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The Nonparametric Test for Detecting Main Effects for Three-Way ANOVA Models [†]

Young Hun Choi [1]

Abstract

When interactions are not present in a three-way layout, the limiting null distribution of the F statistic for testing main effects when applied to the rank-score transformed data is the same as the limiting null distribution of the usual F statistic when applied to the normal data. The simulation results exhibit that the rank transform test is robust with respect to significance level and powerful for testing main effects in a three-way factorial experiment.

Key Words : Three-way layout; Rank-scored statistic; Robustness; Power.

1. INTRODUCTION

Since the rank transform technique is viewed as a useful tool for developing nonparametric procedure to solve problems, a number of papers have discussed the rank transform's appropriateness in various analysis of variance of experimental designs(Hora and Conover 1984, Akritas and Arnold

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[1] Department of Statistics, Hanshin University, Osan, 447-791, Korea.

1994, etc.). These papers have shown that the rank transform procedure is fairly proper for many given circumstances but still controversial under some hypotheses and situations. Meanwhile due to complication of theoretical development for the rank transform method, the theory for tests based on ranks beyond a two-way layout has not yet been provided.

So major concern of this paper is primarily to study the asymptotic theory of the rank transformed statistic, computed on rank scores, for testing for main effects in a three-way layout without interactions. In addition simulation study in terms of the Type I error and power properties of the rank transformed F test and the usual analysis of variance F test is considered to see that the theoretical results agree with computer generated simulation results to extend the application of rank transformation.

2. DEFINITIONS FOR THE TEST STATISTIC

The model discussed is a three-way layout with I, J, K treatments and N replications. Let X_{ijkn}, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ and $n = 1, \ldots, N$, be independent random variables and let $F_{ijk}(x)$ denote the continuous distribution function of X_{ijkn}. Now the hypotheses of our interest are

$$\begin{aligned} H_0 &: F_{ijk} = F_{ik} \text{ for } j = 1, \ldots, J \\ H_1 &: F_{ijk} \neq F_{ik} \text{ for at least one } j = 1, \ldots, J. \end{aligned} \tag{2.1}$$

To define the test statistic, let R_{ijkn} be the rank of X_{ijkn} among the $M = IJKN$ random variables. A rank score $a_M(R_{ijkn})$ can be generated from the score function ϕ defined on $[0, 1]$ in two ways:

$$a_M(i) = \phi\left(\frac{i}{M+1}\right), \text{ for } 1 \leq i \leq M, \quad M = IJKN \tag{2.2}$$

or

$$a_M(i) = E[\phi(U_M^{(i)})] \tag{2.3}$$

where $U_M^{(i)}$ is ith order statistic in a random sample of size M taken on a random variable uniformly distributed on $[0, 1]$. Further the sum of scores for all X_{ijkn} having the index j, (i, k), (i, j, k) will be denoted by $S_{.j..}$, $S_{i.k.}$, $S_{ijk.}$ respectively and the sum of all scores will be denoted by S.

The rank-score transformed statistic is now defined as

$$\chi_N^2 = \frac{\frac{\sum_{j=1}^{J}(S_{.j..} - \bar{S})^2}{(J-1)N}}{\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{n=1}^{N}\left[a_M(R_{ijkn}) - \frac{S_{ijk.}}{N}\right]^2}{J(N-1)}}, \quad \text{where } \bar{S} = \frac{S}{J}.$$

3. ASYMPTOTIC DISTRIBUTION OF F_N

In this section we will develop three theorems, which are related to the limiting distribution of the rank-scored statistic χ_N^2 for main effects. Let $\chi_N^2 = [\sigma^2 Q_N]/[D_N(J-1)]$. Then Theorem 1 gives the limiting chi-squared distribution of the numerator, Q_N, of χ_N^2. Meanwhile Theorem 2 implies that the denominator, D_N, of χ_N^2 converges in probability to σ^2. Hence Theorem 3 combines to give the main results that χ_N^2 converges in distribution to a chi-squared random variable with $J-1$ degrees of freedom divided by $J-1$.

Theorem 1. If the score function ϕ has a bounded second derivative and rank scores $a_M(R_{ijkn})$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ and $n = 1, \ldots, N$, are generated by (2.2) or (2.3), then under the null hypothesis (2.1) the statistic

$$Q_N = \frac{(J-1)\sum_{j=1}^{J}(S_{.j..} - \bar{S})^2}{JN\sigma^2} \tag{3.1}$$

has a limiting distribution that is chi-squared with $J-1$ degrees of freedom as N goes to infinity, where $\sigma^2 = \lim_{N\to\infty} var(S_{.j..})/N$ exists.

