A Single Server Queue Operating under N-Policy with a Renewal Break down Process

Chang-Ouk Kim*
Kyung-Sik Kang**

Abstract

본 연구는 써버의 고장을 허용하는 단수써버 Queueing 시스템의 확률적 모델을 제시한 것으로, 써버는 N 제어 정책에 의하여 작동되며, 도착은 Stationary compound poisson에 의하여 이루어지고, 서비스 시간에 대한 분포는 Erlang에 의하여 발생하며, 수리시간에 대한 분포는 평균이 일정한 분포에 의하여 생성되는 경우를 고려 하였다. 또한 고장간격 시간은 일정한 평균을 가진 임의의 분포를 가진 Renewal process에 의한다고 가정하였고, 완료 시간의 개념은 재생과정의 적용방법에 의하여 유도할 수 있으며, 시스템 크기의 확률 생성 함수의 값이 구해진 다는 것을 제시 하였다.

1. Introduction

In many realistic queueing situations, customers arrive in groups rather than individually and a server is subject to breakdown. If the breakdown is unpredictable in nature and the server cannot operate immediately after a breakdown occurs, the customers waiting in front of the server cannot be served during the breakdown period and the following repair period. in such a case, it is important to understand how the breakdown will affect the performance of the system. This phenomenon can be observed, for example, in fault-tolerant computer system[10] and in database management system[9][15]. Also, an important model is a production-inventory system[14] where the server is a facility that produces items and the requests for demands are arriving customers.

* School of Industrial Engineering, Purdue University, West Lafayette, IN 47907, U.S.A., Tel: 317-746-2308, Fax: 317-494-1299, E-mail: kimco@gilbreth.ecn.purdue.edu
** Department of Industrial Engieering, Myong Ji university, San 38-2, Nam-Dong, Yong in, Kyunggi-do,449-728, KOREA.

Server breakdown models has been investigated by many authors with different assumptions. White and Christie first studied this model, who considered the case with exponential sevice, repair, and inter-breakdown time distributions. Their results were extended by Gaver[4], Keilson[8] and Jaiswal[7] to models with general service and repair time distrivutions but exponential inter-breakdown time distribution. For the variants of these model, Shogan[14] deals with a $M^X/E^k/1$ queue where the server is available or not, and Sumita et al.[15] also worked on the case of breakdowns occurred according to nonhomogeneous poisson process. Ibe and Trivedi[6] considered a queueing model of unreliable polling system and obtained an approximate mean delay of customers in the system. An important assumption of the articles mentioned above is that inter-breakdown time is exponentially distributed. Recently, Federgruen and Green[3] derived a bound and approximations for the system size and mean waiting tim for an M/G/1 queue whose unreliable server alternates between on(operational) periods and off(failed) periods of arbitrary random duration. Sengupta[13] generalized their result so that the arrival rate and service time could be dependent on on-off periods. But his result is also approximation.

In this paper, we consider a single server, first come, first served unreliable queueing system operating under N control policy. We extend the previous result of Gaver[4]. Our model allows the presence of general inter-breakdown time distribution. This model generally provides satisfactory model to many realistic situation such as inventory systems, manufacturing cells, and computer systems.

The rest of the paper is organized as follows. In Section 2, we describe the model considered in this paper in detail, and the Laplace-Stieltjes transform(LST) of completion time distribution is derived. The completion time, first defined by Gaver[4], is the period that elapsed between the instant at which the service of n-th arriving customer begins and at which service of the next(n+1)-th arriving customer may begins(does begin, provided an arrival exists). This period is simply the n-th arrival customer's service time if there are no server breakdowns. In Section 3, a recursive calculation method for the steady state system size is proposed using the method of regenerative process[7], and then the probability generating function(p.g.f.) of the system size is derived.

2. Completion Time

In this section, we present the Laplace-Stieltjes transform(LST) of completion time distribution. As mentioned previously, the completion time for n-th arriving customer is the time interval elapsed from the customer start service until (n+1)-th arriving customer may enter(does enter, if it has arrived) service. Before the analysis, we first describe the queueing system considered in this paper in detail. The queueing system has the following characteristics:

(a) When the server is operational, the system behaves as an $M^X/E^k/1$ queue operating under N control policy; that is, customers arrive according to a compound poisson process with rate λ, service process follows an Erlang distribution with mean $1/\mu$ and shape parameter k. the server begins service only when the system size(the number of customers begins in the system) builds up to or exceed a pre-assigned number, say N.

