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SOME GENERALIZATIONS OF M-FINITE BANACH SPACES

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ABSTRACT. We will show that let X and Y be M -finite Banach spaces with canonical M-decompositions $X \cong \prod_{i=1}^{r\infty} X_i^{n_i}$ and $Y \cong \prod_{j=1}^{\tilde{r}\infty} \tilde{Y}_j^{m_j}$, respectively and M and N nonzero locally compact Hausdorff spaces. Then $I : C_0(M,X) \to C_0(N,Y)$ is an isometrical isomorphism if and only if $r = \tilde{r}$ and there are permutation and homeomorphisms and continuous maps such that $I = I_{N,Y}^{-1} \circ I_{\omega^{-1}} \circ (\prod_{i=1}^{r} I_{t_i,u_i}) \circ I_{M,X}$.

1. Introduction

Throughout of this note X be a nonzero Banach space and M a nonempty locally compact Hausdorff space. We say that a Banach space X is M-finite if $Z(X)$ is finite -dimensional. It follows that $Z(\prod_{i=1}^{n\infty} X_i) \cong \prod_{i=1}^{n\infty} Z(X_i)$ so that finite product is M-finite Banach spaces are also M-finite. In particular, $\prod_{i=1}^{n\infty} X_i$ is M-finite if the centralizers of the X_i are one-dimensional.

Let X be a Banach space. For $T \in Mult(X)$, we say that $S \in Mult(X)$ is an adjoint for T if $a_S = \tilde{a}_T$. If T adjoint, then this operator is uniquely determined, it will be denoted by $T^\star$. $Z(X)$, the centralizer of X is the set of those multipliers T for which an adjoint $T^\star$ exist.

In this note, we will prove that some results of M-structure of $C_0(M,X)$ and Banch space with the local centralizer norming system and some generalization of M-finite Banach spaces.

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2. Preliminaries

Definition 2.1. A function module is a triple $(K, (X_k)_{k\in K}, \tilde{X})$, where K is a nonempty compact Hausdorff space, $(X_k)_{k\in K}$ a family of Banach spaces and X a closed subspace of $\prod_{k\in K}^{\infty} X_k$ such that the following conditions are satisfied :

(1) $hx \in X$ for x in X and h in CK $((hx)(k) = h(k)x(k))$. (i.e., X is a CK-module)

(2) $k \to \|x(k)\|$ is an upper semicontinuous function for every x in X.

(3) $X_k = \{x(k) | x \in X\}$ for every $k \in K$.

(4) $\{k | k \in K, X_k \neq \{0\}\}^{-} = K$ (i.e. dense)

Definition 2.2. Let X be a Banach space and $R_i = [\rho_i, (K_i, (X_k^i)_{k\in K_i, X_i})]$, $(i = 1, 2)$ function module representations of X.

(1) We say that R_1 is finer than $R_2 (R_2 < R_1)$ if there are continuous map t from K_1 onto K_2 and a family of isometric isomorphisms $S_l : X_1|_{t^{-1}(l)} \to X_l^2$. (all $l \in K_2$; for the definition of $X_1|_{t^{-1}(l)}$) such that $S \circ \rho_1 = \rho_2$ (where $(S_{x_1})(l) = S_l(x_1|_{t^{-1}(l)})$) for $x_1 \in X_1$ and $l \in K_2$):

$$\begin{array}{ccl} & & X_1 \hookrightarrow \prod_{k\in K_1}^{\infty} X_k^1 \\ & \rho_1 \nearrow & \\ X & & \downarrow S \\ & \rho_2 \searrow & \\ & & X_2 \hookrightarrow \prod_{l\in K_2} X_l^2 \end{array}$$

(2) R_1 and R_2 are said to be equivalent $(R_2 \approx R_1)$ if $R_2 < R_1$ and, in addition, the mapping t in (1) is a homeomorphism.

Lemma 2.3.[5]. *The idempotent elements of $Z(X)$ are just the M-projections of X.*

Lemma 2.4.[4]. *Let X be a Banach space such that $Z(X)$ is one-dimensional. Then all M-summands of X are trivial and the trivial representation of X (in $\prod_{k\in K}^{\infty} X_k$ with $K = \{1\}$ and $X_1 = X$) is a maximal function module representation.*

Lemma 2.5.[4]. *(1) Suppose that X has no nontrivial M-ideals. Then the M-ideals of $C_0(M, X)$ are the subspaces $Y_C = \{f | f \in C_0(M, X), f|_C = 0\}$ ($C \subset M, C$ a closed subset).*

(2) If X has no nontrivial M-summands, then the M-summands of $C_0(M, X)$ are the subspaces Y_C, where $C \subset M, C$ closed and open.

