Journal of the Korean Institute
of Industrial Engineers
Vol.22, No.3, September, 1996

스트레스 한계가 있는 램프시험의 최적설계: 지수수명분포의 경우

Time-Censored Ramp Tests with Stress Bound for Exponential

배도선*, 전영록**, 차명수***

Do Sun Bai*, Young Rok Chun** and Myung Su Cha***

Abstract

This paper considers ramp tests for exponential lifetime distribution when there are limitations on test stress and test time. The inverse power law and a cumulative exposure model are assumed. Maximum likelihood (ML) estimators of model parameters and their asymptotic covariance matrix are obtained. The optimum ramp test plans are also found which minimize the asymptotic variance of the ML estimator of the log mean life at design constant stress. For selected values of the design parameters, tables useful for finding optimal test plans are given. The effect of the pre-estimates of design parameters is studied.

1. INTRODUCTION

Accelerated life tests (ALTs) are used to obtain information quickly on the lifetime distribution of materials or products. The test items are run at higher-than-usual levels of stress to induce early failures. The test data obtained at the accelerated conditions are analyzed in terms of a model, and then extrapolated to the specified design stress to estimate the lifetime distribution.

One way of applying stress to the test item is a progressive stress scheme which allows the stress setting of an item to be increased

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* 한국과학기술원 산업공학과
** 경남대학교 산업공학과
*** 경성대학교 산업공학과

continuously in time. An ALT with linearly increasing stress is called a ramp test. Ramp tests are commonly used for various materials and products; for example, fatigue testing [17], capacitors [6, 19], insulations [7, 18], and integrated circuits [4] etc.

Statistical theory for progressive stress ALTs has been studied by several authors. Yurkowski et al.[21] surveyed early statistical methods. Allen [1] suggested a statistical model for units having exponential lives under progressive stress and presented methods of estimating the parameters. Yin and Sheng [20] derived a lifetime distribution of an item under progressive stress, and studied the properties of the maximum likelihood (ML) estimators of parameters under a ramp test when the test items have exponential lifetime distribution. Nilsson [15] considered the problem of estimation the parameters of Weibull distribution under a ramp test. Bai et al. [2] considered the design of ramp tests with two ramp rates for Weibull distribution under Type I censoring, and determined low ramp rate and proportion of test items allocated to low ramp rate minimizing the asymptotic variance of the ML estimator of a specified percentile at design stress.

In practice, however, there are cases where the stress can not be increased indefinitely in a ramp test because: i) too high stress may cause failure modes other than that under consideration and/or ii) the test equipment may not be able to provide such a high stress. This paper considers time-censored ramp tests for items with exponential lifetime distribution when there is a stress upper bound. ML estimators of model parameters and their asymptotic covariance matrix are obtained. The optimum ramp test plans are found which minimize the asymptotic variance of the ML estimator of the log mean life at design constant stress. For selected values of design parameters, tables useful for finding optimal test plans are given. The effects of the pre-estimates of design parameters are investigated.

Notation

$s(t)$	stress at time t
s_d	design stress
s_h	upper bound of test stress
δ	standardized design stress ($\delta = s_d/s_h$)
η	censoring time
k	ramp rate
ζ	$=s_h/k$
τ	$=\min(\zeta, \eta)$
ξ	$=\zeta/\eta$
$\theta(s)$	mean of exponential distribution at stress s
α, β	parameters of the inverse power law
$\Psi(\cdot)$	distribution function of standard smallest extreme value distribution; $\Psi(x)=1-\exp(-\exp(x))$

2. The Model

Test Procedure

In a ramp test, the stress on an item is a linear function of test time t, say $s(t)=kt$, where $k(>0)$ is the ramp rate. When there is a stress bound s_h, the test procedure is as follows:

1. All test items are placed on a ramp test with ramp rate k.
2. The test is continued until all test items fail or a prespecified censoring time η.
3. If the stress reaches s_h before η, it remains constant at that level.

Let $\zeta=s_h/k$. Two test situations where $\zeta \leq \eta$ (stress reaches s_h before censoring time η) and $\zeta > \eta$ (stress does not reach s_h by censoring time η) are possible. Figure 1 depicts these test situations.

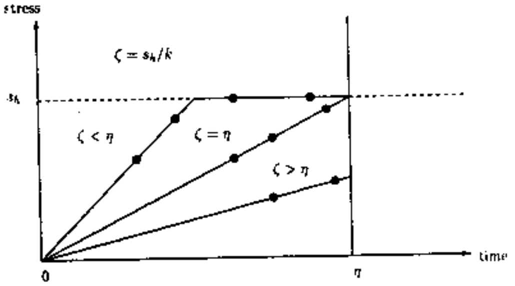

Figure 1. Ramp test situations under stress bound
(• denote failure times)

Assumptions

The following assumptions are made:

1. At any constant stress s, the lifetime of a test item follows an exponential distribution with mean $\theta(s)$.
2. $\theta(s)$ has the inverse power law relationship with stress; that is,

$$\theta(s)=e^{\alpha}/s^{\beta}, \tag{1}$$

where α and β are unknown constants.