(b) When the server is breakdown, repair is started and service resumption takes place as soon s the repair period ends with no loss of service involved(preemptive-resume discipline).

(c) No breakdown occurs when the server is idle. If the server is operational, the breakdown occurs according to a renewal process with the inter-breakdown time of independent and identically but otherwise arbitrary distribution.

(d) The duration of a repair is arbitrary distributed with a finite mean.

For further analysis, some notations are introduced.

$\{A_n\}_{n=1}^{\infty} =$ sequence of inter-arrival time random variables

$\{S_n\}_{n=1}^{\infty} =$ sequence of service time random variables

$\{F_n\}_{n=1}^{\infty} =$ sequence of inter-breakdown time random variables

$\{R_n\}_{n=1}^{\infty} =$ sequence of repair time random variables

$\{C_n\}_{n=1}^{\infty} =$ sequence of completion time random variables

$M(t) =$ equilibrium renewal process associated with $\{F_n\}_{n=1}^{\infty}$

We define that, for any random variable B, $B(x)$, and b^n are the cumulative distribution function(c.d.f.) and n-th moment, respecitively. Also, let $\tilde{V}(\alpha)$ be the Laplace-Stieltjes transform(LST) of a continuous random variable V, and $G_D(z)$ be probability generating function of a discrete random variable D.

If we assume that the system reaches steady state, we can observe that, from the view point of n-th arriving customer, equilibrium renewal process

$M(S_n)$ represents the number of breakdowns experienced by n th arriving customer whose service time equals to S_n. Thus, for preemptive-resume discipline, the completion time for n-th arriving customer is the service time plus sum of the repair times of the breakdowns occurred during this service time. Therefore, the completion time for n-th arriving customer is given by

$$C_n = S_n + \sum_{i=0}^{M(S_n)} R_i$$

The tacit assumption that S_n, R_n and F_n are sequence of i.i.d. random variables implies that C_n is also a sequence of i.i.d. random variables. In particular, C_n approaches C for $n \geq 1$ in steady state. Hence, the LST of C conditioning on $M(S)$ is

$$\begin{aligned} E[\exp(-\alpha C) \mid S=x, M(S)=n] &= \exp(-\alpha x)\cdot E[\exp(-\alpha \sum_{i=0}^{n} R_i)] \\ &= \exp(-\alpha x)\cdot [\tilde{R}(\alpha)]^n \end{aligned}$$

because of independence of R_i. Unconditioning with respect to $M(S)$, we get

$$\begin{aligned} E[\exp(-\alpha C) \mid S=x] &= \sum_{n=0}^{\infty} \exp(-\alpha x)\cdot [\tilde{R}(\alpha)]^n \Pr\{M(x)=n\} \\ &= \exp(-\alpha C) G_{M(x)}[\tilde{R}(\alpha)] \end{aligned}$$

Since the service time is Erlang distributed with mean $1/\mu$ and shape parameter k, E[exp($-\alpha C$)] can be obtained by unconditioning on service time S as follows.

$$\begin{aligned} E[\exp(-\alpha C)] &= \int_0^{\infty} E[\exp(-\alpha x) \mid S=x] dS(X) \\ &= \int_0^{\infty} \exp(-\alpha x) G_{M(x)}[\tilde{R}(\alpha)] \frac{\mu^k x^{k-1} exp(-\mu x)}{(k-1)!} dx \\ &= \int_0^{\infty} G_{M(x)}[\tilde{R}(\alpha)] \frac{\mu^k x^{k-1}}{(k-1)!} \exp\{-(\mu+\alpha)x\} dx \qquad (2.1) \end{aligned}$$

