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The General Moment of Non-central Wishart Distribution

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Abstract

We obtain the general moment of non-central Wishart distribution, using the J-th moment of a matrix quadratic form and the $2J$-th moment of the matrix normal distribution. As an example, the second moment and kurtosis of non-central Wishart distribution are also investigated.

Key Words : Kronecker product; Commutation matrix; Vec operator; Moments of matrix; Quadratic form; Non-central Wishart distribution.

1. INTRODUCTION

Let random matrix Y $(p \times n)$ be distributed as $N_{p,n}(\mu, \Sigma, \Phi)$ $(\mu \neq 0)$, where Σ and Φ are $(p \times p)$ and $(n \times n)$ positive semidefinite matrices, respectively (in fact, $E(Y) = \mu$ and $Cov(vecY, vecY) = \Phi \otimes \Sigma$). For $\Phi = I_n$, YY' is said to have a non-central Wishart distribution with scale matrix Σ and degrees

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of freedom parameter n, where I_n be an $(n \times n)$ identity matrix. Later X denotes Y when $\mu = 0$. Von Rosen (1988) obtained the moments of arbitrary order of matrix X, and using these calculated the second order moment of the quadratic form where A is an $(n \times n)$ arbitrary non-random matrix. Neudecker and Wansbeek (1987) also obtained the second order moment of YAY' by calculating the expectation of $XAX'CXBX'$, where $X = Y - \mu$ and A, B, and C are arbitrary non-random matrices. Tracy and Sultan (1993) obtained the third order moment of XAX' using the sixth moment of matrix X. Kang and Kim (1995) obtained the J-th moment of matrix quadratic form using the $2J$-th moment of matrix X.

In this article, we improve results of Kang and Kim (1995), and present the general moment of non-central Wishart distribution using the J-th moment of matrix quadratic form and the main results(Theorem 3.2 and Theorem 3.3) of von Rosen (1988).

In Section 2, we summarize some preliminary results for commutation matrix and vec operator. The theorems and corollarys concerning the J-th moment of XAX' are presented in Section 3. The J-th moment of non-central Wishart distribution is obtained in Section 4. Also, in Example 4.1, we establish the complete form of second moment of non-central Wishart distribution.

2. SOME KNOWN RESULTS

Definition 1. Let $A = (a)_{ij}$ be an $(m \times n)$ matrix, and $B = (b)_{kl}$ be a $(p \times q)$ matrix. Then the Kronecker product $A \otimes B$ of A and B is defined as

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

It is an $(mp \times nq)$ matrix.

Definition 2. For a $(p \times n)$ matrix A, let $vecA$ denote the pn-vector obtained by '**vectorizing**' A; that is, $vecA = [a_1', a_2', \cdots, a_n']'$ if $A = [a_1, a_2, \cdots, a_n]$, where a_i is a p-vector.

Commutation matrix $I_{m,n}$ is an $(mn \times mn)$ matrix containing mn blocks of order $(m \times n)$ such that the (ij)th block has an 1 in its (ji)th position and zeroes elsewhere. One has

$$I_{m,n} = \sum_{i=1}^{n}\sum_{j=1}^{m}(H_{ij} \otimes H'_{ij}),$$

where H_{ij} is an $(n \times m)$ matrix with a 1 in its (ij)th position and zeroes elsewhere, and can be written as $H_{ij} = \mathbf{e_i e_j'}$, where $\mathbf{e_i}(\mathbf{e_j})$ is the $i(j)$th unit column vector of order $n(m)$. For example,

$$I_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Some preliminary results of commutation matrix, Kronecker product and vec operator are the following:

$$I_m \otimes I_n = I_{mn}, \tag{2.1}$$

where I_m is an $(m \times m)$ identity matrix.

$$I_{m,n}^{-1} = I'_{m,n} = I_{n,m}. \tag{2.2}$$

$$\begin{aligned}&(I_n \otimes I_{n,n} \otimes I_n)(I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n)\\ &= (I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n)(I_n \otimes I_n \otimes I_{n,n}).\end{aligned} \tag{2.3}$$

For $(m \times n)$ matrix A,

$$I_{n,m} vecA = vecA'. \tag{2.4}$$

For A and B conformable matrices,

$$(A \otimes B)' = A' \otimes B'. \tag{2.5}$$

$$trI_{m,n} = 1 + d(m-1, n-1), \tag{2.6}$$

where $d(m,n)$ is the greatest common divisor of m and n $(d(n,0) = d(0,n) = n)$.

