j=1}^{h} \|f(t_i') - f(t_{ij}'')\|\mu([a_i', b_i'] \cap [a_j'', b_j'']) < \frac{\varepsilon}{K}$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \leq j \leq h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ.
Hence

$$\begin{aligned}
&\|\sum_{i=1}^{k}\sum_{j=1}^{h}[(g\circ f)(t_i') - (g\circ f)(t_j'')]\mu([a_i', b_i'] \cap [a_i'', b_j''])\| \\
&\leq \sum_{i=1}^{k}\sum_{j=1}^{h} \|(g\circ f)(t_i') - (g\circ f)(t_j'')\|\mu([a_i', b_i'] \cap [a_i'', b_j'']) \\
&\leq K\sum_{i=1}^{k}\sum_{j=1}^{h} \|f(t_i') - f(t_j'')\|\mu([a_i', b_i'] \cap [a_i'', b_j'']) < \varepsilon
\end{aligned}$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \leq j \leq h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ. Thus $g \circ f$ is Mcshane integrable by Lemma 3.1. □ □

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