A Note on a Construction of t-norm

Dug Hun Hong*

This paper is supported by Catholic University of Taegu-Hyosung Special Research Grant.

ABSTRACT

In this paper, we consider a generalized problem of Fullér's [Fuzzy Sets and Systems 45 pp. 299-303, 1992] open question and prove it.

I. Introduction

As defined in [3], by a fuzzy number we mean a fuzzy subset ξ of the real line with a unimodal, upper semicontinuous membership function such that there exists a unique real number m satisfying $\xi(m) = \sup_x \xi(x) = 1$. The number $m = m(\xi)$ is called the modal value of ξ. Now suppose that a sequence of fuzzy numbers $\xi_1, \xi_2, \cdots, \xi_n$ and a t-norm T (see[1]) are given. The T-sum $\xi_1 + \xi_2 + \cdots + \xi_n$ and the T-arithmetic mean $(\xi_1 + \xi_2 + \cdots + \xi_n)/n$ are the fuzzy numbers defined by

$$(\xi_1 + \xi_2 + \cdots + \xi_n)(z) := \sup_{x_1 + \cdots + x_n = z} T(\xi_1(x_1), \cdots, \xi_n(x_n))$$

and

$$\frac{1}{n}(\xi_1 + \xi_2 + \cdots + \xi_n)(z) := (\xi_1 + \xi_2 + \cdots + \xi_n)(nz),$$

respectively (see[2]). For a fuzzy number ξ and any subset D of the real numbers, the quantity

$$\mathrm{Nes}(\xi \mid D) := 1 - \sup_{x \notin D} \xi(x)$$

is considered to measure the necessity of ξ belonging to D (see[10]). If D is an interval (a, b) we also write $(a < \xi < b)$ instead of Nes $(\xi \mid D)$. Assume now that a sequence $\xi_1, \xi_2, \cdots$ of fuzzy numbers and a t-norm T are given and denote by m_n the modal value of the T-arithmetic mean $(\xi_1 + \xi_2 + \cdots + \xi_n)/n$. Following Fullér (see[4]), we say that $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ obeys the law of large numbers if for all $\varepsilon > 0$ the quantity Nes $(m_n - \varepsilon < (\xi_1 + \xi_2 + \cdots + \xi_n)/n < m_n + \varepsilon)$ tends to 1 for $n \to \infty$. In [4], Fullér proves a law of large numbers for sequences of symmetric triangular fuzzy numbers with common spread if $\lim_{n\to\infty} m_n$ exists and $T(u, v) \le H_0(u, v) := uv/(u + v - uv)$ for all $0 \le u, v \le 1$. Recently Triesch[10] , Hong[7] and Hong and Kim [6] generalize Fullér's result to sequences of L-R fuzzy numbers with bounded spreads if T is a Archimedian t-norm. Fullér proposed the following open question at the end of [4]: Suppose we are given a continuous t-norm such that $H_0 < T <$ 'min' and a sequence $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ of symmetric tringular fuzzy numbers with common spread. Does this sequence obey the law of large numbers?

In this paper we consider a generalized problem of this open question and solve it.

*School of Mechanical and Automotive Engineering, Catholic University of Taegu-Hyosung

II. Preliminary

Definition 1[9]. A t-norm T is a function from $[0, 1] \times [1, 0]$ to $[0, 1]$ which verifies the following conditions:

(i) $T(1, x) = x$ (boundary condition).
(ii) $T(x, y) \leq T(z, t)$ whenever $x \leq z$, $y \leq t$ (monotonicity).
(iii) $T(x, y) = T(y, x)$ (symmetry).
(iv) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity).

A t-norm T is Archimedean, if it is continuous (in each both variables) and $T(x, x) < x$ for all x in $(0, 1)$. A t-norm is strict if it is continuous, and strictly increasing, in both variables on $(0, 1] \times (0, 1]$. It is noted that H_0 is an Archimedean t-norm.

Definition 2. Let J be a finite or countable set. Let $\{T_i \mid i \in J\}$ be a collection of t-norms and $\{(a_i, b_i) \mid i \in J\}$ a collection of disjoint intervals in $[0, 1]$. We call ordinal sum of t-norms $\{T_i \mid i \in J\}$ to the following t-norm:

$$T(x, y) = \begin{cases} a_i + (b_i - a_i) T_i\left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i}\right) & \\ \qquad \text{whenever } (x, y) \in (a_i, b_i)^2 = (a_i, b_i) \times (a_i, b_i), & \\ \min(x, y) & \text{otherwise.} \end{cases}$$

The following theorem gives a general classification of continuous t-norms[8].

Theorem 1. Let T be a continuous t-norm. Then T is Archimedean or T-min or T is an ordinal sum of Archimedean t-norms.