Proof. First of all from a normality condition for a linear rank statistic of $H\acute{a}jek's(1968)$ Theorem 2.1, σ^2 may be expressed as $\sigma^2 = [(J-1)/IJK]\sum_{i=1}^{I}\sum_{k=1}^{K} var[l_{ik}(X_{i1k1})]$, where $l_{ik}(x) = \sum_{l=1}^{I}\sum_{m=1}^{K}\int[u(x-y) - F_{lm}(y)]\phi'[H(y)]dF_{ik}(y)$, $u(x) = 1$ for $x \quad 0$, 0 for $x < 0$, $H(x) = \sum_{i=1}^{I}\sum_{k=1}^{K} F_{ik}(x)/IK$. Thus $\sigma^2 > 0$ exists and using the assumption of a bounded second derivative for ϕ, it follows that each $S_{.j..}/\sqrt{N}$, $j = 1, \ldots, J$, converges in distribution to a normal random variable. Namely $Var(S_{.j..}) = N\sigma^2 > 0 \to \infty$ as $N \to \infty$, which entails $\{S_{.j..}/\sqrt{N} - E(S_{.j..}/\sqrt{N})\}/\sigma \xrightarrow{d} N(0,1)$.

Further consider the vector $\underline{S}^{o'} = [S_{.1..} - \bar{S}, \ldots, S_{.J-1..} - \bar{S}]$. Then the covariance matrix of $\underline{S}^o$ is $N\sigma^2\underline{C}$, where $\underline{C} = \| c_{j,j'} \|$, $c_{j,j'} = 1$ if $j = j'$ and $c_{j,j'} = -1/(J-1)$ if $j \neq j'$ for $j, j' = 1, \ldots, J-1$. It means that the diagonal elements of the covariance matrix of $\underline{S}^o$ are $N\sigma^2$ since the diagonal elements of $\underline{C}$ are 1 and that the off-diagonal elements of the covariance matrix of $\underline{S}^o$ are $-N\sigma^2/(J-1)$ since the off-diagonal elements of $\underline{C}$ are $-1/(J-1)$. In addition since $\underline{C}$ is positive definite symmetric, we have for any $J-1$ element column vector $\underline{\theta}$, $Var(\underline{\theta}'\underline{S}^o) = N\sigma^2\underline{\theta}'\underline{C}\ \underline{\theta} > 0$ so that $\underline{\theta}'\underline{S}^o$ satisfies asymptotic normality. In other words since $\underline{\theta}'\underline{S}^o$ is itself a linear rank statistic, every linear combination of $\underline{S}^o/\sqrt{N}$ converges to normality. Then *Hájek* and *Sidák's*(1967) Theorem V.2.1 may be applied to show that $\underline{S}^o/\sqrt{N}$ converges to a $J-1$ variate normal random vector.

Secondly suppose the quadratic form $Q^o(\underline{x}) = \underline{x}'\underline{C}^{-1}\underline{x}/\sigma^2$ with $\underline{x}$ being a $J-1$ variate normal random vector of mean $\underline{0}$ and covariance matrix $\sigma^2\underline{C}$. Then from Graybill's (1976) Theorem 4.4.3 $Q^o(\underline{x})$ has a chi-square distribution with $J-1$ degrees of freedom. Therefore the limiting distribution of $Q^o(\underline{S}^o/\sqrt{N})$ is chi-squared with $J-1$ degrees of freedom since $\underline{S}^o/\sqrt{N}$ converges in distribution to a $J-1$ variate normal random vector with covariance $\sigma^2\underline{C}$.

Finally Lemma 2 of Hora and Conover(1984) shows that $Q^o(\underline{S}^o/\sqrt{N})$ is identical to Q_N of (3.1) since

$$\begin{aligned} Q^o\left(\frac{\underline{S}^o}{\sqrt{N}}\right) &= \frac{\left(\frac{\underline{S}^o}{\sqrt{N}}\right)'\underline{C}^{-1}\left(\frac{\underline{S}^o}{\sqrt{N}}\right)}{\sigma^2} \\ &= \frac{1}{N}\cdot\frac{1}{\sigma^2}\cdot\underline{S}^{o'}\underline{C}^{-1}\underline{S}^o \\ &= \frac{1}{N}\cdot\frac{1}{\sigma^2}\cdot\frac{J-1}{J}\sum_{j=1}^{J}(S_{.j..}-\overline{S})^2 \\ &= \frac{(J-1)\sum_{j=1}^{J}(S_{.j..}-\overline{S})^2}{JN\sigma^2} \\ &= Q_N. \end{aligned}$$

Hence Q_N will converge in distribution to a chi-squared random variable with $J-1$ degrees of freedom.