Lemma 2.6. *Suppose that $(K, (X_k)_{k\in K}, X)$ and $(L, (Y_l)_{l\in L}, Y)$ are function modules such that $Z(X) = \{M_h | h \in CK\}$ and $Z(Y) = \{M_h | h \in CL\}$. Then for every isometric isomorphism $I : X \to Y$ there are a homeomorphism $\tilde{t} : L \to K$ and a family of isometric isomorphisms $S_l : X_{\tilde{t}(l)} \to Y_l$ (all $l \in L$) such that $(Ix)(l) = S_l(x(\tilde{t}(l)))$ for $x \in X$ and $l \in L$. In particular, X and Y are isometrically isomorphic only if K and L are homeomorphic and the families $(X_k)_{k\in K}$ and $(Y_l)_{l\in L}$ contain the same spaces (modulo isometric isomorphism).*

Proof. Let $\tilde{R}$ be the identical representation of X in $\prod_{k\in K}^{\infty} X_k$ and R the representation $[I, (L, (Y_l)_{l\in L}, Y)]$ of X. Since the algebras $Z_\rho(X)$ are Z(X) for both representations it follows that $R \approx \tilde{R}$. With S as in Definition 2.2. we have $S\tilde{\rho} = I$ so that (since $\tilde{\rho} = Id)S = I$. The assertion follows with $t, (S_l)_{l\in L}$ as in Definition 2.2. and $\tilde{t} = t^{-1}$.

Lemma 2.7. *Let X and Y be Banach spaces, M and N locally compact Hausdorff spaces. Further suppose that $t : N \to M$ is a homeomorphism and that $u : N \to [X, Y]_{iso}$ is a continuous map ($[X, Y]_{iso}$ denotes the set of isometric isomorphisms from X to Y, provided with the strong operator topology). Then $I_{t,u} : C_0(M, X) \to C_0(N, Y)$ defined by $(I_{t,u}f)(w) = [u(w)]f(t(w))$ (all $f \in C_0(M, X), w \in N$) is an isometric isomorphism.*

Proof. For $f \in C_0(M, X), w \in N$ and $\epsilon > 0$ choose a neighborhood W of w_0 such that $\|f(t(w)) - f(t(w_0))\| \leq \epsilon$ and $\|\{u(w) - u(w_0)\}[f(t(w_0))]\| \leq \epsilon$ for $w \in W$. It follows that

$$\begin{aligned}&\|(I_{t,u}f)(w) - (I_{t,u}f)(w_0)\| \\ &= \|u(w)[f(t(w)) - f(t(w_0))] + [u(w) - u(w_0)][f(t(w_0))]\| \leq 2\epsilon\end{aligned}$$

for these w so that $I_{t,u}f$ is continuous at w_0. Since t^{-1} maps compact sets into compact sets and $\|u(w)\| \leq 1$ for every $w \in N, I_{t,u}f$, vanishes at infinity, i.e., $I_{t,u}$ is well-defined. It is clear that $I_{t,u}$ is linear and isometric and it remains to show that $I_{t,u}$ has an inverse. We note that $u^{-1} : N \to [Y,X]_{iso}, u^{-1}(w) = (u(w))^{-1}$ is continuous for $w_0 \in N, y_0 \in Y$ and for $\epsilon > 0$ choose $x_0 \in X$ such that $u(w_0)x_0 = y_0$ and a neighborhood W of w_0 such that $\|u(w)x_0 - u(w_0)x_0\| \leq \epsilon$ for $w \in W$ it follows that

$$\begin{aligned}&\|u^{-1}(w)y_0 - u^{-1}(w_0)y_0\| \\ &= \|u(w)(u^{-1}(w)x_0 - x_0)\| = \|u(w_0)x_0 - u(w)x_0\| \leq \epsilon\end{aligned}$$

for these w so that $w \mapsto u^{-1}(w)y_0$ is continuous. By the first part of the proof this implies that $I_{\tilde{t},\tilde{u}}g(\tilde{t} = t^{-1}, \tilde{u} = u^{-1} \circ t^{-1})$ is contained in $C_0(M,X)$ for every $g \in C_0(N,Y)$, $I_{\tilde{t},\tilde{u}}$ which is defined similarly to $I_{t,u}$. It is obvious that $I_{\tilde{t},\tilde{u}}$ is an inverse of $I_{t,u}$.

