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Reliability Expression for Complex System

Seong Cheol Lee[1)]

Abstract

In this paper, we present a algebraic technique for computing system reliability for complex system. The method was originally developed as an aid to fault tree analysis but it applies to general problems of reliability assessment. A success expression which directly gives the reliability expression is formed and simplified by the procedure. Several algorithms and examples are illustrated.

1. Introduction

The idea of considering probabilistic problem, associated with reliability engineering in the Boolean domain is not new and can be traced back to Hurley(1963), who explored the relationship between two algebras and described how conversion from one to the other could be achieved. Essentially, his procedure was based on the use of probability map (P-map) to identify overlap terms. Suitable corrections were then made to the probabilistic expression. More recently, various authors have recognised that if the algebraic expression is rendered disjoint probability events before interpret as a probabilistic expression, then the Boolean approach becomes more tractable in teams of programming implementation.

First, we discussed a general purpose method for producing reliability expressions by algebraic techniques.

Next, we propose a algorithm which gives a minimum reliability expression. A system given by an oriented or non-oriented network can be transformed into a series or parallel system by taking its paths or cut sets as its terms (Lee, 1993-a). Problems related to the reliability structure of the system are based on Lee(1993-a).

Section 2 proposes algebraic technique for modifying the success expression to yield reliability expression. In section 3, we discussed analysis of reliability block diagrams by Boolean technique. Section 4, main part of this paper, proposed a substitutionary decomposition method for computing system reliability by a Boolean Expression.

1) Full-time Lecturer, Dept. of Mathematics, Inha Univ., Inchon, Korea

Notation

B - domain	Boolean domain.
P - domain	Probability domain.
a, b, c, . . .	Boolean variable.
A, B, C. . .	Probabilistic variable.
S	success expression of the system.
R	reliability expression of the system.
x_{ij}	branch variable j of i.
P_i	success expression of path i.
$\overline{P}_i$	failure expression of path i.
m	number of paths.
n_i	number of branches in path i.
p_A	reliability of component A.
q_A	unreliability of component A.

2. Algebraic Technique

The system success expression S is given by the union of all the system path: that is

$$S = \bigcup_{i=1}^{m} P_i$$

$$= P_1 + \overline{P}_1 P_2 + \overline{P}_1\ \overline{P}_2 P_3 + \overline{P}_1\ \overline{P}_2 \ldots \overline{P}_{m-1} P_m \tag{1}$$

Each P_i in (1) is a product term of the form

$$P_i = \prod_{j=1}^{n_i} x_{ij}, \qquad i = 1, 2, \ldots, m \tag{2}$$

When (2) is substituted in (1), and all $\overline{P}_i$ are expanded using DeMorgan's Law, the product terms in the resulting sum-of-product expression lose their disjointness. The disjointness is retained if $\overline{P}_i$ are expanded with the following relation based on (1), instead of DeMorgan's Law;

$$\overline{\prod_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A}_j = \overline{A}_1 + A_1\ \overline{A}_2 + \ldots + A_1 A_2 \ldots A_{n-1}\ \overline{A}_n. \tag{3}$$

The use of (1) for the union and complement operation does not destroy commutative property of these operations;

$$A\cup B=A+\overline{A}B=A+B=B+A=B+\overline{B}A=B\cup A \tag{4}$$

$$\overline{AB}=\overline{A}+A\overline{B}=\overline{B}+B\overline{A}=\overline{BA} \tag{5}$$

For logical variables A, B, C we have:

$$(\overline{AB})\cup(AC)=(AC)\cup(\overline{AB})=AC\cup(\overline{A}+A\overline{B})=A\overline{B}C \tag{6}$$

$$(\overline{AB})\cup(\overline{AC})=(\overline{A}+A\overline{B})\cup(\overline{A}+A\overline{C})=\overline{A}+A\overline{BC} \tag{7}$$

$$\overline{A}\cup(\overline{AC})=\overline{A}\cup(\overline{A}+A\overline{C})=\overline{A} \tag{8}$$

Relations (6)-(8) along with the simple relations like $A\overline{A}=0$, $AA=A$ are used to simplify it. Final simplified sum-of-product expression for success gives the simplified reliability expression.

The technique illustrated above is put in the form of an algorithm which gives a minimum reliability expression.

Algorithm

Step 1. Enumerate all the m paths of the system and arrange these in a sequence $P_1, P_2, \ldots, P_m$.

Step 2. Write S in a more convenient form given by (1).

$$S=P_1+\overline{P}_1(P_2+\overline{P}_2(P_3+\overline{P}_3(\ldots(P_{m-1}+\overline{P}_{m-1}(P_m))\ldots))). \tag{9}$$

Step 3. Substitute expression of P_m in (9); let $j=m-1$.