The following two lemmas are necessary in the proof of Theorem 2 which will be provided next.

Lemma 1. Define $V_{ikN} = Var[a_M(R_{ijkn})]$ and $C_{ikN} = Cov[a_M(R_{ijkn}), a_M(R_{ij'kn'})]$, where $i = 1,\cdots,I$, $j,j' = 1,\cdots,J$, $k = 1,\cdots,K$, $n,n' = 1,\cdots,N$ and $(j,n) \neq (j',n')$. Then under the null hypothesis (2.1)

$$\sigma_N^2 = Var\left(\frac{S_{.j..}}{\sqrt{N}}\right) = \frac{J-1}{J}\sum_{i=1}^{I}\sum_{k=1}^{K}(V_{ikN} - C_{ikN}).$$

Proof. Primarily it can be shown that

$$Var(S_{.j..}) = \sum_{i=1}^{I}\sum_{k=1}^{K} Var(S_{ijk.}) + \sum_{\substack{i=1 \\ i\neq i'}}^{I}\ \sum_{\substack{i'=1 \\ or}}^{I}\ \sum_{\substack{k=1 \\ k\neq k'}}^{K}\ \sum_{k'=1}^{K} Cov(S_{ijk.}, S_{i'jk'.}), \tag{3.2}$$

$$Cov(S_{.j..}, S_{.j'..}) = \sum_{i=1}^{I}\sum_{k=1}^{K} Cov(S_{ijk.}, S_{ij'k.}) + \sum_{\substack{i=1 \\ i\neq i'}}^{I}\ \sum_{\substack{i'=1 \\ or}}^{I}\ \sum_{\substack{k=1 \\ k\neq k'}}^{K}\ \sum_{k'=1}^{K} Cov(S_{ijk.}, S_{i'j'k'.}). \tag{3.3}$$

After substituting the facts that $Var(S_{.j..}) = N\sigma_N^2$, $Cov(S_{.j..}, S_{.j'..}) = -N\sigma_N^2/(J-1)$, and $Cov(S_{ijk.}, S_{i'j'k'.}) = Cov(S_{ijk.}, S_{i'jk'.})$ under the null hypothesis into (3.2) and (3.3), we can obtain the following result by subtracting (3.3) from (3.2). Namely

$$\frac{J}{J-1}N\sigma_N^2 = \sum_{i=1}^{I}\sum_{k=1}^{K}[Var(S_{ijk.}) - Cov(S_{ijk.}, S_{ij'k.})]. \tag{3.4}$$

Meanwhile under the null hypothesis it follows that

$$\begin{aligned} Var(S_{i.k.}) &= \sum_{j=1}^{J} Var(S_{ijk.}) + \sum_{\substack{j=1 \\ j\neq j'}}^{J}\ \sum_{j'=1}^{J} Cov(S_{ijk.}, S_{ij'k.}) \\ &= JVar(S_{ijk.}) + J(J-1)Cov(S_{ijk.}, S_{ij'k.}). \end{aligned} \tag{3.5}$$

Summing (3.5) with respect to i and k gives

$$\sum_{i=1}^{I}\sum_{k=1}^{K} Var(S_{i.k.}) = J\sum_{i=1}^{I}\sum_{k=1}^{K} Var(S_{ijk.}) + J(J-1)\sum_{i=1}^{I}\sum_{k=1}^{K} Cov(S_{ijk.}, S_{ij'k.}). \tag{3.6}$$

Therefore adding (3.6) with (3.4) multiplied by $J(J-1)$ generates

$$N\sigma_N^2 = \sum_{i=1}^{I}\sum_{k=1}^{K} Var(S_{ijk.}) - \sum_{i=1}^{I}\sum_{k=1}^{K} Var(S_{i.k.})/J^2. \tag{3.7}$$