To obtain closed form of (2.1), define another transform[2] by

$$\begin{aligned}\psi[\tilde{R}(\alpha),\zeta] &= \int_0^\infty E[\exp(-\alpha C) \mid S=x]\exp(-\zeta x)dx \\ &= \int_0^\infty \exp\{-(\alpha+\zeta)x\} G_{M(x)}[\tilde{R}(\alpha)]dx \end{aligned} \tag{2.2}$$

Differentiating (2.2) k-1 times with respect to ζ, we have

$$\frac{\partial^{k-1}\psi[\tilde{R}(\alpha),\zeta]}{\partial\zeta^{(k-1)}} = (-1)^{k-1}\int_0^\infty \exp\{-(\alpha+\zeta)x\}x^{k-1}G_{M(x)}[\tilde{R}(\alpha)]dx \tag{2.3}$$

Using (2.1) and (2.3), we get

$$\begin{aligned}E[\exp(-\alpha C)] &= \int_0^\infty G_{M(x)}[\tilde{R}(\alpha)]\frac{\mu^k x^{k-1}}{(k-1)!}\exp\{-(\mu+\alpha)x\}dx \\ &= \frac{(-1)^{k-1}\mu^k}{(k-1)!}\cdot \left| \frac{\partial^{k-1}\psi[\tilde{R}(\alpha),\zeta]}{\partial\zeta^{(k-1)}} \right|_{\zeta=\mu} \end{aligned} \tag{2.4}$$

Furthermore, $G_{M(x)}[\tilde{R}(\alpha)]$ can be rewritten as

$$\begin{aligned}G_{M(x)}[\tilde{R}(\alpha)] &= \sum_{n=0}^\infty [\tilde{R}(\alpha)\Pr\{M(x)=n\} \\ &= \sum_{n=0}^\infty [\hat{R}(\alpha)]^n\{F_n^*(x)-F_{n+1}^*(x)\} \\ &= 1+\sum_{n=0}^\infty [\tilde{R}(\alpha)]^{n-1}[\tilde{R}(\alpha)-1]F_n^*(x) \end{aligned} \tag{2.5}$$

where $F^*_n(t)$ denoted the n-th convolution of $F(t)$.
By substituting (2.5) into (2.2), we get $\psi[\tilde{R}(\alpha),\zeta]$ as follows.

$$\begin{aligned}\psi[\tilde{R}(\alpha),\zeta] &= \int_0^\infty [\exp-(\alpha+\zeta)x]\cdot(1+\sum_{n=1}^\infty[\tilde{R}(\alpha)]^{n-1}[\tilde{R}(\alpha)-1]F^*(x))dx \\ &= \frac{1}{\alpha+\zeta}+\frac{1}{\alpha+\zeta}\sum_{n=1}^\infty[\tilde{R}(\alpha)]^{n-1}[\tilde{R}(\alpha)-1]\ \tilde{F}_n(\alpha+\zeta) \end{aligned} \tag{2.6}$$

where $\tilde{F}_n(\zeta)=\int_0^{\infty}\exp(-\zeta x)d\tilde{F}_n^*(x)$

Now, we prove the following theorem.

Theorem 2.1] If service process is Erlang- k distribution and inter-breakdown time is arbitrary distributed with probability function F(·), then the Laplace-Stieltjes transform of the completion time distribution satisfies

$$E[\exp(-aC]=\frac{\mu^k}{(k-1)!}\cdot\left[\frac{\partial^{k-1}}{\partial\zeta^{k-1}}\left(\frac{1}{a+\zeta}+\frac{[1-\tilde{F}(\zeta)][\tilde{R}(a)-1]}{\bar{f}\zeta^2[1-\tilde{R}(a)\tilde{F}(\zeta)]}\right)\right]_{\zeta=u} \tag{2.7}$$

Proof) In an equilibrium renewal process[2],

$$\tilde{F}_n(\zeta)=\frac{1-\tilde{F}(\zeta)}{\bar{f}\zeta}[\tilde{F}(\zeta)]^{n-1}$$

Therefore, from (2.6), we get

$$\begin{aligned}\psi[\tilde{R}(a),\zeta] &=\frac{1}{a+\zeta}+\frac{1}{a+\zeta}\sum_{n=1}^{\infty}[\tilde{R}(a)]^{n-1}[\tilde{R}(a)-1]\frac{1-\tilde{F}(\zeta)}{\bar{f}\zeta}[\tilde{F}(\zeta)]^{n-1}\\ &=\frac{1}{a+\zeta}+\frac{[1-\tilde{F}(\zeta)][\tilde{R}(a)-1]}{\bar{f}\zeta^2[1-\tilde{R}(a)\tilde{F}(\zeta)]}\end{aligned} \tag{2.8}$$

By substituting (2.8) into (2.4), we complete the proof.