For A and B conformable matrices,

$$vec'A'vecB = tr(AB). \tag{2.7}$$

For $(m \times n)$ matrix A and $(p \times q)$ matrix B,

$$vec(A \otimes B) = (I_n \otimes I_{m,q} \otimes I_p)(vecA \otimes vecB). \tag{2.8}$$

For A, B, and C conformable matrices,

$$vec(ACB) = (B' \otimes A)vecC. \tag{2.9}$$

For A, B, C, and D conformable matrices,

$$(AB) \otimes (CD) = (A \otimes C)(B \otimes D). \tag{2.10}$$

For $(m \times n)$ matrix A, $(p \times q)$ matrix B, and $(r \times s)$ matrix C,

$$vec(A \otimes B \otimes C) = (I_n \otimes I_{m,qs} \otimes I_{pr})(I_{mnq} \otimes I_{p,s} \otimes I_r)(vecA \otimes vecB \otimes vecC). \tag{2.11}$$

For detailed proofs, see Magnus and Neudecker (1979), Neudecker and Wansbeek (1983), and Tracy and Sultan (1993).

Neudecker and Wansbeek (1983) have discussed higher order commutation matrices, for which

$$\begin{aligned} I_{xy,z} &= (I_{x,z} \otimes I_y)(I_x \otimes I_{y,z}) = (I_{y,z} \otimes I_x)(I_y \otimes I_{x,z}), \\ I_{x,yz} &= (I_y \otimes I_{x,z})(I_{x,y} \otimes I_z) = (I_z \otimes I_{x,y})(I_{x,z} \otimes I_y). \end{aligned} \tag{2.12}$$

For $X \sim N_{p,n}(0, \Sigma, \Phi)$, the $2J$-th moment is (von Rosen, 1988)

$$\begin{aligned} E(\otimes^{2J} X) &= \sum_{j=0}^{J-1} \sum_{2 \le i_j \le 2J-2j} H(p, J, i_0, \ldots, i_{J-1}) \\ &\quad \times (\otimes^J (vec\Sigma vec'\Phi)) H(n, J, i_0, \ldots, i_{J-1})', \quad J = 1, 2, . \end{aligned} \tag{2,13}$$

or

$$E(\otimes^{2J} X) = \sum_{i=2}^{2J} P(p, i, 2J)\Big(vec\Sigma vec'\Phi \otimes E(\otimes^{2J-2} X)\Big)P(n, i, 2J)', \tag{2.14}$$

where $E(\otimes^0 X) = 1$,

$$H(a, J, i_0, \ldots, i_{J-1}) = \prod_{k=0}^{J-1} (I_{a^{2k}} \otimes P(a, i_k, 2J - 2k))$$

and

$$P(a, b, c) = ((I_{a^{b-2},a} \otimes I_a) I_{a^2,a^{b-2}}) \otimes I_{a^{c-b}}.$$

3. THE J-TH MOMENT OF XAX'

For the proofs of the Thoerem 1 and Theorem 2, we need some notations.

Notation 1. $p_0 = q_0 = 1$,

$$pq(i) = \Big(\prod_{k=0}^{i-1} p_k q_k\Big) q_i, \quad p(J;k) = \prod_{i=1}^{J} p_i / \prod_{j=1}^{k} p_j, \quad q(J;k) = \prod_{i=1}^{J} q_i / \prod_{j=1}^{k} q_j.$$

Notation 2.

$$P_{a;b,c;d} = I_{p^a} \otimes I_{p^b,p^c} \otimes I_{p^d}, \quad N_{a;b,c;d} = I_{n^a} \otimes I_{n^b,n^c} \otimes I_{n^d}.$$

Theorem 1. Let $B_1, B_2, \ldots, B_J$ be the set of matrices, where the size of B_j is $(p_j \times q_j)$. Then

$$vec(\otimes_{i=1}^{J} B_i) = \Big[\prod_{j=1}^{J-1} (I_{pq(j)} \otimes I_{p_j, q(J;j)} \otimes I_{p(J;j)})\Big] \Big(\otimes_{i=1}^{J} vec B_i\Big).$$

Proof. See the proof of Theorem 3.2 in Kang and Kim (1995).