3. For the effect of changing stress in a ramp test, a cumulative exposure model holds. (See, Nelson [12])
4. The lifetimes of test items are statistically independent.

Lifetime Distribution under Ramp Tests

In accelerated life models, the reliability function $R(t;s)=\Pr\{T \geq t;s\}$ at constant stress s has the form

$$R(t;s)=R_0(\psi(s)t), \tag{2}$$

where R_0 is some baseline reliability function and $\psi(s)$ is a life-stress relationship. For the accelerated life model, see Crowder et al.[5].

Assuming the cumulative exposure model, the distribution function of lifetime T under stress $s(t)$ is obtained as

$$F(t) = 1-R_0\left\{\int_0^t \psi(s(u))du\right\}.$$

See Bai and Chun [3] and Nelson [12]. Under the exponential lifetime and inverse power law model, $R_0(t)=e^{-t}$ and $\psi(s)=1/\theta(s)=e^{-\alpha}s^{\beta}$. Thus the distribution function $F(t)$ is

$$F(t) = 1 - exp\left\{ -e^{-\alpha}\int_0^t (s(u))^{\beta} du \right\}.$$

Under the above test procedure, the stress at time t is

$$s(t) = \min(kt, s_h), \quad 0 < t < \eta. \qquad (3)$$

The distribution function of T is derived as

$$F(t) = \begin{cases} 0, & t < 0 \\ 1 - exp(-c_1(t)e^{-\alpha}), & 0 \le t < \zeta \\ 1 - exp\{-(c_1(\zeta) + c_2(t))e^{-\alpha}\}, & \zeta \le t \end{cases} \qquad (4)$$

where $c_1(t) = k^{\beta}t^{\beta+1}/(\beta+1)I_{[0,\zeta]}(t), c_2(t) = s_h^{\beta}(t-\zeta)I_{[\zeta,\infty]}(t)$ and $I_{\chi}(\cdot)$ is an indicator function defined as

$$I_{\chi}(x) = \begin{cases} 1, & \text{if } x \in \chi \\ 0, & \text{otherwise} \end{cases}$$

The probability density function of T is

$$f(t) = \begin{cases} 0, & t < 0 \\ (kt)^{\beta}e^{-\alpha}exp(-c_1(t)e^{-\alpha}), & 0 \le t < \zeta \\ s_h^{\beta}e^{-\alpha}exp\{-(c_1(\zeta) + c_2(t))e^{-\alpha}\}, & \zeta \le t \end{cases} .$$

3. Parameter Estimation

Maximum Likelihood Estimators

Suppose that n sample items are tested, t_i is observation i, and the corresponding log likelihood is $L_i, i = 1, \cdots, n$. Then the log likelihood L_0 for n independent observations is $L_0 = \Sigma_{i=1}^{n} L_i$. This is a function of parameters α and β. The ML estimators $\hat{\alpha}$ and $\hat{\beta}$ are the parameter values that maximize the log likelihood L_0.

Let $\tau = \min(\zeta, \eta)$, $D_1\{t \mid 0 \le t < \tau\}$, and $D_2 = \{t \mid \tau \le t < \eta\}$, and $u_i(t) = I_{D_i}(t)$, $i = 1, 2$. For the case with censoring the following two situations can be considered.

(i) The case of $\zeta \le \eta$

When $\zeta \le \eta$, $\tau = \zeta$ and the ML estimators of α and β are the solution of the following two equations. For detailed derivations, see Appendix.

$$e^{\alpha} = \frac{Q_1 + s_h^{\beta}\{(\beta+1)(Q_2 + n_3\eta) - (n_2 + n_3)\tau\}}{(\beta+1)(n_1 + n_2)}, \qquad (5)$$

$$\begin{aligned} &\{w_1\beta(\beta+1) + w_2\}Q_1 - w_2(\beta+1)Q_3 \\ &\qquad + \{w_3\beta(\beta+1) + w_4(\beta+1) + w_5\}s_h^{\beta} = 0, \end{aligned} \qquad (6)$$

where $n_i = \Sigma_{j=1}^{n} u_i(t_j)$, $i = 1, 2$, and $n_3 = n - n_1 - n_2$, $w_1 = Q_4 + n_2\ln(\tau)$, $w_2 = Q_4 - n_1\ln(\tau)$, $w_3 = -n_2\tau + n_3(\eta - \tau)$, $w_4 = n_1 + n_2$, $w_5 = (n_2 + n_3)\tau$, $Q_1 = \Sigma_{t_i \in D_1} k^{\beta} t_i^{\beta+1}$, $Q_2 = \Sigma_{t_i \in D_2} t_i$, $Q_3 = \Sigma_{t_i \in D_1} k^{\beta} t_i^{\beta+1}\ln(t_i)$, and $Q_4 = \Sigma_{t_i \in D_1}\ln(t_i)$.