Note that the moments of the completion time can be obtained by differentiating (2.7) with respect to a at $a=0$.

Corollary 2.1] The first and second moments of the completion time are given by

$$E[C]=\frac{(-1)^{k-1}\mu^k}{(k-1)!}\left[\frac{\partial^{k-1}}{\partial\zeta^{k-1}}\left(\frac{1}{\zeta^2}+\frac{\bar{r}}{\bar{f}\zeta^2}\right)\right]_{\zeta=u}$$

$$E[C^2]=\frac{(-1)^{k-1}\mu^k}{(k-1)!}\left[\frac{\partial^{k-1}}{\partial\zeta^{k-1}}\left(\frac{2}{\zeta^3}+\frac{2\bar{r}^2}{\bar{f}\zeta^2[1-\tilde{F}(\zeta)]}+\frac{\bar{r}^2}{\bar{f}\zeta^2}\right)\right]_{\zeta=u}$$

3. System Size Distribution

In this section, we first derive the means of idle and busy period, and then obtain the probability generating function of the system size distribution at arbitrary time point by using the method of regenerative process and level crossing argument proposed by Van Hoorn[18] and Tijms[17].

The batch size random variable denoted by X is assumed to be independent with a common density function

$$q_i = \Pr\{X=i\} \qquad I=1,2,\ldots\ldots\ldots$$

Let $X_1, X_2, X_3, \cdot\cdot\cdot\cdot$ denote the successive batch size of customers and let

$$\beta_N = \min\{\ k : X_1+X_2+\cdot\cdot\cdot\cdot\cdot+X_k \geq N\ \}$$

then β_N is the number of batches that causes the system size to exceed the pre-assigned number N.

Now, if we define $\beta_N(i) = \Pr\ \beta_N = i,\ \ i=1,2,\ldots\ldots,N$, then it is easy to show that

$$\beta_N(i) = \begin{cases} 1-\sum_{j=1}^{N-1} q_j & \text{if } i = 1 \\ \sum_{j=1}^{N-i+1} q_j\beta_{N-j}(i-1) & \text{if } 2 \leq i \leq N \end{cases} \tag{3.1}$$

By taking generating function of (3.1), we obtain the following lemma.

Lemma 3.1] Define a generating function

$$\beta_N(z) = \sum_{i=0}^{N} \beta_N(i) z^i$$

then $\beta_N(z)$ is given recursively by

$$\beta_N(z) = (1-\sum_{i=1}^{N-1} q_i)z + z\sum_{i=1}^{N-1} q_i\beta_{N-1}(z) \tag{3.2}$$

Proof) Multiplying each equation of (3.1) by z^{i} and summing over all i, we have

$$\begin{aligned}\beta_N(z) &= (1-\sum_{i=1}^{N-1} q_i)z + z\sum_{i=2}^{N}\sum_{j=1}^{N-i+1} q_j\beta_{N-j}(i-1)z^i \\ &= (1-\sum_{i=1}^{N-1} q_i)z + z\sum_{j=1}^{N-1} q_j\beta_{N-j}(z)\end{aligned}$$

Differentiating (3.2) with respect to z and taking the limit as $Z \to 1$, the expectation of β_N is

$$E[\beta_N] = 1 + \sum_{i=1}^{N-1} q_i E[\beta_{N-i}]$$

In what follows, the LST of idle period distribution is derived by applying the above result.