Corollary 1. Let $C_1, C_2, \ldots, C_J$ be the set of $(p \times p)$ matrices. Then

$$vec(\otimes_{i=1}^{J} C_i) = \Big(\prod_{j=1}^{J-1} P_{2j-1;1,J-j;J-j}\Big)\Big(\otimes_{i=1}^{J} vec C_i\Big).$$

Proof. See the proof of Corollary 3.3 in Kang and Kim (1995).

Lemma 1.

$$P(p,i,J) = P_{1;1,i-2;J-i},$$
$$P(n,i,J)' = N_{1;i-2,1;J-i},$$
$$H(p,J,i_0,\ldots,i_{J-1}) = \prod_{k=0}^{J-1} P_{2k+1;1,i_k-2;2J-2k-i_k}$$

and

$$H(n,J,i_0,\ldots,i_{J-1})' = \prod_{k=0}^{J-1} N_{2J-2k-1;i_{J-1-k}-2,1;2k+2-i_{J-1-k}}.$$

Proof.

$$\begin{aligned} P(p,i,J) &= ((I_{p^{i-2},p} \otimes I_p) I_{p^2,p^{i-2}}) \otimes I_{p^{J-i}} \\ &= ((I_{p^{i-2},p} \otimes I_p)(I_{p,p^{i-2}} \otimes I_p)(I_p \otimes I_{p,p^{i-2}})) \otimes I_{p^{J-i}} \\ &= I_p \otimes I_{p,p^{i-2}} \otimes I_{p^{J-i}}, \end{aligned}$$

using (2.12) and (2.2). Also,

$$\begin{aligned} P(n,i,J)' &= (I_n \otimes I_{n,n^{i-2}} \otimes I_{n^{J-i}})' \\ &= I_n \otimes I_{n^{i-2},n} \otimes I_{n^{J-i}}, \end{aligned}$$

by (2.5) and (2.2).
Applying the above results and (2.1),

$$\begin{aligned} H(p,J,i_0,\ldots,i_{J-1}) &= \prod_{k=0}^{J-1} (I_{p^{2k}} \otimes P(p,i_k,2J-2k)) \\ &= \prod_{k=0}^{J-1} (I_{p^{2k}} \otimes P_{1;1,i_k-2;2J-2k-i_k}) \\ &= \prod_{k=0}^{J-1} (I_{p^{2k}} \otimes (I_p \otimes I_{p,p^{i_k-2}} \otimes I_{p^{2J-2k-i_k}})) \\ &= \prod_{k=0}^{J-1} (I_{p^{2k+1}} \otimes I_{p,p^{i_k-2}} \otimes I_{p^{2J-2k-i_k}}). \end{aligned}$$

Similarly,

$$H(n,J,i_0,\ldots,i_{J-1})' = \prod_{k=0}^{J-1} (I_{n^{2J-2k-1}} \otimes I_{n^{i_{J-1-k}-2},n} \otimes I_{n^{2k+2-i_{J-1-k}}}),$$

using (2.5) and (2.2).

Theorem 2. The J-th moment of matrix quadratic form XAX' is

$$vecE(\otimes^J XAX') = \left(\textstyle\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\right) \times \left[\sum_{i=2}^{2J} P_{1;1,i-2;2J-i}(vec\Sigma vec'\Phi \otimes E(\otimes^{2J-2} X))N_{1;i-2,1;2J-i}\right]\left(\otimes^J vecA\right), \quad (3.1)$$

or

$$\begin{aligned} vecE(\otimes^J XAX') = &\left(\textstyle\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\right) \\ \times &\left[\sum_{j=0}^{J-1} \sum_{k_j=2}^{2J-2j} \left(\prod_{h=0}^{J-1} P_{2h+1;1,k_h-2;2J-2h-k_h}\right)\left(\otimes^J (vec\Sigma vec'\Phi)\right)\right. \\ \times &\left.\left(\prod_{h=0}^{J-1} N_{2J-2h-1;k_{J-1-h}-2,1;2h+2-k_{J-1-h}}\right)\right]\left(\otimes^J vecA\right). \quad (3.2) \end{aligned}$$