Since $Var(S_{ijk.}) = NV_{ikN} + N(N-1)C_{ikN}$ and $Var(S_{i.k.}) = JNV_{ikN} + JN(JN-1)C_{ikN}$, substituting these results into (3.7) simplifies

$$\sigma_N^2 = \frac{J-1}{J}\sum_{i=1}^{I}\sum_{k=1}^{K}(V_{ikN} - C_{ikN}).$$

Lemma 2. Let $A_{ijkn} = a_M(R_{ijkn})$, where $i = 1, \cdots, I$, $j = 1, \cdots, J$, $k = 1, \cdots, K$, $n = 1, \cdots, N$. And define $B_{ijk} = \sum_{n=1}^{N}(A_{ijkn} - \bar{A}_{ijk.})^2/(N-1)$, where $\bar{A}_{ijk.} = \sum_{n=1}^{N} A_{ijkn}/N$. Then under the null hypothesis (2.1)

$$E(B_{ijk}) = V_{ikN} - C_{ikN},$$

where V_{ikN} and C_{ikN} are as denoted in Lemma 1.

Proof. Let $B_{ijkn} = (A_{ijkn} - \bar{A}_{ijk.})^2$, then

$$\begin{aligned}
E(B_{ijkn}) &= E(A_{ijkn} - \bar{A}_{ijk.})^2 \\
&= E[\{(A_{ijkn} - EA_{ijkn}) - (\bar{A}_{ijk.} - E\bar{A}_{ijk.})\}^2] \\
&= Var(A_{ijkn}) + Var(\bar{A}_{ijk.}) - 2Cov(A_{ijkn}, \bar{A}_{ijk.}) \\
&= V_{ikN} + \frac{1}{N^2}[NV_{ikN} + N(N-1)C_{ikN}] - \frac{2}{N}[V_{ikN} + (N-1)C_{ikN}] \\
&= \frac{N-1}{N}(V_{ikN} - C_{ikN}).
\end{aligned}$$

So we can say that $E(B_{ijk}) = E[\sum_{n=1}^{N} B_{ijkn}/(N-1)] = V_{ikN} - C_{ikN}$.

Theorem 2. If the condition $\sum_{i=1}^{M}[a_M(i)]^2/M < k_0$, $M = IJKN$, is met, under the null hypothesis (2.1) the estimator

$$D_N = \frac{J-1}{J^2}\frac{1}{N-1}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{n=1}^{N}\left[a_M(R_{ijkn}) - \frac{S_{ijk.}}{N}\right]^2 \qquad (3.8)$$

is unbiased for $\sigma_N^2 = Var(S_{.j..}/\sqrt{N})$ and D_N converges in probability to σ^2 defined in Theorem 1 as N goes to infinity.

Proof. First the result of Lemma 1 states that

$$\sigma_N^2 = \frac{J-1}{J}\sum_{i=1}^{I}\sum_{k=1}^{K}(V_{ikN} - C_{ikN}).$$

By applying Lemma 2 we can establish that

$$\begin{aligned} E(D_N) &= \frac{J-1}{J^2}\frac{1}{N-1}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{n=1}^{N} E\left[a_M(R_{ijkn}) - \frac{S_{ijk.}}{N}\right]^2 \\ &= \frac{J-1}{J^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(V_{ikN} - C_{ikN}) \\ &= \frac{J-1}{J}\sum_{i=1}^{I}\sum_{k=1}^{K}(V_{ikN} - C_{ikN}). \end{aligned}$$

Hence $E(D_N) = \sigma_N^2$, which satisfies unbiasedness.

Secondly note that

$$\begin{aligned} D_N &= \frac{J-1}{J^2}\frac{1}{N-1}\left\{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{n=1}^{N}[a_M(R_{ijkn})]^2 - \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{n=1}^{N}\left(\frac{S_{ijk.}}{N}\right)^2\right\} \\ &= \frac{J-1}{J^2}\left\{\frac{\sum_{i=1}^{M}[a_M(i)]^2}{N-1} - \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{S_{ijk.}}{\sqrt{N(N-1)}}\right)^2\right\}. \end{aligned} \qquad (3.9)$$