The ML estimator $\hat{\beta}$ has the following properties.

- Replacing s_h of (6) with $k\tau$, we can obtain an equation in β independent of k. Note that $\hat{\beta}$ is indirectly related to k by test data.
- Let $t_i' = t_i/\lambda$, $\tau' = \tau/\lambda$ and $\eta' = \eta/\lambda$, where λ is an arbitrary constant. Substituting $\lambda t_i'$, $\lambda\tau'$ and $\lambda\eta'$ for t_i, τ and η in (6), respectively, we can derive an equation independent of λ. That is, $\hat{\beta}$ is

invariant under scale transformation of the data.

(ii) The case of $\zeta > \eta$

When $\zeta > \eta$, $\tau = \eta$ and $u_2(t)=0$ for all t and thus $n_2=Q_2=0$. The ML estimators of α and β can be obtained from (5) and (6) with $w_1=Q_4$, $w_3=0$, $w_4=n_1$, $w_5=n_3\eta$.

For the case without censoring, the ML estimators can be obtained from (5) and (6) with $n_3=0$, since $1-u_1(t)-u_2(t)=0$ for all t.

Fisher Information Matrix and Asymptotic Covariance Matrix

The Fisher information matrix is obtained by taking expectations of the negative of the second partial derivatives of the log likelihood function with respect to α and β. The Fisher information matrix F_0 from n test items is $F_0=nF$, where F is the Fisher information matrix from a single item and is derived in the Appendix. The results are as follows.

$$F_0 = n \cdot F$$

$$= n \cdot \begin{bmatrix} A_1+A_4 & -A_2-(A_1+A_4)\ln(s_h) \\ -A_2-(A_1+A_4)\ln(s_h) & A_3+2A_2\ln(s_h)+(A_1+A_4)\ln^2(s_h) \end{bmatrix}, \quad (7)$$

where $A_1=\psi_1(z_1)$, $A_2=\{z_2\psi_1(z_1)+\psi_2(z_1)\}/(\beta+1)$, $A_3=\{z_2^2\psi_1(z_1)+2z_2\psi_2(z_1)+\psi_3(z_1)\}/(\beta+1)^2$, $A_4=(1-\psi_1(z_1))\psi_1(z_3)$, $\psi_1(y)=1-e^{-y}$, $\psi_2(y)=\int_0^y \ln(u)e^{-u}du$, $\psi_3(y)=\int_0^y \ln^2(u)e^{-u}du$, $z_1=c_1(\tau)e^{-\alpha}$, $z_2=\alpha+\ln(\beta+1)+\ln(k)-(\beta+1)\ln(s_h)$, and $z_3=c_2(\eta)e^{-\alpha}$.

For the case without censoring, we can obtain the elements of Fisher information matrix by setting $\eta\rightarrow\infty$. Note that when $\eta\rightarrow\infty$, $\psi_1(z_3)\rightarrow 1.0$ and consequently $A_1+A_4=1.0$

The asymptotic covariance matrix of ML estimators $\hat{\alpha}$ and $\hat{\beta}$ is the inverse matrix of F_0. This covariance matrix is used in interval estimation and testing hypotheses for model parameters. For the details on interval estimation and test of hypotheses in life testing, see Mann et al.[9], Lawless [8], and Nelson [13, 14].

Example 1: We illustrate the ML method with artificial data generated under a ramp test with a stress bound. Consider a pair of parallel disk electrodes immersed in oil. The test situation is that voltage at use condition $s_d=20$ (kV), the upper bound of test stress $s_h=40$ (kV), ramp rate $k=20$ (V/sec) and censoring time $\eta=2{,}400$(sec). It is known that the lifetime distribution is exponential, and the lifetime has inverse power law relationship with voltage stress. Under this situation with the values of true parameters in the inverse power law (1) $\alpha=117.5$ and $\beta=10.5$, a sample of size n=50 is generated and are given in Table 1.

From this artifical data ML estinates $\hat{\alpha}$ and $\hat{\beta}$ are obstained. To obtain ML estimate $\hat{\beta}$ we transformed the data with scale factor $\lambda=1{,}000$ and used the bisection method. The estimated values of α, β and $\ln\theta(s_d)$ are 126.7111, 11.4136 and 13.6767, respectively. Using (7), the estimate $\hat{F}_0$ of Fisher information matrix F_0 is obtained, and the estimate $\hat{V}$ of asymptotic covariance matrix for $\hat{\alpha}$ and $\hat{\beta}$ is the inverse matrix of $\hat{F}_0$. The results are

$$\hat{F}_0 = 50\begin{bmatrix} 1.0000 & -10.6052 \\ -10.6052 & 112.4801 \end{bmatrix},$$

$$\hat{V} = \begin{bmatrix} 210.7434 & 19.8699 \\ 19.8699 & 1.8736 \end{bmatrix}.$$

95% confidence intervals for model parameters α, β and $\ln\theta(s_d)$ are (98.26,155.16), (8.73, 14.10) and (11.77, 15.58), respectively.