Step 4. In the resulting expression substitute expression of P_i and simplify it using (6)-(8). Let $j \leftarrow j-1$.

Step 5. Repeat step 4 if $j \geq 1$.

Step 6. Replace logical variables by their reliabilities to get the required reliability expression.

Example 1. For illustration, consider the 5-component bridge system shown in Fig. 1 which is also solved in [8].

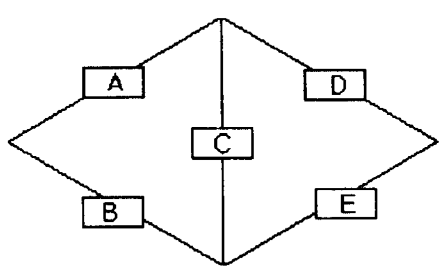


Fig.1 5-component bridge system

Step 1. Paths are $P_1=\{A, D\}$, $P_2=\{B, E\}$, $P_3=\{A, C, E\}$,

$P_4=\{B, C, D\}$.

Step 2. $S=P_1+\overline{P}_1(P_2+\overline{P}_2(P_3+\overline{P}_3(P_4)))$

Step 3, 4, 5.

$$S=AD+\overline{AD}(BE+\overline{BE}(ACE+\overline{ACE}(BCD)))$$

$$=AD+\overline{AD}BE+\overline{AD}\,\overline{BE}ACE+\overline{AD}\,\overline{BE}\,\overline{ACE}BCD$$

$$=AD+\overline{A}B\overline{D}E+A\overline{B}C\overline{D}E+\overline{A}BCD\overline{E}.$$

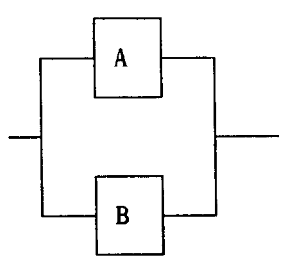
Step 6.

$$R=p_Ap_D+q_Ap_Bq_Dp_E+p_Aq_Bp_Cq_Dp_E+q_Ap_Bp_Cp_Dq_E$$

This algorithm is very simple; is equally efficient for simple and complex system. Further simplification is required using Boolean algebra techniques to get a minimum reliability expression.

3. Reliability Block Diagrams by Boolean Expression

A reliability block diagram for complex systems is often analyzed by applying the series-parallel product laws or where this is not possible, by using a conditional probability result (Bayes theorem). In both cases, the analysis is conducted in the probabilistic domain and, for complex systems, is lengthy. An alternative method is to consider the component reliability parameters as Boolean variables rather than probabilistic variables and to treat the whole problem as if it were Boolean. This has the advantage of allowing the use of powerful Boolean reduction theorems to contain the size of the problem. Consider the parallel system shown in Figure 2. Figure 2a shows the reliability block diagram and Fig.2b the truth-table corresponding to the successful and unsuccessful path.



(a) Reliability block diagram

A	B	R
0	0	0
0	1	1
1	0	1
1	1	1

(b) Truth table

Fig 2. 2-component parallel system.

For the purpose of illustrating the problem however, a tabular solution is presented based on the truth table.
The S of the reliability block diagram can be derived from Fig.2b ;

$$S=\overline{A}B+A\overline{B}+AB \quad . \tag{10}$$

Alternatively, the truth-table can be transcribed onto a P-map (see Fig.3a) and various 1-entries grouped to form a simpler but equivalent expression. Fig.3b shows one possibility in which the grouping is disjoint. This produces an alternative expression as

$$S=A+\overline{A}B \quad . \tag{11}$$

Fig.3c shows yet another possibility - non disjoint. In this case S is

$$S=A+B-AB \quad . \tag{12}$$

Of these three forms of S, (11) is preferable to either (10) or (12). In the B- domain, (10) is known as a canonical expression and is to be avoided for a realistic number of variables. Equation (12) is also complicated by the need to identify and remove the overlap terms and, again for a practical example, is not viable. Equation (11) has the useful properties of being both reduced (non-canonical) and disjoint.

B \ A	0	1
0	0	1
1	1	1

Fig.3 (a) P - map

B \ A	0	1
0	0	1
0	1	1

(b) disjoint grouping

B \ A	0	1
0	0	1
1	1	1

(c) non-disjoint grouping

Consider again the reliability block diagram shown in Fig. 2a. Define a Boolean expression called the Path Set expression, to be the logical OR between all successful paths (minimal tie-sets) through the reliability block diagram. For Fig. 2a, this expression is :

$$\text{Path set} = a+b$$
$$= a+\overline{a}b \text{ (by reduction rule)} \tag{13}$$

From (13), We derived (11) very easily.
Now we compare and contrast the procedure just outlined with the more traditional approach.