Now by examining Lemma 5 of Hora and Conover(1984), we can derive the following asymptotic property that

$$\lim_{N\to\infty} D_N = \frac{J-1}{J^2}\left\{IJK\int_0^1[\phi(t)]^2dt - \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left[\int\phi[H(x)]dF_{ijk}(x)\right]^2\right\}, \qquad (3.10)$$

where $H(x) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} F_{ijk}(x)/(IJK)$. So D_N converges in probability to a constant. For instance in the case of ranks where $\phi(t) = t$ and $\phi[H(x)] = H(x)$, the right hand side of equation (3.10) becomes $[(J-1)/J^2]\cdot\{IJK/3 - \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}[\int HdF_{ijk}]^2\}$ which is obviously a constant. In particular $0 < D_N < [(J-1)IJKNk_0]/[J^2(N-1)]$ because from $\sum_{i=1}^{M}[a_M(i)]^2/M < k_0$ the first term of equation (3.9) can be reexpressed as

$$\begin{aligned} &\frac{J-1}{J^2(N-1)}\frac{\sum_{i=1}^{M}[a_M(i)]^2}{M} < \frac{J-1}{J^2(N-1)}k_0 \\ \Rightarrow\quad &\frac{J-1}{J^2(N-1)}\sum_{i=1}^{M}[a_M(i)]^2 < \frac{(J-1)IJKN}{J^2(N-1)}k_0. \end{aligned}$$

Finally when we combine the above results, it suffices to say that D_N converges in probability to σ^2.

Theorem 3. If the score function ϕ has a bounded second derivative and rank scores $a_M(R_{ijkn})$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ and $n = 1, \ldots, N$, are generated by (2.2) or (2.3), then under the null hypothesis (2.1) the statistic χ^2_N converges in distribution to a chi-squared random variable with $J-1$ degrees of freedom divided by $J-1$.

Proof. From (3.1) and (3.8), it follows that $\chi^2_N = [\sigma^2 Q_N]/[D_N(J-1)]$. In addition Theorem 1 reveals that Q_N has a limiting distribution that is chi-squared with $J-1$ degrees of freedom, whereas Theorem 2 indicates that D_N converges in probability to a constant σ^2 defined in Theorem 1. Consequently χ^2_N converges in distribution to a chi-squared random variable with $J-1$ degrees of freedom divided by $J-1$. Simply

$$\chi^2_N = \frac{\sigma^2 \cdot Q_N}{D_N \cdot (J-1)} \to \frac{\sigma^2 \cdot \chi^2_{J-1}}{\sigma^2 \cdot (J-1)} \to \frac{\chi^2_{J-1}}{J-1}.$$

4. SIMULATION RESULTS

To explore the properties of the proposed test statistic, a simulation was conducted. The purpose of this simulation study is to investigate robustness and power properties between the usual parametric analysis of variance test and the rank transform test. Consider a balanced $2 \times 2 \times 2$ fixed effects linear model :

$$\begin{aligned} X_{ijkn} &= \mu + \alpha_i + \beta_j + \gamma_k + \epsilon_{ijkn}, \\ & \quad \text{for } i,\ j,\ k = 1, 2, \quad n = 1, 2, \ldots, N, \end{aligned}$$

where μ, representing the overall mean, is equal to 0; the nuisance parameter α_i represents the main effect of the ith level of factor A; β_j the effect of the jth level of factor B; γ_k the effect of the kth level of factor C; and ϵ_{ijkn} represents independent observation sampled from standard normal, exponential, double exponential and uniform distributions. The process was carried for $N = 2, 4, 10, 20, 30, 50$ observations per cell. However results for $N = 2, 4$ are presented here since increases in sample size beyond $N = 10$ contribute to fast power inflations near to 1 in almost all situations so that we can not sharply contrast the difference of power for tests under the given conditions.

To test for the null hypothesis of no main effects $H_0 : \beta_j = 0$ for all j, F and FR will denote the usual ANOVA test statistic and F test on the ranks of the data respectively. The A, B and C main effects are formed by setting $\alpha_1 = c$, $\alpha_2 = -c$, $\beta_1 = c$, $\beta_2 = -c$ and $\gamma_1 = c$, $\gamma_2 = -c$ respectively, where c takes the values from 0.25 to 1.50 by 0.25. The procedure of this study is to draw random deviates first, then add constants corresponding to the effect size to yield the given effects. Next the usual ANOVA F statistic is computed. After comparing the critical values at the significance level of 0.05, we count the proportion of rejections for the given conditions of effets. Finally each sample is ranked in overall and the procedure is repeated. For each situation 10,000 repetitions are employed in the experiment. The pseudo random deviates for standard normal, exponential, double exponential and uniform distributions are generated using C programs based on the Box-Muller transformation(Box and Muller 1958), the inverse transform method(Marsaglia 1961), the composition method and the linear congruential method(Law and Kelton 1991) respectively.