Definition 2.8. *A function module property is a rule P which assigns to every function module $(K,(X_k)_{k\in K},X)$ such that $Z(X) = \{M_h | h \in CK\}$ a subset $P(K,(X_k)_{k\in K},X)$ of $K^*(= \{k | k \in K, X_k \neq \{0\}\})$ such that the following holds : If $(K,(X_k)_{k\in K},X)$ and $(L,(Y_l)_{l\in L},L)$ are function modules such that $Z(X) = \{M_h | h \in CK\}$ and $Z(Y) = \{M_g | g \in CL\}$ and if $I : X \to Y$ is an isometric isomorphism, then $t(P(L,(Y_l)_{l\in L},Y)) = P(K,(X_k)_{k\in K},X)$ where t is the homeomorphism from L to K.*

Definition 2.9. *(1) Let $X_1, \cdots, X_n$ be Banach spaces and $\omega : \{1,\cdots,n\} \to \{1,\cdots,n\}$ a permutation. By $I_\omega : \prod_{i=1}^{n\infty} X_i \to \prod_{i=1}^{n\infty} X_{\omega(i)}$ we denote the isometric isomorphism*

$$(x_1,\cdots,x_n) \mapsto (x_{\omega(1)},\cdots,x_{\omega(n)})$$

(2) For Banach spaces $Y, Y_1, \cdots, Y_n$ and locally compact Hausdorff spaces M, there are natural isometric isomorphisms

$$C_0(M,Y^n) \cong C_0(nM,Y), \quad C_0(M,\prod_{i=1}^{n\infty} Y_i) \cong \prod_{i=1}^{n\infty} C_0(M,Y_i)$$

where nM is the disjoint union of n copies of M. Thus, for every M-finite Banach space $X \cong \prod_{i=1}^{r_\infty} \tilde{X}_i^{n_i}$ there is a natural isometric isomorphism (which will be denoted by $I_{M,X}$) from $C_0(M,X)$ onto $\prod_{i=1}^{r_\infty} C_0(n_iM, X_i)$.

Lemma 2.10. *Let $X_1, \cdots, X_r, Y_1, \cdots, Y_{\tilde{r}}$ be nonzero Banach spaces such that $Z(X_i) = Id, Z(Y_j) = Id$ for $i = 1, \cdots, r, j = 1, \cdots, \tilde{r}$ and $X_i \ncong X_{i'}$ if $i \neq i'$ and $Y_j \ncong Y_{j'}$, if $j \neq j'$. Further suppose that $M_1, \cdots, M_r$ and $N_1, \cdots, N_{\tilde{r}}$ are nonempty locally compact Hausdorff spaces and*

$$\hat{I} : \prod_{i=1}^{r_\infty} C_0(M_i, X_i) \to \prod_{j=1}^{\tilde{r}_\infty} C_0(N_j, Y_j)$$

is an isometric isomorphism. Then $r = \tilde{r}$ and there are a permutation $\omega : \{1, \cdots, r\} \to \{1, \cdots, r\}$ and homeomorphisms $t_i : N_{\omega(i)} \to M_i$ and continuous maps

$$u_i : N_{\omega(i)} \to [X_i, Y_{\omega(i)}]_{iso} \quad (i = 1, \cdots, r)$$

such that $\hat{I} = I_{\omega^{-1}} \circ (\prod_{i=1}^{r} I_{t_i,u_i})$ (I_{t_i,u_i} as in Lemma 2.7.):

$$\begin{array}{ccc} \prod_{i=1}^{r_\infty} C_0(M_i, X_i) & \xrightarrow{\hat{I}} & \prod_{j=1}^{\tilde{r}_\infty} C_0(N_j, Y_j) \\ \prod_{i=1}^{r} I_{t_i,u_i} \searrow & & \nearrow I_{\omega^{-1}} \\ & \prod_{i=1}^{r_\infty} C_0(N_{\omega(i)}, Y_{\omega(i)}) & \end{array}$$

Proof. We will prove that for every $i_0 \in \{1, \cdots, r\}$ there is an $j_0 \in \{1, \cdots, \tilde{r}\}$ such that $X_{i_0} \cong Y_{j_0}$. For simplicity we will regard the $C_0(M_i, X_i) = J_i$ (the $C_0(N_j, Y_j) = J_j^*$) as subspaces of

$$\prod_{i=1}^{r_\infty} C_0(M_i, X_i) \qquad (\text{of} \prod_{j=1}^{\tilde{r}_\infty} C_0(N_j, Y_j)).$$