Proof. As a result of Theorem 3.4 in Kang and Kim (1995),

$$vecE(\otimes_{i=1}^{J} XAX') = \left[\prod_{d=1}^{J-1}(I_{p^{2d-1}} \otimes I_{p,p^{J-d}} \otimes I_{p^{J-d}})\right]$$
$$\times \quad \left[\sum_{j=2}^{2J}(I_p \otimes I_{p,p^{j-2}} \otimes I_{p^{2J-j}})\left(vec\Sigma vec'\Phi \otimes E(\otimes^{2J-2}X)\right)\right.$$
$$\times \quad \left.(I_n \otimes I_{n^{j-2},n} \otimes I_{n^{2J-j}})\right]\left(\otimes^J vecA\right).$$

By virtue of Notation 2,

$$vecE(\otimes_{i=1}^{J} XAX') = \left(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\right)$$
$$\times \left[\sum_{j=2}^{2J} P_{1;1,j-2;2J-j}\left(vec\Sigma vec'\Phi \otimes E(\otimes^{2J-2}X)\right)N_{1;j-2,1;2J-j}\right]\left(\otimes^J vecA\right). \quad (3.3)$$

Using (2.14), (3.3) becomes (3.1). Also,

$$E(\otimes^{2J-2X}) = \sum_{j=2}^{2J-2} P_{1;1,j-2;2J-2-j}(vec\Sigma vec'\Phi \otimes E(\otimes^{2J-4}X))N_{1;j-2,1;2J-2-j}.$$

Then (3.3) is

$$vecE(\otimes^J XAX') = \left(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\right)\left[\sum_{i=2}^{2J} P_{1;1,i-2;2J-i} \times\right.$$
$$\left\{vec\Sigma vec'\Phi \otimes \left(\sum_{j=2}^{2J-2} P_{1;1,j-2;2J-2-j}(vec\Sigma vec'\Phi \otimes\right.\right.$$
$$\left.\left.\left.E(\otimes^{2J-4}X))N_{1;j-2,1;2J-2-j}\right)\right\}N_{1;i-2,1;2J-i}\right]\left(\otimes^J vecA\right)$$
$$= \left(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\right)\left[\sum_{i=2}^{2J}\sum_{j=2}^{2J-2}\left\{P_{1;1,i-2;2J-i}P_{3;1,j-2;2J-2-j} \times\right.\right.$$
$$\left.\left.((\otimes^2 vec\Sigma vec'\Phi) \otimes E(\otimes^{2J-4}X))N_{1;j-2,1;2J-2-j}N_{1;i-2,1;2J-i}\right\}\right]$$
$$\times\left(\otimes^J vecA\right),$$

using (2.10). We show that (3.3) becomes (3.2), by recursive manner.

Corollary 2. The J-th moment of Wishart distribution is

$$vecE(\otimes^J XX') = \left(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\right)$$
$$\times \Big[\sum_{i=2}^{2J} P_{1;1,i-2;2J-i}(vec\Sigma vec' I_n \otimes E(\otimes^{2J-2} X))N_{1;i-2,1;2J-i}\Big]\left(\otimes^J vecI_n\right), \quad (3.4)$$

or

$$vecE(\otimes^J XX') = \left(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\right)$$
$$\times\Bigg[\sum_{j=0}^{J-1}\sum_{k_j=2}^{2J-2j}\left(\prod_{h=0}^{J-1} P_{2h+1;1,k_h-2;2J-2h-k_h}\right)\left(\otimes^J(vec\Sigma vec' I_n)\right)$$
$$\times\left(\prod_{h=0}^{J-1} N_{2J-2h-1;k_{J-1-h}-2,1;2h+2-k_{J-1-h}}\right)\Bigg]\left(\otimes^J vecI_n\right).$$

Proof. Substituting I_n for A and Φ in Theorem 2, it is clear. This result is the vectorizing of general moment of Wishart distribution.
For the calculation of the general moment of non-central Wishart distribution, we need Corollary 2.