For the robustness analysis, four different situations are considered. (1) All effects are null; $\alpha = \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. (2) One main effect is nonnull; $\alpha = c$, $\gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. (3) Two main effects are nonnull; $\alpha = \gamma = c$, $\alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. (4) Another construction in which two main effects are nonnull; $\alpha = 1.5c$, $\gamma = 0.5c$, $\alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. Meanwhile for the power analysis, another four different situations are considered. (1) One main effect is nonnull; $\beta = c$, $\alpha = \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. (2) Two main effects are nonnull; $\alpha = \beta = c$, $\gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. (3) Three main effects are nonnull; $\alpha = \beta = \gamma = c$, $\alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. (4) Another construction in which three main effects are nonnull; $\alpha = 1.5c$, $\beta = c$, $\gamma = 0.5c$, $\alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$. The primary points of focus in this study are (a) types of population distributions, (b) number of observations per cell, (c) effect size and (d) number of nonnull effects in the model, in order to provide a more rigorous examination of Type I error and power of the rank transformed ANOVA test in the context of a $2 \times 2 \times 2$ fixed effects design.

Table 1 and Figure 1 represent that Type I error rates are very similar in all situations regardless of types of tests and populations, sample size, effect size and the fashion in which main effects are constructed. Especially as can be expected, the parametric F test and the rank transformed FR test produce a little conservative error rates under the double exponential distribution and a little liberal error rates under the uniform distribution. However the degree of conservativeness and liberality is very slight so that we can say the parametric F test and the rank transformed FR test are robust without regard to types of populations. This result agrees with that of Sawilowsky, Blair and Higgins(1989).

In general Table 2 and Figure 2 show that powers of FR test are high as compared with those of F test. Particularly the gain of powers of FR test becomes high in heavy tailed distributions such as exponential and double exponential ones. Further note that increases in effect size and sample size are associated with increases in powers of FR test.

It is also found that although increase in the number of main effects contributes to power reduction of FR test, the difference is very slight. Moreover when three main effects exist, the manner in which main effects are constructed does not seem to affect the power of FR test. For reference powers of tests for $N = 1$ also indicate very similar pattern like Table 2 and have 60 percent level of powers for $N = 2$ on the average, even though results are not given here. Accordingly we can indicate that FR statistic appears to maintain much of its substantial power and is more desirable.

In essence our simulation study illustrates that rank transform test is robust with respect to significance level and has the superior power of testing main effects in a $2 \times 2 \times 2$ factorial experiment.

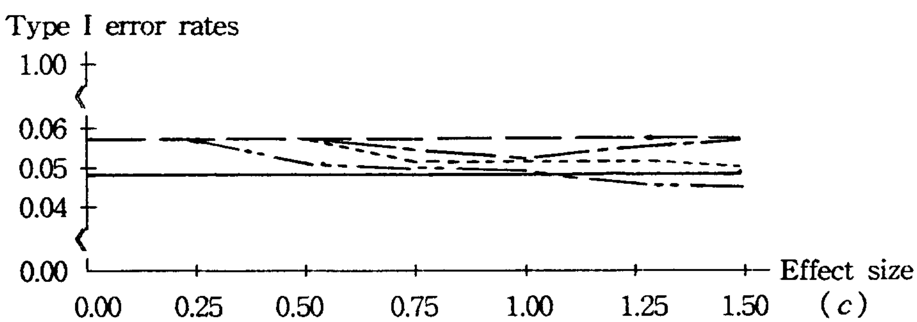

Figure 1. Estimated Type I error rates of test statistics for $n = 2$ when the error terms have exponential distribution

—— (0) all cases of F

— — (1) FR : $\alpha = \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

- - - - (2) FR : $\alpha = c,\ \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

—- (3) FR : $\alpha = \gamma = c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

—-- (4) FR : $\alpha = 1.5c,\ \gamma = 0.5c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

Table 1. Type I error rates of tests for main effects for
(0) all cases of F
(1) $\alpha = \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$
(2) $\alpha = c,\ \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$
(3) $\alpha = \gamma = c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$
(4) $\alpha = 1.5c,\ \gamma = 0.5c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