Example 2. Usually, the reliability block diagram shown in Fig 4. would be solved by applying the series, parallel laws.

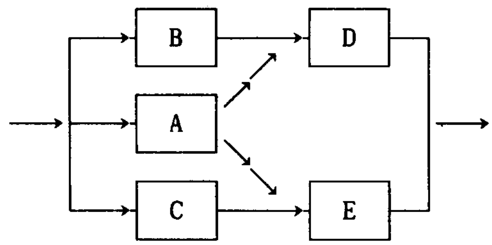


Fig4. series/parallel system

This is the classic example to illustrate the conditional probability result Bayes Theorem. Choose A to the keystone :

$$S=A(D+E-DE)+\overline{A}(BD+CE-BCDE). \tag{14}$$

Alternatively :

$$\text{Path set } = bd + ad + ae + ce$$

$$= bd+\overline{b}ad+\overline{b}ae+b\overline{d}ae+\overline{b}ce+b\overline{d}ce$$

$$= bd+\overline{b}ad+\overline{db}ae+b\overline{d}ae+\overline{ab}ce+a\overline{db}ce+\overline{a}b\overline{d}ce.$$

The ($a\overline{bd}ce$) term is absorbed by the ($\overline{db}ae$) term and the expression becomes disjoint. Also the ($\overline{db}ae$) and ($b\overline{d}ae$) terms reduce to a single term ($\overline{d}ae$).
Therefore :

$$S=BD+A\overline{B}D+A\overline{D}E+\overline{AB}CE+\overline{A}BC\overline{D}E. \tag{15}$$

$$R=p_Bp_D+p_Aq_Bp_D+p_Aq_Dp_E+q_Aq_Bp_Cp_E+q_Ap_Bp_Cq_Dp_E. \tag{16}$$

4. Decomposition Method by Boolean Expression

In my earlier paper(Lee, 1993-a), a system given by a network is decomposed into subsystems according to up-and-down states of its keystone element. In this section the decomposition method is used for a Σ or Π-system instead of one given by a network. The method is not different, although it requires transforming the network in to a suitable Boolean expression.

If system S given by a Boolean expression, then S can be decomposed into 1-subsystem $S(x)$ and 0-subsystem $S(\overline{x})$ according to up-and-down states of its arbitrary variable x. Its reliability is given by following Theorem.

Theorem Reliability of system S is :

$$R(S) = R(S(x)) \times R(x) + R(s(\overline{x})) \times R(\overline{x}) \tag{17}$$

Proof is given by Premo(1963).

Algorithm for Σ-system S.

Step 1. Repeat 1-substituted decomposition until 1-subsystem $S(\ldots x_{ij})$ has all s-independent terms.

Step 2. 1) If 'no absorbable term occurs in $S(\ldots x_{ij})$' OR if 'some absorbable terms occur in $S(\ldots \bar{x}_{ij})$' AND if '$S(\ldots \bar{x}_{ij})$ is composed of s-independent terms', then

$$R(S) = R(S) + R(S(\ldots x_{ij})) \times R(\ldots x_{ij}) + R(S(\ldots \bar{x}) \times R(\ldots \bar{x}) \quad (18)$$

2) If 'some absorbable terms occur in $S(\ldots . x_{ij})$' AND if '$S(\ldots . \bar{x}_{ij})$ is not composed of s-independent terms', then

$$R(S) = R(S) + R(S(\ldots x_{ij})) \times R(\ldots x_{ij}). \quad (19)$$

Repeat step 1 for $S(\ldots \bar{x}_{ij})$

Step 3. 1) If 'some keystone is one in $S(\ldots \bar{x})$', namely,

$$S(\ldots \bar{x}_{ij'})) = S(\ldots \bar{x}_{ik}\ \bar{x}_{ik+1}\ \bar{x}_{ik+2} \ldots \bar{x}_{ij}). \quad (20)$$

then repeat step 1 for $S(\ldots \bar{x}_{ik})$.

2) If 'all keystones are zero ', then

$$S(\ldots \bar{x}_{ij}) = S(\bar{x}_{i1}\ \bar{x}_{i2} \ldots \bar{x}_{ij}). \quad (21)$$

Example 3. System S in Fig.5 can be transformed into a Σ-system

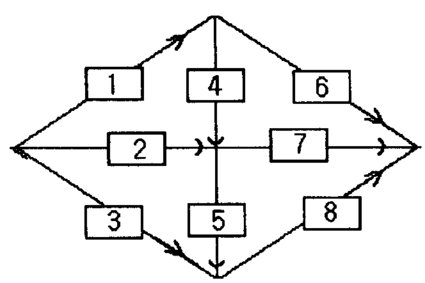


Fig.5 system S given by an oriented network.