Table 1. Data under a ramp test with stress bound

($ID=0$ if $t \in D_1$, $=1$ if $t \in D_2$, and $=2$ if $t \geq \eta$)

t (second)	ID	t (second)	ID	t (second)	ID	t (second)	ID	t (second)	ID
1,469.08	0	1,951.96	0	2,300.26	1	1,897.84	0	1,717.69	0
1,609.45	0	2,400.00	2	2,091.61	1	2,141.54	1	1,962.91	0
1,962.61	0	1,940.36	0	2,134.87	1	1,992.86	0	1,884.03	0
2,400.00	2	1,824.03	0	1,974.20	0	1,916.45	0	2,117.28	1
2,054.50	1	2,386.89	1	1,897.88	0	2,400.00	2	2,125.42	1
1,670.63	0	2,218.22	1	1,769.97	0	2,014.09	1	2,346.61	1
2,383.80	1	2,400.00	2	2,400.00	2	2,400.00	2	2,359.80	1
2,227.28	1	2,293.37	1	2,119.29	1	1,965.90	0	1,954.25	0
1,774.78	0	1,835.61	0	2,400.00	2	2,400.00	2	1,746.84	0
2,247.55	1	2,047.78	1	2,073.97	1	2,265.68	1	1,932.42	0

4. Optimum Test Plans

Optimality Criterion and Decision Variable

Minimizing the asymptotic variance of ML estimator of log mean life at design stress s_d is used as the optimality criterion. An optimum ramp test is determined by ramp rate k. The ML estimator of log mean life at s_d is

$$\ln\hat{\theta}(s_d) = \hat{\alpha} + \hat{\beta}x,$$

where $x = -\ln(s_d)$. The corresponding asymptotic variance is

$$\text{Asvar}(\ln\hat{\theta}(s_d)) = \text{Asvar}(\hat{\alpha}) + 2x\text{Ascov}(\hat{\alpha}, \beta) + x^2\text{Asvar}(\hat{\beta}). \quad (8)$$

Optimum Ramp Tests

From the Fisher information matrix (7), formula (8) can be written as

$$\text{Asvar}(\ln\hat{\theta}(s_d)) = \frac{(A_1+A_4)ln^2(\delta) - 2A_2\ln(\delta) + A_3}{(A_1+A_4)A_3 - A_2^2}, \quad (9)$$

where $\delta \equiv s_d/s_h$ is the standardized design stress.

Let p_d and p_h be the probabilities that an item will fail η at s_d and at s_h, respectively. Then we have

$$p_d = 1 - exp\{-s_d^{\beta}\,\eta\, e^{-\alpha}\},$$

$$p_h = 1 - exp\{-s_h^{\beta}\,\eta\, e^{-\alpha}\}.$$

Let $\xi \equiv \zeta/\eta$. Then z_1, z_2 and z_3 defined in (7) can be characterized in terms of p_h, ξ and

β as

$$z_1 = \begin{cases} -\xi\ln(1-p_h)/(\beta+1), & \text{if } 0 < \xi < 1, \\ -\xi^{-\beta}\ln(1-p_h)/(\beta+1), & \text{if } \xi \geq 1 \end{cases}$$

$$z_2 = -\ln(-\ln(1-p_h)) + \ln(\beta+1) - \ln(\xi),$$

$$z_3 = \begin{cases} -(1-\xi)\ln(1-p_h), & \text{if } 0 < \xi < 1 \\ 0, & \text{if } \xi \geq 1 \end{cases}$$

and β can be expressed in terms of p_d, p_h and δ as $\beta = \{\ln(-\ln(1-p_d)) - \ln(-\ln(1-p_h))\}/\ln(\delta)$.

Thus Asvar($\ln\hat{\theta}(s_d)$) can be expressed in terms of p_d, p_h, δ and ξ. Further, let a and b be

$$a = \Psi^{-1}(p_h)$$
$$b = \Psi^{-1}(p_h) - \Psi^{-1}(p_d). \qquad (10)$$

a and b are standardized log censoring time and slope, respectively; see Kielpinski and Nelson [11]. Then, Asvar($\ln\hat{\theta}(s_d)$) can also be expressed in terms of a, b, δ and ξ. It is more convenient to represent Asvar ($\ln\hat{\theta}(s_d)$) in terms of a and b than p_d and p_h when p_d and p_h are close to zero.