		Normal		Exponential		Double-Exp.		Uniform	
c	Statistic	$n=2$	$n=4$	$n=2$	$n=4$	$n=2$	$n=4$	$n=2$	$n=4$
0.25	(0) F	0.051	0.050	0.049	0.048	0.046	0.048	0.056	0.056
	(1) FR	0.053	0.052	0.056	0.055	0.053	0.053	0.055	0.054
	(2) FR	0.052	0.052	0.056	0.054	0.055	0.052	0.053	0.055
	(3) FR	0.051	0.049	0.056	0.056	0.053	0.053	0.057	0.059
	(4) FR	0.050	0.051	0.056	0.054	0.052	0.052	0.055	0.057
0.50	(0) F	0.051	0.050	0.049	0.048	0.046	0.048	0.056	0.056
	(1) FR	0.053	0.052	0.056	0.055	0.053	0.053	0.055	0.054
	(2) FR	0.051	0.050	0.056	0.053	0.051	0.051	0.052	0.053
	(3) FR	0.049	0.050	0.057	0.054	0.051	0.052	0.060	0.062
	(4) FR	0.048	0.049	0.051	0.054	0.050	0.051	0.049	0.053
0.75	(0) F	0.051	0.050	0.049	0.048	0.046	0.048	0.056	0.056
	(1) FR	0.053	0.052	0.056	0.055	0.053	0.053	0.055	0.054
	(2) FR	0.049	0.050	0.053	0.054	0.049	0.051	0.052	0.053
	(3) FR	0.050	0.050	0.055	0.054	0.050	0.049	0.060	0.062
	(4) FR	0.046	0.049	0.050	0.053	0.046	0.051	0.043	0.053
1.00	(0) F	0.051	0.050	0.049	0.048	0.046	0.048	0.056	0.056
	(1) FR	0.053	0.052	0.056	0.055	0.053	0.053	0.055	0.054
	(2) FR	0.049	0.050	0.052	0.052	0.048	0.051	0.052	0.053
	(3) FR	0.052	0.052	0.052	0.057	0.052	0.050	0.060	0.062
	(4) FR	0.045	0.050	0.049	0.049	0.044	0.051	0.044	0.055
1.25	(0) F	0.051	0.050	0.049	0.048	0.046	0.048	0.056	0.056
	(1) FR	0.053	0.052	0.056	0.055	0.053	0.053	0.055	0.054
	(2) FR	0.049	0.049	0.052	0.052	0.045	0.049	0.052	0.053
	(3) FR	0.054	0.052	0.055	0.057	0.049	0.049	0.060	0.062
	(4) FR	0.044	0.050	0.047	0.050	0.041	0.050	0.044	0.055
1.50	(0) F	0.051	0.050	0.049	0.048	0.046	0.048	0.056	0.056
	(1) FR	0.053	0.052	0.056	0.055	0.053	0.053	0.055	0.054
	(2) FR	0.052	0.050	0.050	0.052	0.046	0.048	0.052	0.053
	(3) FR	0.057	0.054	0.056	0.056	0.049	0.051	0.060	0.062
	(4) FR	0.042	0.048	0.046	0.050	0.041	0.047	0.044	0.055

Table 2. Power of tests for main effects for
(0) all cases of F
(1) $\beta = c,\ \alpha = \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$
(2) $\alpha = \beta = c,\ \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$
(3) $\alpha = \beta = \gamma = c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$
(4) $\alpha = 1.5c,\ \beta = c,\ \gamma = 0.5c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

