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On Progressing Waves over Sloping Beaches
경사 해안 위를 진행하는 파의 거동

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Abstract□The solution for progressing waves over beaches is obtained for a sloping angle π/2n with n, an integer, based on the solution of Lewy and Stoker for standing waves. The behaviors of the waves are graphically illustrated.

要　旨：Lewy와 Stoker의 정상파 해에 근거하여 경사각이 π/2n인 경사 해안 위를 진행하는 파의 거동 식을 구하였다. 또 이 해에 의거한 파의 거동을 도형으로 표시하였다.

1. INTRODUCTION

Because progressing waves over uniformly sloping beaches exhibit interesting behaviors, the present problem has attracted many prominent mathematicians and scientists around the middle of the twentieth century. Their solutions, mainly dealing with standing waves over sloping beaches, have been successfully obtained in principle, but the solutions do not offer any detailed knowledge of characteristic behaviors of progressing waves over sloping beaches regarding the variations of the amplitude, wavelength and phase speed of the waves. Recently, Chung *et al.* (1995) have worked out in detail the solution for variations of the wave amplitude, wavelength and phase speed through complicated procedures such as analytic continuation for arbitrary slope angles of the beaches. The solution of Lewy (1946) and Stoker (1947, 1957) is confined to the problem for special slope angle of the beach π/2n with n an integer. But their solution is in much simpler form than those for arbitrary angles of the beach. Hence we make use of the solution of Lewy and Stoker and show how to derive characteristic behavior of the progressing waves over uniformly sloping beaches.

2. FORMULATION

A system of plane progressing waves is moving toward the shoreline over a uniformly sloping beach and the slope angle of the beach is π/2n where n is an integer. The wave potential $\Phi(x, y: t)$ satisfies

$$\nabla^2\Phi = 0 \qquad \text{for } x>0 \text{ and } -x\tan\omega \le y \le 0 \qquad (1)$$

$$\Phi_{tt} + g\,\Phi_y = 0 \qquad \text{along } y=0 \qquad (2)$$

$$\Phi_n = 0 \qquad \text{along bottom } y = -x\tan\omega \qquad (3)$$

$$\Phi = \mathrm{Re}A_\infty e^{ky}\, e^{i(kx+\sigma t)} \quad as\ x \to +\infty \qquad (4)$$

where $k=\sigma^2/g$ and A_∞ is real.
If the solution $\Phi(x, y: t)$ of (1) through (4) is found, the surface displacement $\eta(x; t)$ is given by

$$\eta(x;t) = -\frac{1}{g}\,\Phi_t(x, 0; t) \qquad (5)$$

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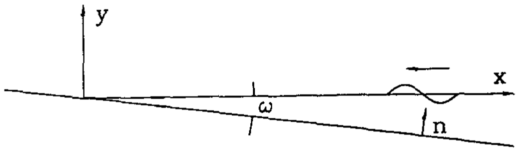

Fig. 1. Schematic diagram.

we write $\Phi\ (x,\ y;\ t)$in the form of

$$\Phi(x,y;t) = \phi_1(x,y)\cos\sigma t - \phi_2(x,y)\sin\sigma t \tag{6}$$

where $\phi_1(x,y)$ and $\phi_1(x,y)$ are two standing wave potentials. The following dimensionless variables are introduced for convenience:

$$(x, y') = k(x,y),\ t' = \sigma t \tag{7}$$

Substituting (6) and (7) in the full problem, we get

$$\nabla^2\phi_j = 0 \quad \text{for } x>0 \text{ and } -x\tan\omega \le y \le 0 \tag{8}$$

$$\phi_{jy} - \phi_j = 0 \quad \text{along } y=0 \tag{9}$$

$$\phi_{jn} = 0 \quad \text{along } y = -x\tan\omega \tag{10}$$

$$\phi_1 = A_\infty e^y\cos x,\ \phi_2 = A_\infty e^y\sin x \quad as\ \ x \longrightarrow +\infty \tag{11}$$

for j=1, 2 after dropping the primes. The present problem is to find ϕ_1 and ϕ_2 for (8) through (11).