4. THE GENERAL MOMENT OF NON-CENTRAL WISHART DISTRIBUTION

Let random matrix Y $(p \times n)$ be distributed as $N_{p,n}(\mu, \Sigma, I_n)$ $(\mu \neq 0)$. We obtain the expectation of vectorizing of $\otimes^J YY'$. Let $M_J(YY') = Evec(\otimes^J YY')$, and, $Y = X + \mu$. Then

$$\begin{aligned} M_J(YY') &= Evec(\otimes^J YY') \\ &= Evec(\otimes^J (X+\mu)(X+\mu)') \\ &= Evec(\otimes^J (XX' + X\mu' + \mu X' + \mu\mu')) \\ &= Evec((\otimes^J XX') + (\otimes^J X\mu') + (\otimes^J \mu X') + (\otimes^J \mu\mu')) \\ &= Evec(\otimes^J XX') + Evec(\otimes^J X\mu') + Evec(\otimes^J \mu X') + Evec(\otimes^J \mu\mu') \end{aligned}$$

Using (2.5), we get

$$\begin{aligned} M_J(YY') &= M_J(XX') + Evec((\otimes^J X)(\otimes^J \mu')) \\ &+ Evec((\otimes^J \mu)(\otimes^J X')) + vec((\otimes^J \mu)(\otimes^J \mu')). \end{aligned} \quad (4.1)$$

By Corollary 2,

$$M_J(XX') = \Big(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\Big)E(\otimes^{2J} X)\left(\otimes^J vecI_n\right). \quad (4.2)$$

Now,

$$\begin{aligned}
&vec((\otimes^J X)(\otimes^J \mu')) \\
&\quad = vec((\otimes^J X) I_{n^J} (\otimes^J \mu')) \\
&\quad = ((\otimes^J \mu) \otimes (\otimes^J X)) vec I_{n^J} \\
&\quad = ((\otimes^J \mu) \otimes (\otimes^J X)) \Big(\prod_{d=1}^{J-1} N_{2d-1;J-d,1;J-d} \Big) \Big(\otimes^J vec I_n \Big),
\end{aligned}$$

using (2.5), (2.9) and Corollary 3.4. Hence

$$\begin{aligned}
&Evec((\otimes^J X)(\otimes^J \mu')) \\
&\quad = \Big[(\otimes^J \mu) \otimes (E(\otimes^J X)) \Big] \Big(\prod_{d=1}^{J-1} N_{2d-1;J-d,1;J-d} \Big) \Big(\otimes^J vec I_n \Big). \quad (4.3)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&Evec((\otimes^J \mu)(\otimes^J X')) \\
&\quad = \Big[(E(\otimes^J X)) \otimes (\otimes^J \mu) \Big] \Big(\prod_{d=1}^{J-1} N_{2d-1;J-d,1;J-d} \Big) \Big(\otimes^J vec I_n \Big). \quad (4.4)
\end{aligned}$$

Also,

$$vec((\otimes^J \mu)(\otimes^J \mu')) = (\otimes^{2J} \mu) \Big(\prod_{d=1}^{J-1} N_{2d-1;J-d,1;J-d} \Big) \Big(\otimes^J vec I_n \Big). \quad (4.5)$$

Substituting (4.2), (4.3), (4.4) and (4.5) for the right-hand side of (4.1), (4.1) is

$$\begin{aligned}
&\Big(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d} \Big) E(\otimes^{2J} X) \Big(\otimes^J vec I_n \Big) \\
&\quad + \Big[(\otimes^J \mu) \otimes (E(\otimes^J X)) \Big] \Big(\prod_{d=1}^{J-1} N_{2d-1;J-d,1;J-d} \Big) \Big(\otimes^J vec I_n \Big) \\
&\quad + \Big[(E(\otimes^J X)) \otimes (\otimes^J \mu) \Big] \Big(\prod_{d=1}^{J-1} N_{2d-1;J-d,1;J-d} \Big) \Big(\otimes^J vec I_n \Big) \\
&\quad + (\otimes^{2J} \mu) \Big(\prod_{d=1}^{J-1} N_{2d-1;J-d,1;J-d} \Big) \Big(\otimes^J vec I_n \Big). \quad (4.6)
\end{aligned}$$