		Normal		Exponential		Double-Exp.		Uniform	
c	Statistic	$n=2$	$n=4$	$n=2$	$n=4$	$n=2$	$n=4$	$n=2$	$n=4$
0.25	(0) F	0.145	0.272	0.177	0.319	0.106	0.173	0.869	0.999
	(1) FR	0.145	0.266	0.256	0.502	0.126	0.220	0.821	0.993
	(2) FR	0.143	0.265	0.234	0.458	0.121	0.215	0.803	0.995
	(3) FR	0.142	0.261	0.224	0.432	0.121	0.217	0.789	0.994
	(4) FR	0.142	0.262	0.215	0.420	0.117	0.210	0.798	0.995
0.50	(0) F	0.426	0.774	0.498	0.787	0.279	0.519	1.000	1.000
	(1) FR	0.420	0.756	0.597	0.910	0.323	0.609	1.000	1.000
	(2) FR	0.405	0.746	0.550	0.877	0.304	0.584	1.000	1.000
	(3) FR	0.399	0.742	0.526	0.856	0.294	0.574	1.000	1.000
	(4) FR	0.392	0.742	0.523	0.844	0.288	0.563	1.000	1.000
0.75	(0) F	0.756	0.982	0.776	0.967	0.514	0.825	1.000	1.000
	(1) FR	0.742	0.978	0.822	0.990	0.563	0.888	1.000	1.000
	(2) FR	0.717	0.975	0.785	0.984	0.527	0.861	1.000	1.000
	(3) FR	0.707	0.971	0.758	0.977	0.511	0.847	1.000	1.000
	(4) FR	0.706	0.972	0.771	0.978	0.506	0.844	1.000	1.000
1.00	(0) F	0.942	0.999	0.917	0.997	0.727	0.962	1.000	1.000
	(1) FR	0.935	0.999	0.927	0.999	0.762	0.983	1.000	1.000
	(2) FR	0.914	0.999	0.907	0.998	0.727	0.972	1.000	1.000
	(3) FR	0.903	0.999	0.889	0.996	0.709	0.965	1.000	1.000
	(4) FR	0.911	0.999	0.908	0.998	0.709	0.963	1.000	1.000
1.25	(0) F	0.993	1.000	0.972	1.000	0.872	0.994	1.000	1.000
	(1) FR	0.991	1.000	0.972	1.000	0.891	0.998	1.000	1.000
	(2) FR	0.985	1.000	0.964	1.000	0.863	0.996	1.000	1.000
	(3) FR	0.976	1.000	0.952	1.000	0.840	0.993	1.000	1.000
	(4) FR	0.982	1.000	0.962	1.000	0.850	0.994	1.000	1.000
1.50	(0) F	1.000	1.000	0.992	1.000	0.948	0.999	1.000	1.000
	(1) FR	1.000	1.000	0.991	1.000	0.953	1.000	1.000	1.000
	(2) FR	0.999	1.000	0.986	1.000	0.938	0.999	1.000	1.000
	(3) FR	0.995	1.000	0.979	1.000	0.920	0.999	1.000	1.000
	(4) FR	0.998	1.000	0.988	1.000	0.929	0.999	1.000	1.000

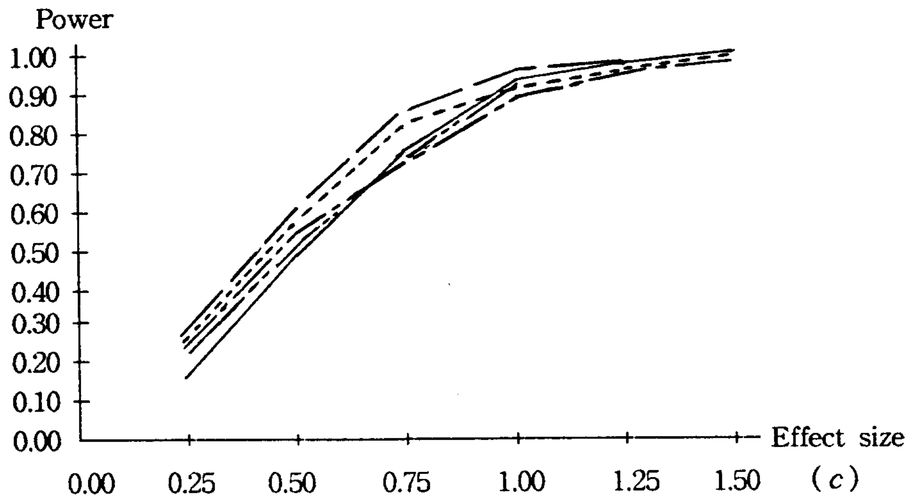

Figure 2. Estimated powers of test statistics for $n = 2$ when the error terms have exponential distribution

(0) all cases of F

(1) FR : $\beta = c,\ \alpha = \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

(2) FR : $\alpha = \beta = c,\ \gamma = \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

(3) FR : $\alpha = \beta = \gamma = c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

(4) FR : $\alpha = 1.5c,\ \beta = c,\ \gamma = 0.5c,\ \alpha\beta = \alpha\gamma = \beta\gamma = \alpha\beta\gamma = 0$

5. CONCLUSIONS

The asymptotic null distribution of F statistic applied to the scores based on ranks for main effects in a three-way layout without interactions is revealed to have the same asymptotic distribution with the parametric F statistic. In conclusion it can be said that under the null hypothesis (2.1) of no main effects, the rank-score transformed F statistic for main effects in a three-way layout converges in distribution to a chi-squared random variable with $J - 1$ degrees of freedom divided by $J - 1$.

The results of the simulation study support the conclusion that the rank transform procedure has definite advantages over the parametric procedure. In other words the rank transform test is quite robust with respect to significance level and powerful than the standard parametric test procedure.

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