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ON THE LOCALLY NILPOTENT SPACES AND (T^*)

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1. Introduction

There are many results on the nilpotent spaces and its applications [1, 5]. The extension problems of the concept of nilpotent space with respect to the P-localization have been studied recently [2]. In this paper, we define the locally nilpotent space as the extension of nilpotent space and investigate the properties of the locally nilpotent space(Theorem 3.1., Theorem 3.2., Theorem 3.4.). We work in the category of the connected CW-complexs with base point and continous maps unless otherwise stated and denote the above category as T.

2. Preliminaries

In this section, we define the locally nilpotent space and soluble space and study the properties of them.

DEFINITION 2.1. A locally nilpotent group is defined as the group whose finitely generated subgroups are nilpotent. We use the symbol LN for the category of locally nilpotent groups and group homomorphisms.

We know that the class of locally nilpotent groups contains the class of all abelian, nilpotent, hypercentral, Fitting, Baer groups. Futhermore, we know that LN is closed under subgroup, quotient

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groups, finite products and of course direct limits. We say that a group π acts nilpotently on a group G if there is given a homomorphism $\alpha : \pi \to \text{Aut}G$ and there exists a finite sequence of subgroups of G

$$\{e\} = G_n \subset G_{n-1} \subset \cdots \subset G_2 \subset G_1 = G$$

such that for each α

(1) G_α is closed under the action of π,
(2) $G_{\alpha+1}$ is normal in G_α and $\frac{G_\alpha}{G_{\alpha+1}}$ is abelian, and
(3) the induced action on $\frac{G_\alpha}{G_{\alpha+1}}$ is trivial.

DEFINITION 2.2. A space $X(\in T)$ is said to be a nilpotent space if

(1) $\pi_1(X)$ is a nilpotent group,
(2) The action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$.

And we denote the category of nilpotent spaces and continuous maps as T_N. In this paper, we extend the concept of nilpotent space like this ;

DEFINITION 2.3. A space $X(\in T)$ is said to be a locally nilpotent space if

(1) $\pi_1(X)$ is a locally nilpotent group,
(2) The action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$.

And we denote the category of locally nilpotent spaces and continuous maps as T_{LN}. For any nilpotent groups their finitely generated subgroups are also nilpotent groups, thus the category T_N is the full subcategory of T_{LN}.

DEFINITION 2.4. A group action G on H is called to be soluble if there exists a lower abelian series of H ; there exists a chain $\{e\} = H_n \subset H_{n-1} \subset H_{n-2} \subset \cdots \subset H_1 = H$ such that

(1) H_α is closed under the action,
(2) For each α, $H_{\alpha+1}$ is a normal subgroup in H_α and $\frac{H_\alpha}{H_{\alpha+1}}$ is abelian.

DEFINITION 2.5. A space $X(\in T)$ is said to be a soluble space if

(1) $\pi_1(X)$ is a soluble group,
(2) The action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is soluble for all $n \geq 2$.

And we denote the category of soluble spaces and continuous maps as T_S. We know that the category of T_N is the full subcategory of T_S.

Example. BS_3 is a nonnilpotent soluble space where B is the Milnor's classifying space and S_3 is the symmetric group. Generally, for a group G and a fixed $g \in G$, we denote by $[g, G]$ the subgroup of G generated by all commutators $[g, a] = g^{-1}a^{-1}ga(a \in G)$. In fact, $[g, G]$ is a normal subgroup in G since $[g, a]^b = [g, b]^{-1}[g, ab]$ for each $a, b \in G$ (where $a^b = b^{-1}ab$).

DEFINITION 2.6. We say that a space $X(\in T)$ satisfies condition (T^*) if for all $g, t \in \pi_1(X)$ either $g[g, \pi_1(X)] = t[t, \pi_1(X)]$ or $g[g, \pi_1(X)] \cap t[t, \pi_1(X)] = \phi$.