III. Results

Frist we consider constructing a t-norm.

Theorem 2. For any continuous t-norm $T <$ 'min', there exists a continuous t-norm T' such that $T < T <$ 'min'.

Proof. Let T be a continuous t-norm $T <$ 'min'. Since $T <$ 'min', there exists a number $a \in (0, 1)$ such that $T(a, a) < a$. Now using the continuity of T we can choose a small positive number δ such that whenever $(x, y) \in (a - \frac{\delta}{2}, a + \frac{\delta}{2})^2$, $\min(x, y) - T(x, y) > \delta > 0$ and $1 - \frac{\delta}{2} > a > \frac{\delta}{2}$. Define T' as follows:

$$T'(x, y) = \begin{cases} a - \dfrac{\delta}{2} + \delta T\left(\dfrac{x - (a - \frac{\delta}{2})}{\delta}, \dfrac{y - (a - \frac{\delta}{2})}{\delta}\right) & \\ \qquad \text{for } (x, y) \in (a - \frac{\delta}{2}, a + \frac{\delta}{2})^2, & \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Then it is clear that T' is a continuous t-norm by Theorem 1 and $T'(x, y) \geq T(x, y)$ for $(x, y) \notin (a - \frac{\delta}{2}, a + \frac{\delta}{2})^2$. We also note that for $(x, y) \in (a - \frac{\delta}{2}, a + \frac{\delta}{2})^2$,

$$\begin{aligned} \min(x, y) - T'(x, y) &= \min(x, y) - (a - \tfrac{\delta}{2}) - \delta T\left(\frac{x - (a - \frac{\delta}{2})}{\delta}, \frac{y - (a - \frac{\delta}{2})}{\delta}\right) \\ &\leq \min(x, y) - (a - \tfrac{\delta}{2}) \\ &\leq (a + \tfrac{\delta}{2}) - (a - \tfrac{\delta}{2}) \\ &= \delta. \end{aligned}$$

Hence, for $(x, y) \in (a - \frac{\delta}{2}, a + \frac{\delta}{2})^2$,

$$\begin{aligned} T'(x, y) - T(x, y) &= \min(x, y) - T(x, y)) - (\min(x, y) - T'(x, y)) > \delta - \delta \\ &= 0. \end{aligned}$$

This completes the proof of theorem.

Now let T' be the t-norm defined in Theorem 2. Let ξ_i, $i \in N$, be symmetric triangular fuzzy numbers with common modal value 0 and common spread 1. Then Fullér's[4] theorem states that the membership function $(\eta_n/n)(x)$ where $\eta_n = \xi_1 + \cdots + \xi_n$ converges pointwise (as $n \to \infty$) to the function μ given by

$$\mu(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

at least on $(-1, 0) \cup (0, 1)$. But it is easy to check that $(\eta_n/n)(z) = 1 - |z|$ if $|z| \in [0, a - \frac{\delta}{2}] \cup [a + \frac{\delta}{2}, 1]$ and hence $\lim_{n \to \infty} (\eta_n/n)(z) = 1 - |z|$ if $|z| \in [0, a - \frac{\delta}{2}] \cup [a + \frac{\delta}{2}, 1]$. If T is an Archimedean t-norm, we can also show that

$$\lim_{n \to \infty} \frac{\eta_n}{n}(z) = \begin{cases} a - \frac{\delta}{2} & \text{if } |z| \in (a - \frac{\delta}{2}, a + \frac{\delta}{2}), \\ 1 - |z| & \text{if } |z| \in [0, a - \frac{\delta}{2}] \cup [a + \frac{\delta}{2}, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Hence we proved the so-called "*generalized Fullér's open question*": Let T_0 be an Archimedean t-norm. Suppose we are given a t-norm such that $T_0 < T <$ 'min' and a sequence $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ of symmetric triangular fuzzy numbers with common spread. Does this sequence obey the law of large numbers?

Ⅳ. Concluding remarks

Hong[5] already solved Fullér's open problem on H_0. We have solved a generalized problem on Archimedean t-norm. As a special case the open problem can be solved, in another manner than by Hong.

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홍 덕 헌(Dug Hun Hong)
1974. 3~1981. 2 : 경북대학교 수학과
1981. 3~1983. 2 : 경북대학교 수학과 대학원
1985. 3~1990. 8 : Univ. of Minnesota(M.S., Ph.D)
1991. 3~1993. 3 : 효성여자대학교 통계학과 전임강사
1993. 4~1996. 2 : 대구효성가톨릭대학교 통계학과 조교수
1996. 3~현재 : 대구효성가톨릭대학교 기계자동차공학부 조교수