3. SOLUTION

We introduce the following linear homogeneous problem for standing waves:

$$\nabla^2\psi_j = 0 \quad \text{for } x>0 \text{ and } -x\tan\omega \le y \le 0 \tag{12}$$

$$\psi_{jy} - \psi_j = 0 \quad \text{along } y=0 \tag{13}$$

$$\psi_{jn} = 0 \quad \text{along } y = 0 - x\tan\omega \tag{14}$$

for j=1, 2 where ψ_1 and ψ_2 are 90 degrees out of phase at $x=+\infty$. We seek to construct the solutions for ϕ_1 and ϕ_2 with ψ_1 and ψ_2. The solutions of ψ_1 and ψ_2 are given by Lewy and Stoker (1957) as

$$\psi_1(x,y) = \mathrm{Re}\frac{\pi}{(n-1)\sqrt{n}}\sum_{j=1}^{n} C_j e^{z\beta_j},\quad (z = x+iy) \tag{15}$$

$$\psi_2(x,y) = \mathrm{Re}\sum_{j=1}^{n} a_j\left[e^{z\beta_j}\int_{i\infty}^{iz\beta_j}\frac{e^{it}}{t}\,dt - i\pi e^{z\beta_j}\right] \tag{16}$$

where

$$a_j = \frac{C_i}{(n-1)!\sqrt{n}}$$

$$\beta_j = e^{i\pi(\frac{j}{n}+\frac{1}{2})}$$

$$C_j = e^{i\pi(\frac{n+1}{4}-\frac{1}{2}j)}\cot\frac{\pi}{2n}\cot\frac{2\pi}{2n}\cdots\cot\frac{(\frac{1}{2}j-1)\pi}{2n}$$

$$C_1 = \bar{C}_n$$

Equation (16) has a logarithmic singularity at $x=0$. Further we have the following relations:

$$\int_{i\infty}^{iz\beta_j}\frac{e^{it}}{t}dt = O(z^{-1}) \text{ for } \mathrm{Re}\, iz\beta_j < 0 \text{ and } \mathrm{Im}\, iz\beta_j \le 0$$
$$= 2\pi i - O(z^{-1}) \text{ for } \mathrm{Re}\, iz\beta_j > 0 \text{ and } \mathrm{Im}\, iz\beta_j \le 0 \tag{17}$$

for large $|z|$. Hence (16) is further written as

$$\psi_2(x,y) = \mathrm{Re}\left[-i\pi\sum_{j=1}^{n} a_j e^{z\beta_j}\right] + O(x^{-1})$$
$$\text{for } iz\beta_i < 0 \text{ and } \mathrm{Im}\, iz\beta_j \le 0$$
$$= \mathrm{Re}\left[i\pi\sum_{j=1}^{n} a_j e^{z\beta_j}\right] + O(x^{-1})$$
$$\text{for } iz\beta_j > 0 \text{ and } \mathrm{Im}\, iz\beta_j \le 0$$
$$\text{for large } x \tag{18}$$

The constant C_n is written in form of $C_n = |C_n|e^{i\alpha}$. Then $C_1 = \bar{C}_n = |C_n|e^{-i\alpha}$. It follows from (15), (17) and (18) that

$$\psi_1(x,y) = \frac{|C_n|\pi}{(n-1)\sqrt{n}}\Big[e^y\cos(x-\alpha) +$$
$$+ \cot\frac{\pi}{2n}e^{-(x\sin 2\omega - y\cos 2\omega)}\cos(x\cos 2\omega$$
$$+ y\sin 2\omega - \frac{\pi}{2} - \alpha)$$
$$+ \cot\frac{\pi}{2n}\cot\frac{2\pi}{2n}e^{-(x\sin 4\omega - y\cos 4\omega)}\cos(x\cos 4\omega$$
$$+ y\sin 4\omega - \pi - \alpha) + \cdots$$
$$+ e^{-(x\sin 2\omega + x\cos 2\omega)}\cos(x\cos 2\omega - y\sin 2\omega - \alpha)$$
$$+ \cot\frac{\pi}{2n}e^{-(x\sin 4\omega + y\cos 4\omega)}\cos(x\cos 4\omega$$
$$- y\sin 4\omega - \frac{\pi}{2} - \alpha) + \cdots\Big] + O(x^{-1}) \quad \text{for large } x \tag{19}$$