LEMMA 2.7, [3]. *Let G be an arbitrary group. If $b \in a[a, G]$ $(a, b \in G)$ then $b[b, G] \subset a[a, G]$.*

Proof. Let c be an arbitrary element from $b[b, G]$. There exist elements $h_1 \in [a, G]$ and $h_2 = \prod_{i=1}^m [b, g_i]^{\epsilon_i} \in [b, G](\epsilon_i = \pm 1)$ such that $b = h_1 a$ and $c = h_2 b$. We have $h_2 = \prod_{i=1}^m [h_1 a, g_i]^{\epsilon_i} = \prod_{i=1}^m ([h_1, g_i])^a [a, g_i]^{\epsilon_i} \in [a, G]$ since $[a, G]$ is normal in G. Consequently, $c = h_2 h_1 a \in a[a, G]$ as desired.

THEOREM 2.8. *If $X \in T_{LN}$ then X satisfies the condition (T^*).*

Proof. Since $\pi_1(X)$ is a locally nilpotent group, suppose $c \in a[a, \pi_1(X)] \cap b[b, \pi_1(X)]$ for some $a, b, c \in \pi_1(X)$. We only show that $a[a, \pi_1(X)] = b[b, \pi_1(X)]$. By Lemma 2.7,

$$c[c, \pi_1(X)] \subset a[a, \pi_1(X)] \cap b[b, \pi_1(X)] \cdots\cdots\cdots\cdots\cdots (*)$$

Clearly, $c = h^{-1}a$ for some $h = \prod_{i=1}^m [a, g_i]^{\epsilon_i} \in [a, \pi_1(X)]$ $(g_i \in \pi_1(X), \epsilon_i = \pm 1)$. Let $G_1 =< a, g_1, \cdots, g_m >$. Since $a = hc$, $h \equiv \prod_{i=1}^m [h, g_i]^{\epsilon_i}$ modulo $[c, G_1]$, that is, $h = \prod_{i=1}^m [h, g_i\ ]^{\epsilon_i}$ in

$\frac{G_1}{[c,G_1]}$. However, since the latter group is nilpotent it follows that $h = 1$ in $\frac{G_1}{[c,G_1]}$ and $h \in [c, G_1]$. Therefore, $a = hc \in c[c, \pi_1(X)]$ and by Lemma 2.7, $a[a, \pi_1(X)] \subset [c, \pi_1(X)]$. It follows from (*) that $a[a, \pi_1(X)] = c[c, \pi_1(X)]$. Similary, $b[b, \pi_1(X)] = c[c, \pi_1(X)]$ and consequently, $a[a, \pi_1(X)] = b[b, \pi_1(X)]$.

LEMMA 2.9. *The following conditions are equivalent.*

(1) *X satisfies the condition (T^*).*
(2) *For each $a \in \pi_1(X)$,$a[a, \pi_1(X)] \subset b[b, \pi_1(X)] \Rightarrow$ $a[a, \pi_1(X)] = b[b, \pi_1(X)]$.*
(3) *For each $a, b \in \pi_1(X)$, $h \in [a, \pi_1(X)] \Rightarrow [ah, \pi_1(X)] = [a, \pi_1(X)]$.*

Proof. Trivially (1) $\Rightarrow$ (2). By Lemma 2.7, (2) $\Rightarrow$ (3). Suppose that (3) holds and $c \in a[a, \pi_1(X)] \cap b[b, \pi_1(X)]$. Then $c = ah$ for some $h \in [c, \pi_1(X)]$ and $[c, \pi_1(X)] = [a, \pi_1(X)]$. Hence, $c[c, \pi_1(X)] = ah[a, \pi_1(X)] = a[a, \pi_1(X)]$. Similarly, $c[c, \pi_1(X)] = b[b, \pi_1(X)]$ so that $a[a, \pi_1(X)] = b[b, \pi_1(X)]$, which complete the proof.

Theorem 2.8. and Lemma 2.9. are induced from Dokuchaev's result [3] in the point of view algebraic topology.

3. Main Theorems

In this section, we study some important properties of locally nilpotent space.