Thus the design problem becomes: Given the values of a, b and δ, find the value ξ^* which minimizes Asvar($\ln\hat{\theta}(s_d)$). Optimum ramp rate k^* is

$$k^* = s_h/(\xi^*\eta). \qquad (11)$$

Since the optimum ramp test depends on design parameters a and b, we must obtain their values from past experiences, similar data, or a preliminary test. ξ^* can be found through an optimization procedure such as Powell algorithm [16]. We have computed optimum ramp test plans for various values of a, b and δ Table 2 shows ξ^*, probability p_R of failure of a test item by η and asymptotic variance when δ=0.1(0.2)0.5. In Table 2, the selected values of a are −2.0, 0.0, 2.0 and 4.0, and the corresponding values of p_h are 0.127, 0.632, 0.999 and 1.000, respectively. The values of b selected are $b=a+c$, where c=*4.0 (2.0)12.0*, and corresponding values of p_d are 0.018149, 0.002476, 0.000335, 0.000045 and 0.000006, respectively. For the selected values of a and b and corresponding values of p_h and p_d, see, Meeker [10].

Example 2 :Assume a ramp test of a pair of parallel disk electrodes immersed in oil. The voltage across the electrodes is increased linearly with time. The lifetime follows the exponential distribution and has the inverse power law relationship with voltage stress. The stress upper bound and the design constant stress are 40 kV and 20 kV, respectively. Suppose that about 2.0% of test items fail within 10(hour) of testing under 20 kV and 99.9% under 40 kV. From (10), preestimates of a and b are about 2.0 and 6.0, respectively, and δ=0.5. Table 2 gives $\xi^* \simeq$ 0.9095 and $k^* = 40/(\xi^*\eta) = 40/(0.9095 \times 10.0) = 4.398$ (kV/hour), the asymptotic variance of $\ln\theta(s_d)$ is about 44.2 and the probability that a test item will fail by η=10.0(hour) under ramp test

Table 2. Optimum ramp test palns

δ	a \ c	4.0	6.0	8.0	10.0	12.0
0.10	-2.00	2.1217[1] .0174[2] .0610[3]	6.0878 .0021 .5265	.8898 .0472 1.3566	.8753 .0424 2.5942	.8691 .0390 4.3875
	0.00	6.0815 .0158 .0715	.8892 .3005 .1929	.8740 .2748 .3666	.8677 .2553 .6169	.8653 .2397 .9565
	2.00	.9155 .9180 .0373	.8864 .9001 .0687	.8746 .8819 .1121	.8697 .8643 .1690	.8681 .8475 .2413
	4.00	1.2949 .9931 .0311	1.1380 .9971 .0538	1.0616 .9984 .0818	1.0182 .9990 .1150	.9936 .9992 .1533
0.30	-2.00	2.0042 .0159 .0706	.9250 .0384 .5748	.8957 .0338 1.5673	.8863 .0306 3.2710	.8863 .0282 5.8597
	0.00	.9264 .2502 .0804	.8953 .2246 .2185	.8856 .2057 .4542	.8829 .1910 .8111	.8831 .1792 1.3123
	2.00	.9046 .8383 .0369	.8899 .8124 .0749	.8854 .7877 .1308	.8847 .7649 .2079	.8857 .7441 .3089
	4.00	1.0301 .9972 .0280	.9892 .9983 .0491	.9712 .9984 .0758	.9597 .9983 .1082	.9524 .9980 .1465
0.50	-2.00	1.3779 .0137 .0872	.9293 .0278 .6913	.9061 .0251 2.0587	.9002 .0230 4.5109	.8996 .0213 8.3399
	0.00	.9299 .1873 .0956	.9058 .1714 .2838	.8998 .1580 .6203	.8992 .1471 1.1445	.9005 .1381 1.8954
	2.00	.9095 .7445 .0442	.9015 .7160 .0950	.9002 .6891 .1726	.9013 .6649 .2825	.9032 .6431 .4296
	4.00	.9733 .9966 .0284	.9587 .9965 .0502	.9511 .9959 .0786	.9472 .9951 .1140	.9452 .9941 .1570

1) Optimum ξ^*
2) Failure Probability p_R
3) Asvar(ln $\theta(s_D)$) × 10^{-3}

is about 0.74.

Effects of Pre-Estimates

To use an optimum test plan, we need information about the values of design parameters a and b. Incorrect choice of them gives a poor estimate of log mean life at design constant stress. The effects of incorrect pre-estimates of a and b in terms of asymptotic variance ratio V'/V^* ($\geq$ 1.0) were studied, where V' and V^* are Asvar(ln $\hat{\theta}(s_d)$) under incorrect preestimates of a and b and that evaluated at true values, respectively.