Let $i_0 \in \{1, \cdots, n\}$ be arbitrary. Since images of M-summands under isometric isomorphisms are also M-summands, $\hat{I}(J_{i_0})$ must be an M-summand in $\prod_{j=1}^{\tilde{r}\infty} J_j^*$ and thus of the form $\prod_{j=1}^{\tilde{r}}(J_j^* \cap \hat{I}(J_{i_0}))$. Since $\hat{I}(J_{i_0})$ is nonzero, there must be a $j_0 \in \{1, \cdots, \tilde{r}\}$ such that $J^* = J_{j_0}^* \cap \hat{I}(J_{i_0})$ is a nonzero M-summand in $J_{j_0}^*$. By Lemma 2.4. and Lemma 2.5, there is a closed and open subset C^* of N_{j_0} such that

$$J^* = \{f | f \in C_0(N_{j_0}, Y_{j_0}), f|_{C^*} = 0\} \cong C_0(N_{j_0} \backslash C^*, Y_{j_0}).$$

Similarly we obtain a closed and open subset C of M_{i_0} such that

$$\hat{I}^{-1}(J^*) \cong C_0(M_i \backslash C, X_{i_0}).$$

Hence $C_0(M_i \backslash C, X_{i_0}) \cong C_0(N_{j_0} \backslash C^*, Y_{j_0})$ and since $M_{i_0} \backslash C \neq \emptyset \neq N_{j_0} \backslash C^*, X_{i_0} \cong Y_{j_0}$. Since the X_i and the Y_j are pairwise not isometrically isomorphic, the map $\omega : \{1, \cdots, r\} \to \{1, \cdots, \tilde{r}\}, \omega(i_0) = j_0$ is well-defined and bijective (so that, in particular, $r = \tilde{r}$). We have $\hat{I}(J_i) = J_{\omega(i)}^*$ (since $\hat{I}(J_i) \cap J_j^* = \{0\}$ for $j \neq \omega(i)$, i.e., there are homeomorphisms $t_1 : N_{\omega(i)} \to M_i$ and continuous maps $u_i : N_{\omega(i)} \to [X_i, Y_{\omega(i)}]_{iso}$ such that $\hat{I}|_{J_i} = I_{t_i,u_i}$. It is clear that $\hat{I} = I_{\omega^{-1}} \circ (\prod_{i=1}^{r} I_{t_i,u_i})$.

3. Main Results

Theorem 3.1. *Let X and Y be M-finite Banach spaces with canonical M - decompositions $X \cong \prod_{i=1}^{r\infty} \tilde{X}_i^{n_i}$ and $Y \cong \prod_{j=1}^{\tilde{r}\infty} \tilde{Y}_j^{m_j}$, respectively and M and N nonzero locally compact Hausdorff spaces. Then $I : C_0(M, X) \to C_0(N, Y)$ is an isometrical isomorphism if and only if $r = \tilde{r}$ and there are a permutation $\omega : \{1, \cdots, r\} \to \{1, \cdots, r\}$ and homeomorphisms $t_i : m_\omega(i)N -> n_iM$ and continuous maps $u_i : m_{\omega(i)}N \to [X_i, Y_{\omega(i)}]_{iso}(i = 1, \cdots, r)$ such that*

$$I = I_{N,Y}^{-1} \circ I_\omega^{-1} \circ (\prod_{i=1}^{r} I_{t_i,u_i}) \circ I_{M,X} (I_{t,u} \textit{as in Lemma 2.7.}):$$

$$\begin{array}{ccc} C_0(M,X) & \xrightarrow{I} & C_0(N,Y) \\ I_{M,X}\downarrow & & \uparrow I_{N,Y}^{-1} \\ \prod_{i=1}^{r\infty} C_0(n_iM, X_i) & & \prod_{j=1}^{\tilde{r}\infty} C_0(m_iN, Y_j) \\ \prod_{i=1}^{r} I_{t_i,u_i} \searrow & & \nearrow I_{\omega^{-1}} \\ & \prod_{i=1}^{r\infty} C_0(m_{\omega(i)}N, Y_{\omega(i)}) & \end{array}$$