Theorem 3.1] Let I be the idle period random variable. Then LST of the idle period distribution is given by

$$\tilde{I}(\alpha) = \sum_{i=1}^{N-1} \beta_N(i) \left\{ \frac{\lambda}{\lambda + \alpha} \right\}^i \tag{3.3}$$

Proof)See Appendix A.1

By differentiation (3.3) with respect to α and taking limit as $\alpha \to 0$, we have the mean idle period after simple algebraic manipulation

$$E[I] = \frac{E[\beta_N]}{\lambda} \tag{3.4}$$

Next, we consider the mean of busy period. Lee and Srinivasan[5] showed that when N control policy is employed, the mean busy period is

$$E[B] = \frac{E[\theta_N]E[S]}{1 - \lambda E[X]E[S]} \tag{3.5}$$

where θ_N denotes the number of customers existing in the system when service starts. With the property that β_N is stopping time and Wald's lemma[12], E $[\theta_N]$ is represented as

$$E[\theta_N] = E[\beta_N]E[X] \tag{3.6}$$

Since service interaction discipline is preemptive-resume and breakdown process follows renewal process, the completion time is i.i.d random variable. Therefore, we can replace service time with completion time in (3.5). Let BC be the cycle time, then from (3.4), (3.5) and (3.6), it is clear that the mean busy cycle time is given by

$$E[BC]=E[I]+E[B]=\frac{E[\beta_N]}{\lambda(1-\lambda E[X]E[C])}$$

We now show that the steady state distribution of the number of customers(system size) in the system can be obtained from simple linear recursive equations. Our derivation follows the arguments in Tijms[17]. Let(i,j) be a state where

$$i=\begin{cases}0 & \text{if } \textit{the server is idle}\\ 1 & \text{if } \textit{the server is busy}\end{cases}$$
$$j = \textit{the number of customers in the system}$$

and let

T_{ij} = *amount of time in state*(i,j) *during the busy cycle BC*
U_j^i = exp*ected amount of time during which i customers exist in a service time S, given that j customers are present at the start of this service*($U_j^i=0$, if $i<j$)
P_j^i = *steady state probability that the system is in state*(i,j)

then, by the mean value theorem of regenerative process[12]

$$P_{ij}=\frac{E[T_{ij}]}{E[BC]} \tag{3.7}$$

First, we derive the expected amount of time in state $(0,j), j = 0,1,2.....$

Lemma 3.2] Let $Z(t)$ be a queue size process and $\Upsilon_t(0,j) = \Pr\{Z(t)=(0,j)\}$, then

$$E[T_{0j}]=\frac{1}{\Upsilon}\sum_{n=0}^{j} q_n^*(j)$$

where $q_n^*(\cdot)$ denotes the n-th convolution of batch size distribution $q(\cdot)$.

Proof) See Appendix A.2.

By Lemma 3.2 we have

$$P_{oj} = \frac{E[T_{oj}]}{E[BC]} = \frac{(1-\rho)\sum_{n=1}^{j} q_n^*(j)}{E[\beta_N]} \qquad \text{if} \quad 0 \le j \le N-1 \tag{3.8}$$

$$0 \qquad \text{if} \quad j \ge N$$

where $\rho = \lambda E[X]E[C]$

Now, we consider the expected amount of time in state $(1, j), j = 0, 1, 2, \ldots$. Let $E[N_j]$ be the number of service completion epochs at which customer served leaves j other customer behind in the system during a busy cycle (0, BC], $j \ge$, then E[T_{1j}] has a following recursive
relation[17]

$$E[T_{1j}] = \sum_{i=N}^{j} \theta_N(i) U_j^i + \sum_{i=0}^{j} E[N_i] U_j^i \quad j \ge N \tag{3.9}$$

where $\theta_N(i) = P_r \theta_N = i, \quad i \ge N$

Note that if we use level crossing argument[17], we have

$$E[N_j] = E[A_j]$$

where $E[A_j]$ denotes the expected number of batches which see j order customers on arrival.