For each pair *(a,b)* of true values we computed the ratios V'/V^* for misspecifications ± 1 and ± 2 in both a and b, and found the maximum and the minimum of the ratios. Table 3 shows the maximum and the minimum of the ratios and the corresponding deviations from a and b when $\delta=0.5$. We can see from Table 3 that: (i) sensitivity to misspecification is more acute when b is small (i.e., p_d is large); (ii) when a is misspecified +1 or +2 and b is misspecified -1 or -2, the ratios are maximum except the pair *(a,b)* of which the maximum ratio is small (e.g., less than 1%).

The Case without Time-Censoring

In this case, $A_1+A_4=1.0$ in formula (7). The asymptotic variance (8) can be expressed as

$$\text{Asvar}(\ln \hat{\theta}(s_d)) = 1+\frac{(A_2-\ln(\delta))^2}{A_3-A_2^2}. \qquad (12)$$

Since $\tau = \zeta \doteq s_H/k$, A_2 can be written as

$$A_2 \doteq \frac{-\ln(\nu(\zeta))\psi_1(\nu(\zeta))+\psi_2(\nu(\zeta))}{\beta+1}, \quad (13)$$

Table 3. Effects of incorrect pre-estimates of a and b (δ=0.5)

True Values		Misspecification of a and b by ±1								Misspecification of a and b by ±2							
		Max. Ratio	Deviation Yielding Max. Ratio		Min. Ratio	Deviation Yielding Min. Ratio		Max. Ratio	Deviation Yielding Max. Ratio		Min. Ratio	Deviation Yielding Min. Ratio	
a	b		a	b		a	b		a	b		a	b
-2	2	1.147	+1	-1	1.087	+1	+1	2.446	+2	-2	1.342	+2	+2
-2	4	1.422	+1	-1	1.002	-1	+1	1.531	+2	-2	1.003	-2	+2
-2	6	1.002	-1	-1	1.000	-1	+1	1.005	+2	-2	1.000	-2	+2
-2	8	1.000	-1	-1	1.000	-1	+1	1.000	-2	-2	1.000	-2	+2
-2	10	1.000	-1	+1	1.000	-1	-1	1.000	-2	+2	1.000	+2	-2
0	4	1.399	+1	-1	1.002	+1	+1	1.513	+2	-2	1.003	+2	+2
0	6	1.001	+1	-1	1.000	-1	+1	1.012	+2	-2	1.000	+2	+2
0	8	1.000	-1	-1	1.000	-1	+1	1.001	+2	-2	1.000	-2	+2
0	10	1.000	-1	+1	1.000	+1	+1	1.000	+2	-2	1.000	-2	-2
0	12	1.000	-1	+1	1.000	-1	-1	1.000	+2	+2	1.000	+2	-2
2	6	1.017	+1	-1	1.000	-1	-1	4.117	+2	-2	1.001	-2	+2
2	8	1.005	+1	-1	1.000	-1	-1	1.239	+2	-2	1.000	-2	+2
2	10	1.002	+1	-1	1.000	-1	+1	1.083	+2	-2	1.000	-2	+2
2	12	1.002	+1	-1	1.000	-1	+1	1.046	+2	-2	1.000	-2	+2
2	14	1.001	+1	+1	1.000	-1	+1	1.031	+2	-2	1.000	-2	+2
4	8	1.283	+1	-1	1.011	-1	-1	7.788	+2	-2	1.016	-2	-2
4	10	1.184	+1	-1	1.008	-1	-1	2.834	+2	-2	1.013	-2	-2
4	12	1.120	+1	-1	1.007	-1	-1	2.386	+2	-2	1.010	-2	+2
4	14	1.077	+1	-1	1.006	-1	+1	2.116	+2	-2	1.009	-2	+2
4	16	1.058	+1	-1	1.005	-1	+1	1.925	+2	-2	1.008	-2	+2

where $\nu(\zeta)=c_1(\zeta)e^{-\alpha}$. Since $A_3-A_2^2$ is determinant of the Fisher information matrix and is positive, the value ζ^* satisfying $A_2=\ln(\delta)$ gives a minimum of (12). Thus, if pre-eatimates of α and β are given, we can obtain the optimum ramp rate $k^*=s_h/\zeta^*$. Differentiating A_2 with respect to ζ gives $A_2'=-\psi_1(\nu(\zeta))/\{\zeta(\beta+1)\}<0$ for all $\zeta>0$, and $\lim_{\zeta\to 0}A_2=0$ and $\lim_{\zeta\to\infty}A_2=-\infty$. Therefore, the value of ζ satisfying $A_2=\ln(\delta)$ is unique.