$$\psi_2(x, y) = \frac{|C_n|\pi}{(n-1)\sqrt{n}}\Big[e^y \sin(x-\alpha) + \cot\frac{\pi}{2n}e^{-(x\sin 2\omega - y\cos 2\omega)}\sin(x\cos 2\omega + y\sin 2\omega - \frac{\pi}{2} - \alpha) + \cot\frac{\pi}{2n}\cot\frac{2\pi}{2n}e^{-(x\sin 4\omega + y\cos 4\omega)}\sin(x\cos 4\omega + y\sin 4\omega - \pi - \alpha) + \cdots + e^{-(x\sin 2\omega + x\cos 2\omega)}\sin(x\cos 2\omega - y\sin 2\omega - \alpha) + \cot\frac{\pi}{2n}e^{-(x\sin 4\omega + x\cos 4\omega)}\cos(x\cos 4\omega - y\sin 4\omega - \frac{\pi}{2} - \alpha) + \cdots\Big] + O(x^{-1}) \quad \text{for large } x \tag{20}$$

Since $\phi_1(x, y)$ and $\phi_2(x, y)$ are the solution for a linear homogeneous problem, any linear combination of ϕ_1 and ϕ_2 is also a solution from the linearity principle. By proper combinations of ϕ_1 and ϕ_2, we obtain

$$\phi_1 = A_\infty\Big[e^y\cos x + \cot\frac{\pi}{2n}e^{-(x\sin 2\omega - y\cos 2\omega)}\cos(x\cos 2\omega + y\sin 2\omega - \frac{\pi}{2}) + \cot\frac{\pi}{2n}\cot\frac{2\pi}{2n}e^{-(x\sin 4\omega - y\cos 4\omega)}\sin(x\cos 4\omega + y\sin 4\omega - \pi) + \cdots + e^{-(x\sin 2\omega + x\cos 2\omega)}\cos(x\cos 2\omega - y\sin 2\omega) + \cot\frac{\pi}{2n}e^{-(x\sin 4\omega + y\cos 4\omega)}\cos(x\cos 4\omega - y\sin 4\omega - \frac{\pi}{2}) + \cdots\Big] + O(x^{-1}) \tag{21}$$

$$\phi_2 = A_\infty\Big[e^y\sin x + \cot\frac{\pi}{2n}e^{-(x\sin 2\omega - y\cos 2\omega)}\sin(x\cos 2\omega + y\sin 2\omega - \frac{\pi}{2}) + \cot\frac{\pi}{2n}\cot\frac{2\pi}{2n}e^{-(x\sin 4\omega - y\cos 4\omega)}\sin(x\cos 4\omega + y\sin 4\omega - \pi) + \cdots + e^{-(x\sin 2\omega + y\cos 2\omega)}\sin(x\cos 2\omega - y\sin 2\omega) + \cot\frac{\pi}{2n}e^{-(x\sin 4\omega + y\cos 4\omega)}\cos(x\cos 4\omega - y\sin 4\omega - \frac{\pi}{2}) + \Big] + O(x^{-1}) \tag{22}$$

Equations (21) and (22) agree with (27) and (28) in Chung *et al.* (1995). We combine the right sides of (21) and (22) into cosine and sine waves, respectively, as Chung and Lim (1991). Then we substitute those single waves in (6) and return to the original variables to get

$$\Phi(x, 0; t) = A_0(x)\cos\left[kx + \beta(x) + \sigma t\right] + O(x^{-1}) \tag{23}$$

$$\eta(x; t) = \eta_0(x)\sin\left[kx + \beta(x) + \sigma t\right] + O(x^{-1}) \tag{24}$$