Also, by the property that poisson arrivals see time average[20], it follows that

$$\begin{aligned} E[A_j] &= \sum_{i=0}^{j} \lambda \sum_{n=j+1-i}^{\infty} q_n E[T_{0i} + T_{1i}] \\ &= \lambda \sum_{i=0}^{j} \{E[T_{0i}] + E[T_{1i}]\} \sum_{n=j+1-i}^{\infty} q_n \end{aligned} \tag{3.10}$$

Therefore, substituting (3.10) into (3.9), we get

$$E[T_{1j}] = \sum_{i=N}^{j} \theta_N(i) U_j^i + \lambda \sum_{i=N}^{j} \sum_{k=0}^{i} \{E[T_{0k}] + E[T_{1k}]\} \sum_{n=j+1-i}^{\infty} q_n U_j^i \quad j \ge 1$$

By(3.7), the following linear recursive equation is derived.

$$P_{1j}=\frac{\sum_{i=N}^{j}\theta_n(j)U_j^i}{E[BC]}+\lambda\sum_{i=N}^{j}\sum_{k=0}^{i}[P_{0k}+P_{1k}]\sum_{n=i+1-k}^{\infty}q_nU_j^i \quad j\geq 1 \tag{3.11}$$

combining (3.8)and(3.11),the steady state probability can be obtained recursively as follows.

$$\begin{aligned}
P_0 &= P_{00}+P_{10}=\frac{1-\rho}{E[\beta_N]} \\
P_{0j} &= P_0\sum_{n=1}^{j}q_n(j) && 1\leq j\leq N-1 \\
P_{0j} &= 0 && j\geq N \\
P_{1j} &= \lambda P_0\sum_{i=N}^{j}\theta_N(j)U_j^i+\lambda\sum_{i=1}^{j}\sum_{k=0}^{i}[P_{0k}+P_{1k}]\sum_{n=i+1-k}^{\infty}q_nU_j^i && j\geq 1
\end{aligned} \tag{3.12}$$

Let us define the probability generating function of the steady state probability by

$$P(z)=\sum_{i=0}^{\infty}P_i=\sum_{i=0}^{\infty}[P_{0i}+P_{1i}]z^i$$

then, after some tedious algebraic calculation, we obtain the following theorem,

Theorem 3.2] The probability generating function of the system size is given by

$$P(z)=\frac{P_0(1-z)\left(\Pi+\frac{\theta_N(z)\{1-C^{-}(\lambda(1-G_X(Z)))\}+C^{-}(\lambda(1-G_X(z)))-G_X(z)}{1-G_X(z)}\right)}{C^{-}(\lambda(1-G_X(z)))-z} \tag{3.13}$$

where $\Pi=\sum_{i=1}^{N-1}\sum_{j=1}^{i}q_j^*(i)z^i$

4. Concluding Remarks

In this research, we have established the probability generating function of a single server queue operating under N control policy where the server is subject to breakdown whose process follows a renewal process. Compared to the previous research, two main contributions are distinguished. First, we treated general repair and inter-breakdown time distributions and derived the exact closed from of the probability generating function of system size. Also, the steady state probability can be obtained computationally using recursive formula if the service time is deterministic or generalized Erlang distribution[17]. the mean and variance of the system size distribution were not derived in this paper, but these measures can be obtained from (3.13) by regular derivation routine.

References

[1] Y. Baba, On the $M^X/G/1$ Queue with Vacation time, Operations Research Letter, 5(1986) 93-98.

[2] D. Cox, Renewal Theory, (Methuen & Co., London, 1962).

[3] A.Federgruen, and L. Green, Queueing Systems with Service Interruption, Operations Research, 34(1986)753-767

[4] D.P. Gaver, A Waiting Line with Interrupted Service, Including $M^X/G/1$ Priorities, Journal of Royal Statistics Society, B24(1962)73-90.

[5] H. Lee and M.M. Srinivasan, Control Policies for the Queueing System, Management Science, 35(1989)708-720.

[6] O.C.Ibe and K.S. Trivedi, Two Queues with Alternating Service and Server Breakdown, Queueing System, 7(1990)253-268.

[7] N.K. Jaiswal, Preemptive Resume Priority Queue, Operation Research, 9(1961)732-742.

[8] J.Keilson, Queues Subject to Service Interruption , Annals of Mathematical Statistics, 33(1962)1314-1322.

[9] V.F. Nicola , A Single Server Queue with Mixed Types of Interruption , Acta Information,23(1986)465-486.

[10] V.F. Nicola et al, Queueing Analysis of Fault-Talerant Computer Systems, IEEE transaction. on Software Engineering, 13(1987)363-375.