Example 3 : Consider a ramp test of an insulating fluid whose breakdown time follows the exponential distribution and has the inverse power law relationship with voltage stress. Suppose that s_d=20(kV), s_h=40(kV), and preestimated values of α and β from a preliminary test or past experiences are 25.0 and 8.0, respectively. We used the bisection method to solve the equation $A_2=\ln(\delta)$ and obtained optimum ramp rate k^*=1.407(kV/hour).

5. The Case of No Stress Bound

In this section, we consider a special case where there is no stress bound, i.e., $s_h \to \infty$. Then, $\tau = \eta$ and $u_2(t)=0$ for all t. The ML estimators are the same as those of the case where there is a stress bound and $\zeta > \eta$.

Define $B_1 \equiv A_1 \ln(s_h) + A_2 = \{z_4 \psi_1(z_1) + \psi_2(z_1)\}/(\beta+1)$, and $B_2 \equiv A_1 \ln^2(s_h) + 2A_2 \ln(s_h) + A_3 = \{z_4^2 \psi_1(z_1) + 2z_4 \psi_2(z_1) + \psi_3(z_1)\}/(\beta+1)^2$, where $z_4 = z_2 + (\beta+1)\ln(s_h) = \alpha + \ln(\beta+1) + \ln(k)$. Then B_1 and B_2 do not depend on $\ln(s_h)$ and since $z_3=0$ when $\zeta > \eta$, $A_4=0$. The Fisher information matrix is then

$$F = \begin{bmatrix} A_1 & -B_1 \\ -B_1 & B_2 \end{bmatrix}. \tag{14}$$

Asymptotic variance (8) is now

$$\mathrm{Asvar}(\ln \hat{\theta}(s_d)) = \frac{x^2 A_1 + 2xB_1 + B_2}{A_1 B_2 - B_1^2}. \tag{15}$$

If pre-estimates of α and β are given, the optimum ramp rate k^* which minimizes (15) can be determined by using a numerical method.

If, in addition, there is no time-censoring, a closed form solution k^* can be found. The ML estimators can be obtained from equations (5) and (6) with $n_2 = n_3 = Q_2 = 0$ since $u_1(t)=1$ for all t. The results coincide with those of Yin and Sheng [20]. The Fisher information matrix is obtained by taking limit $\lim_{\tau \to \infty} \psi_1(z_1) = 1$, $\lim_{\tau \to \infty} \psi_2(z_1) = -\gamma$ and $\lim_{\tau \to \infty} \psi_3(z_1) = \pi^2/6 + \gamma^2$ of B_1 and B_2, where γ is the Euler's constant. That is,

$$F = \begin{bmatrix} 1 & (\gamma - z_4)/(\beta+1) \\ (\gamma - z_4)/(\beta+1) & \{\pi^2/6 + (\gamma - z_4)^2\}/(\beta+1)^2 \end{bmatrix} \tag{16}$$

The asymptotic variance (8) can then be rewritten as

$$\mathrm{Asvar}(\ln \hat{\theta}(s_d)) = \frac{6}{\pi^2}\{x^2(\beta+1)^2 - 2x(\gamma - z_4)(\beta+1) + (\gamma - z_4)^2\} + 1. \tag{17}$$

(17) is a quadratic function of z_4 and only z_4 involves k. Therefore, the value of k minimizing (17) is

$$k^* = \frac{1}{\beta+1} s_d^{\beta+1} e^{-\alpha+\gamma}. \tag{18}$$

Example 3 :(continued) Suppose that $\alpha = 25.0$, $\beta = 8.0$, $s_d = 20$(kV) and there is no stress upper bound. When censoring time η is given as 10.0(hour), we used three-point equal-interval search method to find optimum ramp rate k^* minimizing (15) and obtained $k^* = 2.510$ (kV/hour).

For the case without censoring, k^* of (18) is obtained as 1.407(kV/hour). This result is the same as that of the case with a stress bound. For some the other examples, we obtained the same results. Thus, it appears that a stress bound does not affect optimum ramp rate when there is no time censoring. However, we were unable to show that solution of the equation $A_2 = \ln(\delta)$ is equal to k^* of (18).

Appendix

(ⅰ) Likelihood Equations

The log likelihood L for a single observation t is

$$L = u_1(t)\{-\alpha + \beta\ln(kt) - c_1(t)e^{-\alpha}\}$$
$$+u_2(t)\{-\alpha + \beta\ln(s_h) - [c_1(\tau) + c_2(t)]e^{-\alpha}\} \quad \text{(A1)}$$
$$-(1-u_1(t)-u_2(t))\{c_1(\tau)+c_2(\eta)\}e^{-\alpha}$$