THEOREM 3.1. *For finite set $\{X_\alpha\}_{\alpha \in M}$ where $X_\alpha \in T_{LN}$, $\prod_{\alpha \in M} X_\alpha$ also satisfies the condition (T^*).*

Proof. Let G be the group $\bigoplus_{\alpha \in M} \pi_1(X_\alpha)$. Let P_α be the projection of G on $\pi_1(X_\alpha)$. We know that each X_α satisfies the condition (T^*) by Theorem 2.8. Suppose that $a[a, G] \subset [b, G]$ for some $a, b \in G$, put $P_\alpha(a) = a_\alpha$, $P_\alpha(b) = b_\alpha$ then for $a_\alpha[a_\alpha, \pi_1(X_\alpha)] \subseteq b_\alpha[b_\alpha, \pi_1(X_\alpha)]$, $a_\alpha[a_\alpha, \pi_1(X_\alpha)] = b_\alpha[b_\alpha, \pi_1(X_\alpha)]$ by Lemma 2.9. Thus $a[a, G] = b[b, G]$. Hence $\prod_{\alpha \in M} X_\alpha$ satisfies the condition (T^*).

THEOREM 3.2. *For finite set* $\{X_\alpha\}_{\alpha\in M}$ *where* $X_\alpha \in T_{LN}$ *then* $\prod_{\alpha\in M} X_\alpha \in T_{LN}$.

Proof. Since the finite direct sum of locally nilpotent groups is also locally nilpotent group, thus $\bigoplus_{\alpha\in M}\pi_1(X_\alpha)$ is a locally nilpotent group. Since the action $\pi_1(X_\alpha)\times\pi_n(X_\alpha)\to\pi_n(X_\alpha)$ is nilpotent and the action $\bigoplus_{\alpha\in M}\pi_1(X_\alpha)\times\bigoplus_{\alpha\in M}\pi_n(X_\alpha)\to\bigoplus_{\alpha\in M}\pi_n(X_\alpha)$ is componentwise action and the action is nilpotent. Thus our proof is completd.

Remark. Proof of Theorem 3.1 will also be made like this ; since $\prod_{\alpha\in M} X_\alpha$ is the locally nilpotent by Theorem 3.2 and Theorem 2.8, $\prod_{\alpha\in M} X_\alpha$ satisfies the condition (T^*).

LEMMA 3.3. *A finite group* $\pi_1(X)$ *is nilpotent if and only if* X *satisfies condition* (T^*).

Proof. The proof is a little modification of Dokuchaev's Theorem 3 [3].

THEOREM 3.4. *For* $X, Y(\in T_{LN})$ *and* $\pi_1(X)$, $\pi_1(Y)$ *are finite, then* X *is homotopy equivalent to* Y *if* $f: X\to Y$ *is an acyclic map.*

Proof. By Lemma 3.3 and Theorem 2.8, $\pi_1(X)$, $\pi_1(Y)$ is a nilpotent group. Thus $X, Y\in T_N$. From the fact that $f: X\to Y$ is an acyclic map and the classical homotopy exact sequence of fibration : $F_f\to X\xrightarrow{f} Y$, we know that $\pi_1(f)$ is an epimorphism. Furthermore $H_1(F_f)\cong\frac{\pi_1(F_f)}{[\pi_1(F_f),\pi_1(F_f)]}=0$ where [,] means the commutator subgroup and F_f is the homotopy fiber of f. Thus $P\pi_1(X)$ is perpect normal subgroup of $\pi_1(X)$. Since $X\in T_N$, $P\pi_1(X)$ is trivial. Thus $\pi_1(f)$ is an epimorphism. By use of the Hurewicz Theorem [4] inductively, $\pi_i(F_f)=0$. Thus f is a weak homotopy equivalence. By the Whitehead Theorem [4], f is a homotopy equivalence.

THEOREM 3.5. *For* $X\in T_{LN}$, *if* $\pi_1(X)$ *is finite then* $X\in T_N$.

Proof. By Theorem 2.8, the property that X satisfies the condition (T^*) and by Lemma 3.3 our proof is completed.

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