[11] V.Ramaswami, The N/G/1 Queue and its Detailed Analysis, Advance in Applied Probability, 12(1980)222-261.

[12] S.M. Ross,Stochastic Processes, (John Wiley, New York, 1983).

[13] B. Sengupta, A Queue with Service Interruptions in Alternating Random Environment, Operations Research, 38(1990)308-318.

[14] A.W.Shogan, A Single Server Queue with Arrival Rate Dependent on erver Breakdown, Naval Research Logistic Research, 26(1979)487-497.

[15] U. Sumita et al, Analysis of Effective Service Time with Age Dependent Interruptions and its Application to Optimal Rollback Policy for Database Management, Queueing System, 4(1989)193-212

[16] J. Teghem, Control of the Service Process in a Queueing System, European Journal of Operational Research, 23(1986)141-158.
[17] H.C. Tijms, Stochastic Modeling and Analysis: A Computational Approach, (John Wiley, New York,1986).
[18] M.H. Van Hoorn, Algorithms for the State Probabilities in a General Class of Single Server Queueing Systems with Group Arrivals, Management Science, 27(1981)
[19] H.C. White and L.C. Christie, Queueing with Preemptive Priorities or with Breakdown, Operations Research, 6(1958)79-95.
[20] R.W. Wolfe, Poisson Arrivals See Time Average, Operations Research, 30(1982)223-231.

Appendix

A.1 Proof of theorem 3.1

By conditioning on β_N, we obtain the probability that the idle period ends in time interval $(t, t+\Delta t)$ as follows.

$$\begin{aligned}\Pr\{t \le I \le t+\Delta t\} &= \sum_{i=1}^{N} \Pr\{t \le I \le t+\Delta t \mid \beta_N = i\}\beta_N(i) \\ &= \sum_{i=1}^{N} \Pr\left\{t \le \sum_{j=1}^{i} Aj \le t+\Delta t\right\}\beta_N(i) \\ &= \sum_{i=1}^{N} Y_i(t)\Delta t \beta_N(i)\end{aligned}$$

where $Y_i(t)$ is Erlang distribution with shape parameter i and mean $1/\lambda$. As $\Delta t \to 0$, we have

$$dI(t) = \sum_{i=1}^{N} Yi(t) dt \beta_N(i)$$

Since inter-arrival time is exponential distribution $Y_i(t)$ is Erlang-i distribution, the LST of the idle period distribution is

$$\begin{aligned}\tilde{I}(\alpha) &= \int_0^{\infty} \exp(-\alpha t) dI(t) = \int_0^{\infty} \exp(-\alpha t) \sum_{i=1}^{N} Yi(t) dt \beta_N(i) \\ &= \sum_{i=1}^{N} \beta_N(i) \int_0^{\infty} \exp(-\alpha t) Yi(t) dt = \sum_{i=1}^{N} \beta_N(i) \left\{\frac{\lambda}{\lambda+\alpha}\right\}^{i}\end{aligned}$$

A.2 Proof of lemma 3.2

Proof)Let I_t be an indicator function defined by

$$I_t=\begin{cases}1 & \text{if } \textit{the number} \text{ if } \textit{customers are } j \text{ and } \textit{the server is idle at time } t\\ 0 & \textit{otherwise}\end{cases}$$

then it is easy to check that

$$E[T_{0j}]=E[\int_0^\infty I_t dt]=\int_0^\infty E[I_t]dt$$

Since $E[I_t]=\Pr\{I_t=1\}=\gamma_t(0,j)$, we have

$$\begin{aligned}E[T_{0j}] &= \int_0^\infty \gamma_t(0,j)\,dt=\int_0^\infty \sum_{n=1}^{j} q_n^*(j)\frac{\exp(-\lambda t)(\lambda t)^n}{n!}\,dt\\ &= \sum_{n=1}^{j}\frac{q_n^*(j)}{n!}\int_0^\infty \exp(-\lambda t)(\lambda t)^n dt\\ &=\frac{1}{\lambda}\sum_{n=1}^{j} q*n(j)\end{aligned}$$