The first partial derivatives of L with respect to model parameters are

$$\partial L/\partial\alpha = u_1(t)\{-1+c_1(t)e^{-\alpha}\}+u_2(t)\{-1+[c_1(\tau)+c_2(t)]e^{-\alpha}\} \quad \text{(A2)}$$
$$+(1-u_1(t)-u_2(t))\{c_1(\tau)+c_2(\eta)\}e^{-\alpha}$$

$$\partial L/\partial\beta = u_1(t)\{\ln(kt)-c_1(t)e^{-\alpha}(\ln(kt)-1/(\beta+1))\}$$
$$u_2(t)\{\ln(s_h)-[G_1+c_2(t)\ln(s_h)]e^{-\alpha}\} \quad \text{(A3)}$$
$$-(1-u_1(t)-u_2(t))\{G_1+c_2(\eta)\ln(s_h)\}e^{-\alpha},$$

where $G_1=c_1(\tau)(\ln(s_h)-1/(\beta+1))$. These expressions, when summed over all n items and set equal to zero, are the likelihood equations. From (A2) and (A3), we can obtain the following two equations

$$(\beta+1)(n_1+n_2)=e^{-\alpha}\{Q_1+s_h^{\beta}[(\beta+1)(Q_2+n_3\eta)-(n_2+n_3)\tau]\}, \quad \text{(A4)}$$

$$(\beta+1)(n_1\ln(k)+Q_4+n_2\ln(s_h))=$$
$$e^{-\alpha}\{\ln(k)Q_1+Q_3-(Q_1+(n-n_1)s_h^{\beta}\tau/(\beta+1)) \quad \text{(A5)}$$
$$+s_h^{\beta}\ln(s_h)[(\beta+1)(Q_2+n_3\eta)-\beta(n_2+n_3)\tau]\}$$

We obtain from (A4) and (A5), after eliminating α, an equation for β in the form of (6). Equation (5) is obtained from (A4).

(ii) Fisher Information Matrix

The second partial derivatives are

$$\partial^2L/\partial\alpha^2=-(\partial L/\partial\alpha)-u_1(t)-u_2(t) \quad \text{(A6)}$$
$$\partial^2L/\partial\alpha\,\partial\beta=-(\partial L/\partial\beta)+u_1(t)\ln(kt)+u_2(t)\ln(s_h) \quad \text{(A7)}$$

$$\partial^2L/\partial\beta^2=-(2/(\beta+1))\cdot(\partial L/\partial\beta)$$
$$+u_1(t)\{-c_1(t)e^{-\alpha}\ln^2(kt)+2\ln(kt)/(\beta+1)\}$$
$$-u_2(t)\{[c_1(t)\ln^2(s_h)+G_2c_2(t)]e^{-\alpha}-2\ln(s_h)/(\beta+1)\} \quad \text{(A8)}$$
$$-(1-u_1(t)-u_2(t))\{c_1(\tau)\ln^2(s_h)+G_2c_2(\eta)\}e^{-\alpha}.$$

where $G_2=\ln^2(s_h)+2\ln(s_h)/(\beta+1)$.

We obtain the following identities through integration by parts.

$$E[u_1(T)e^{-\alpha}c_1(T)\ln^2(kT)] = -e^{-\alpha}c_1(\tau)\ln^2(s_h)\{1-E[u_1(T)]\} \quad \text{(A9)}$$
$$+E[u_1(T)\ln^2(kT)]+2E[u_1(T)\ln(kT)]/(\beta+1),$$

$$E[u_2(T)e^{-\alpha}c_2(T)] = -e^{-\alpha}c_2(\eta)\{1-E[u_1(T)]-E[u_2(T)]\}+E[u_2(T)], \quad \text{(A10)}$$

Using (A9), (A10) and the fact that $E[\partial L/\partial\alpha]=E[\partial L/\partial\beta]=0$, we obtain

$$-E[\partial^2L/\partial\alpha^2]=E[u_1(T)]+E[u_2(T)]$$
$$-E[\partial^2L/\partial\alpha\,\partial\beta]=-E[u_1(T)\ln(kT)]-\ln(s_h)E[u_2(T)] \quad \text{(A11)}$$
$$-E[\partial^2L/\partial\beta^2]=E[u_1(T)\ln^2(kT)]+\ln^2(s_h)E[u_2(T)].$$

Expectations in (A11) are obtained as follows.

$$E[u_1(T)]=\psi_1(z_1),$$
$$E[u_2(T)]=(1-\psi_1(z_1))\,\psi_1(z_3),$$
$$E[u_1(T)\ln(kT)]=\{z_2\psi_1(z_1)+\psi_2(z_1)\}/(\beta+1)+\ln(s_h)E[u_1(T)]$$

$$E[u_1(T)\ln^2(kT)]=\{z_2^2\psi_1(z_1)+2z_2\psi_2(z_1)+\psi_3(z_1)\}/(\beta+1)^2$$
$$+2\ln(s_h)E[u_1(T)\ln(kT)]-\ln^2(s_h)E[u_1(T)].$$

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96년 5월 최초 접수, 96년 7월 최종 수정