$S = x_1x_6 + x_1x_4x_7 + x_1x_4x_5x_8 + x_2x_7 + x_2x_5x_8 + x_3x_8$

$S(x_1) = x_6 + x_4x_7 + x_4x_5x_8 + x_2x_7 + x_2x_5x_8 + x_3x_8$

$S(x_1x_8) = x_6 + x_4x_7 + x_4x_5 + x_2x_7 + x_2x_5 + x_3$

$S(x_1x_8x_4) = x_6 + x_7 + x_5 + x_2x_7 + x_2x_5 + x_3 = x_6 + x_7 + x_5 + x_3$

$$S(x_1x_8\ \bar{x}_4)=x_6+x_2x_7+x_2x_5+x_3$$
$$S(x_1x_8\ \bar{x}_4\ x_2)=x_6+x_7+x_5+x_3$$
$$S(x_1x_8\ \bar{x}_4\ \bar{x}_2)=x_6+x_3$$
$$S(x_1\ \bar{x}_8)=x_6+x_4x_7+x_2x_7$$
$$S(x_1\ \bar{x}_8x_7)=x_6+x_4+x_2$$
$$S(x_1\ \bar{x}_8\ \bar{x}_7)=x_6$$
$$S(\ \bar{x}_1)=x_2x_7+x_2x_5x_8+x_3x_8$$
$$S(\ \bar{x}_1x_2)=x_7+x_5x_8+x_3x_8$$
$$S(\ \bar{x}_1x_2x_8)=x_7+x_5+x_3$$
$$S(\ \bar{x}_1x_2\ \bar{x}_8)=x_7$$
$$S(\ \bar{x}_1\ \bar{x}_2)=x_3x_8$$

Absorbable terms x_2x_7 and x_2x_5 occur in $S(x_1x_8x_4)$; so s-independency of $S(x_1x_8\ \bar{x}_4)$ must be examined. $S(x_1x_8\ \bar{x}_4)$ has s-dependent terms.

No absorbable term occurs in $S(x_1x_8\ \bar{x}_4x_2)$, $S(x_1\ \bar{x}_8x_7)$, $S(\ \bar{x}_1x_2x_8)$.

Hence s-dependent terms does not exist in $S(x_1x_8\ \bar{x}_4\ \bar{x}_2)$, $S(x_1\ \bar{x}_8\ \bar{x}_7)$, $S(\ \bar{x}_1x_2\ \bar{x}_8)$.

Reliability of S is

$$R(S)=\sum_{i=1}^{8}\ (\prod_{x_j\in\{K_i\}} R(x_j))\times R(S(K_i)) \tag{22}$$

where

$$K_i:\ \ x_1x_8x_4,\ x_1x_8\ \bar{x}_4x_2,\ x_1x_8\ \bar{x}_4\ \bar{x}_2,\ x_1\ \bar{x}_8x_7,\ x_1\ \bar{x}_8\ \bar{x}_7,\ \bar{x}_1x_2x_8,\ \bar{x}_1x_2\ \bar{x}_8,\ \bar{x}_1\ \bar{x}_2. \tag{23}$$

$R(S(K_i))$ are known.

Algorithm for Σ-system S can be applied to Π-system exchanging 1-decomposition, 1-subsystem (x_a, $\bar{x}_a$) by 0-decomposition, 0-subsystem($\bar{x}_a$, x_a), respectively.

References

[1] R. E. Barlow, F. Proschan (1965). *Mathematical Theory of Reliability*, John wiley & Sons, Inc., NewYork.

[2] M. L. Shooman (1968). *Probabilistic Reliability, An Engineering Approach*, McGraw-Hill.

[3] R. E. Barlow, F. Proschan (1975). *Statistical Theory of Reliability and Life testing*, Holt, Rinehart & Winston, Inc.

[4] A. F. Premo (1963). The use of Boolean algebra and truth table in the formulation of a mathematical model of success, *IEEE Trans. Reliability*, Vol. R-12, 45-49.

[5] R. B. Hurley (1963). Probability maps, *IEEE Trans. Reliability*, Vol. R-12, 39-44.

[6] S. C. Lee (1991). System diagrams and event trees for coherent structure, *Bulletin of the Institute for Basic Science,* Vol. 12 Inha Univ.

[7] S. C. Lee (1993-a). Probability of system failure of coherent structure, *Bulletin of the Institute for Applied Sience*, Korea Aviation Univ.

[8] S. C. Lee (1993-b). Determination of probability of component or subsystem failure, *Journal of the Korean Society for Quality Control*, Vol 21, No 2.

[9] S. C. Lee (1996). The inclusion-exclusion method applied to system reliability calculation, *Bulletin of the Institute for Basic Sience*, Vol 17 Inha Univ.