The unknown A_0, η_0 and $\beta(x)$ become known if the slope angle ω is given. The local wavelength $\lambda(x)$ and phase speed $c(x)$ of progressing waves over a uniformly sloping beach are given by the following relations according to Chung *et al.* (1995).

$$\frac{c(x)}{c(\infty)} = \frac{\lambda(x)}{\lambda(\infty)} = \frac{1}{\frac{d}{dx}[x + \beta(x)/k]} \tag{25}$$

For example, if $\omega = \pi/6$, then

$$\begin{aligned} A_0(x) &= A_\infty A_1(x) \\ \eta_0(x) &= \eta(\infty)A_1(x) \\ \beta(x) &= -\tan^{-1}\left[\frac{1}{2}\frac{\beta_1(x)\sin\omega_1}{1-\beta_1\sin^2(\omega_1/2)}\right] \\ A_1(x) &= \sqrt{[1+\tau(x)]^2 + 4\tau(x)\sin^2(\omega_1/2)} \\ \tau(x) &= \csc\omega e^{-kx\sin 2\omega} - \omega + \pi/2 \\ \beta_1(x) &= 2\tau(x)/[1+\tau(x)] \end{aligned} \tag{26}$$

The results for $\omega = \pi/6$ are plotted in Figs 2 and 3. Similarly, the results for the case of $\omega = \pi/12$ are computed and shown.

4. DISCUSSION

As seen from Figs. 2 through 5, the amplitude, wavelength and phase speed of the progressing waves first decrease and then increase near the shoreline. The

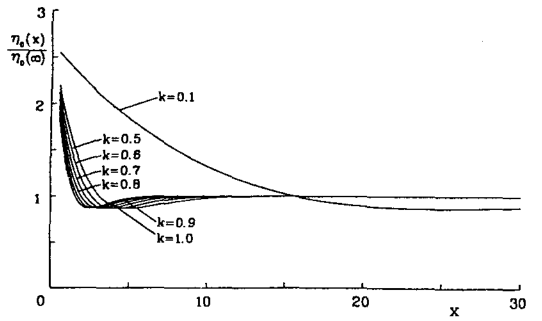

Fig. 2. Variation of wave amplitude for a sloping angle of 30 degrees.

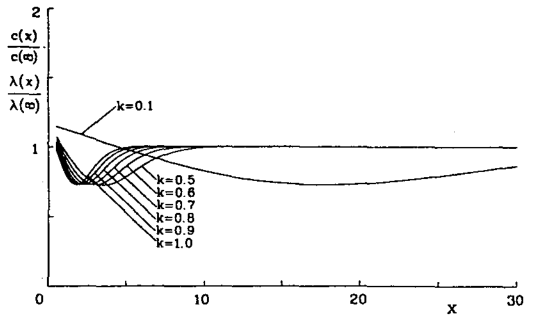

Fig. 3. Variation of phase speed or wavelength for a sloping angle of 30 degrees.

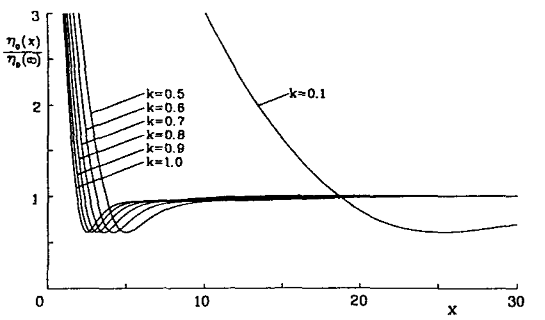

Fig. 4. Variation of wave amplitude for a sloping angle of 15 degrees.

energy of the waves converges to a point as they approach the shoreline. That is why the solution has a logarithmic singularity along the shoreline at x=0, and the wave amplitude becomes infinite there.

Since $\lambda = 2\pi/k$, the wavelength is larger if k is smaller. If the wavelength is larger, the waves feel the bottom earlier. Therefore, the amplitude decreases earlier if k is smaller as seen in Figs. 2 and 3. The wavelength and phase speed behave similarly.