Proof. ($\Longrightarrow$) By Lemma 2.10, we have

$$\hat{I} = I_{N,Y} \circ I \circ I_{M,X}^{-1} : \prod_{i=1}^{r\infty} C_0(n_iM, \tilde{X}_i) \to \prod_{j=1}^{\tilde{r}\infty} C_0(m_jN, \tilde{Y}_i) :$$

$$\begin{array}{ccc} \prod_{i=1}^{r\infty} C_0(M_i, X_i) & \xrightarrow{\hat{I}} & \prod_{j=1}^{\tilde{r}\infty} C_0(N_j, Y_j) \\ I_{M,X}^{-1} \downarrow\uparrow I_{M,X} & & I_{N,Y}^{-1} \downarrow\uparrow I_{N,Y} \\ C_0(M,X) & \xrightarrow{I} & C_0(N,Y) \end{array}$$

From $\hat{I} = I_{\omega^{-1}} \circ \prod_{i=1}^{r} I_{t_i,u_i}$ and the above diagram , we have

$$I_{\omega^{-1}} \circ \prod_{i=1}^{r} I_{t_i,u_i} = I_{N,Y} \circ I \circ I_{M,X}^{-1}$$
$$I_{N,Y}^{-1} \circ I_{\omega^{-1}} \circ (\prod I_{t_i,u_i}) = I \circ I_{M,X}^{-1}$$
$$I_{N,Y}^{-1} \circ I_{\omega^{-1}} \circ (\prod I_{t_i,u_i}) \circ I_{M,X} = I$$

By Lemma 2.7, $I = I_{N,Y}^{-1} \circ I_{\omega^{-1}} \circ (\prod_{i=1}^{r} I_{t_i,u_i}) \circ I_{M,X}$ is an isometric isomorphism.
($\Longleftarrow$) It is clear from Lemma 2.10.

Theorem 3.2. *Let X and Y be M-finite Banach spaces with canonical M - decompositions $X \cong \prod_{i=1}^{r\infty} \tilde{X}_i^{n_i}$ and $Y \cong \prod_{j=1}^{\tilde{r}\infty} \tilde{Y}_j^{m_j}$, respectively, and M and N nonzero locally compact Hausdorff spaces. Then $C_0(M, X) \cong C_0(N, Y)$ if and only if $r = \tilde{r}$, and there is a permutation $\omega : \{1, \cdots, r\} \to \{1, \cdots, r\}$ such that $n_i M \cong m_{\omega(i)} N$ and $\tilde{X}_i \cong \tilde{Y}_{\omega(i)}$ for every $i \in \{1, \cdots, r\}$.*

Proof. ($\Longrightarrow$) This is a consequence of Theorem 3.1.

($\Longleftarrow$) Since X_i and Y_j are not pairwise isometrically isomorphic, the map $\omega : \{1, \cdots, r\} \to \{1, \cdots, \tilde{r}\}, \omega(i_0) = j_0$ is well-defined and bijective, therefore $r = \tilde{r}$. Hence, there are homeomorphism $t_i : m_{\omega(t)} N \to n_i M$ and continuous maps $u_i : m_{\omega(i)} N \to [X_i, Y_{\omega(i)}]_{iso}$ such that $I = I_{N,Y}^{-1} \circ I_\omega^{-1} \circ (\prod_{i=1}^{r} I_{t_i, u_i}) \circ I_{M,X}$. And so $X_i \cong Y_{\omega(t)}$. Thus $\tilde{X}_i \cong \tilde{Y}_{\omega(t)}$ for every $i \in \{1, \cdots, r\}$.

REFERENCES

1. E.M.Alfsen and E.G.Effros, *Structure in real Banach spaces I*, Ann. of Math. **96** (1972).
2. E.M.Alfsen and E.G.Effros, *Structure in real Banach spaces II*, Ann. of Math. **96** (1972).
3. B.Beauzamy, *Introduction to Banach spaces and their Geometry*, North-Holland publishing company-Amsterdam, New York, Oxford. (1982).
4. E.Behrends, *On the Banach-Stone theorem*, Math. Annalen **233** (1978).
5. M.Cambern, *A generalized Banach-Stone theorem*, Proc. of the Amer. Math. Soc. **17** (1966).
6. H.E.Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, Berlin (1974).
7. J.Lindenstrauss and L.Tzafriri, *Classical Banach spaces I, Sequnence space*, Springer-Verlag (1977).
8. J.Lindenstrauss and L.Tzafriri, *Classical Banach spaces II, Function space*, Springer-Verlag (1979).

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