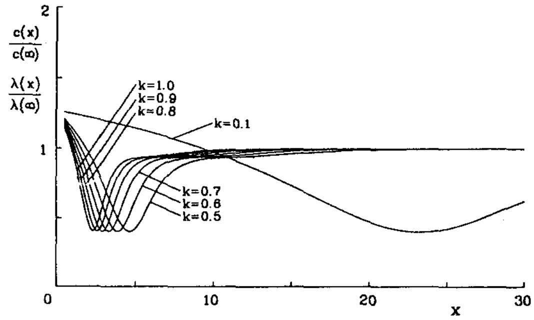

Fig. 5. Variation of phase speed or wavelength for a sloping angle of 15 degress.

One often observes from the shore that approaching waves on a beach whose slope angle is not so mild disappear and are lost halfway to the shoreline. Then sudden big waves surge violently and break near the shoreline. Those peculiar behaviors of progressing waves over a uniformly sloping beach are well illustrated by the variations of the wave amplitude, wavelength and phase speed on a sloping beach.

5. CONCLUDING REMARKS

The present solution is based on the ingenious solution of Lewy and Stoker for slope angle $\omega=2\pi/n$. The present study shows in detail how the solution can be used to see the behaviors of progressing waves over a sloping beach even if the angle is restricted.

The behaviors of the waves over a sloping beach is still a mystery because it is not clear why the amplitude of the waves first decreases as their energy converges.

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APPENDIX. DERIVATION OF ϕ_1 and ϕ_2

We seek the solutions ϕ_1 and ϕ_2 to the present problem via Lewy and Stoker solutions. Since ψ_1 and ψ_2 in (19) and (20) are the solutions of linear homogeneous problem, ϕ_1 and ϕ_2 can be obtained readily from the linearity principle. For simplicity, we consider the problem near $x=+\infty$. In the case ψ_1 and ψ_2 reduce to

$$\psi_1(x,y)=\frac{|C_n|\pi e^y}{(n-1)\sqrt{\pi}}\cos(x-\alpha) \quad \text{(A1)}$$

$$\psi_2(x,y)=\frac{|C_n|\pi e^y}{(n-1)\sqrt{\pi}}\sin(x-\alpha) \quad \text{(A2)}$$

near $x=+\infty$. Because of α and $|C_n|$ in (A1) and (A2), ψ_1 and ψ_2 are different from ϕ_1 and ϕ_2 in (11). In order to delete α and $|C_n|$ from (A1) and (A2), we first expand right sides of (A1) and (A2). Then

$$\psi_1(x,y)=\frac{|C_n|\pi e^y}{(n-1)\sqrt{\pi}}(\cos x\cos\alpha+\sin x\sin\alpha) \quad \text{(A3)}$$

$$\psi_2(x,y)=\frac{|C_n|\pi e^y}{(n-1)\sqrt{\pi}}(\sin x\cos\alpha-\cos x\sin\alpha) \quad \text{(A4)}$$

From the linearity principle, we obtain ϕ_1 and ϕ_2 from the following linear combinations

$$\phi_1=C_1\cos\alpha\,\psi_1-C_1\sin\alpha\,\psi_2=\frac{C_1|C_n|\pi e^y}{(n-1)\sqrt{n}}\cos x \quad \text{(A5)}$$

$$\phi_2=C_1\sin\alpha\,\psi_1+C_1\cos\alpha\,\psi_2=\frac{C_1|C_n|\pi e^y}{(n-1)\sqrt{n}}\sin x \quad \text{(A6)}$$

for some constant C_1. The wave amplitude A_∞ *at* $x=+\infty$ is now given by

$$A_\infty=\frac{C_1|C_n|\pi}{(n-1)\sqrt{n}} \quad \text{(A7)}$$

Therefore, we simply obtained the present solution from the Lewy and Stoker solution. Similarly, we obtain (20) and (21) from (19) and (20).