Now applying (3.4), (2.14) and Lemma 1, we get

$$
\begin{aligned}
& vecE(\otimes^J YY') \\
= \quad & Evec(\otimes^J YY') \\
= \quad & \Big(\prod_{d=1}^{J-1} P_{2d-1;1,J-d;J-d}\Big) \\
\times \quad & \Big[\sum_{i=2}^{2J} P_{1;1,i-2;2J-i}(vec\Sigma vec' I_n \otimes E(\otimes^{2J-2} X))N_{1;i-2,1;2J-i}\Big]\Big(\otimes^J vecI_n\Big) \\
+ \quad & \Big[(\otimes^J \mu) \otimes \Big(\sum_{i=2}^{J} P_{1;1,i-2;J-i}(vec\Sigma vec' I_n \otimes E(\otimes^{J-2} X))N_{1;i-2,1;J-i}\Big)\Big] \\
\times \quad & \Big(\prod_{d=1}^{J-1} N_{2d-1;1,J-d;J-d}\Big)\Big(\otimes^J vecI_n\Big) \\
+ \quad & \Big[\Big(\sum_{i=2}^{J} P_{1;1,i-2;J-i}(vec\Sigma vec' I_n \otimes E(\otimes^{J-2} X))N_{1;i-2,1;J-i}\Big) \otimes (\otimes^J \mu)\Big] \\
\times \quad & \Big(\prod_{d=1}^{J-1} N_{2d-1;1,J-d;J-d}\Big)\Big(\otimes^J vecI_n\Big) \\
+ \quad & (\otimes^{2J} \mu)\Big(\prod_{d=1}^{J-1} N_{2d-1;1,J-d;J-d}\Big)\Big(\otimes^J vecI_n\Big).
\end{aligned}
$$

Example 1. (The second moment of non-central Wishart distribution)

Let $V = vec\Sigma vec' I_n$ $(p^2 \times n^2)$. For $J = 2$,

$$
\begin{aligned}
& vecE(\otimes^2 YY') \\
& = P_{1;1,1;1}[P_{1;1,0;2}(vec\Sigma vec' I_n \otimes vec\Sigma vec' I_n)N_{1;0,1;2} \\
& \quad + P_{1;1,1;1}(vec\Sigma vec' I_n \otimes vec\Sigma vec' I_n)N_{1;1,1;1} \\
& \quad + P_{1;1,2;0}(vec\Sigma vec' I_n \otimes vec\Sigma vec' I_n)N_{1;2,1;0}](vecI_n \otimes vecI_n) \\
& \quad + [(\mu \otimes \mu) \otimes (P_{1;1,0;0} vec\Sigma vec' I_n N_{1;0,1;0})]N_{1;1,1;1}(vecI_n \otimes vecI_n) \\
& \quad + [(P_{1;1,0;0} vec\Sigma vec' I_n N_{1;0,1;0}) \otimes (\mu \otimes \mu)]N_{1;1,1;1}(vecI_n \otimes vecI_n) \\
& \quad + (\mu \otimes \mu \otimes \mu \otimes \mu)N_{1;1,1;1}(vecI_n \otimes vecI_n) \\
& = (I_p \otimes I_{p,p} \otimes I_p)[(V \otimes V) + (I_p \otimes I_{p,p} \otimes I_p)(V \otimes V)(I_n \otimes I_{n,n} \otimes I_n) \\
& \quad + (I_p \otimes I_{p,p^2})(V \otimes V)(I_n \otimes I_{n^2,n})](vecI_n \otimes vecI_n) \\
& \quad + (\mu \otimes \mu \otimes V)(I_n \otimes I_{n,n} \otimes I_n)(vecI_n \otimes vecI_n) \\
& \quad + (V \otimes \mu \otimes \mu)(I_n \otimes I_{n,n} \otimes I_n)(vecI_n \otimes vecI_n) \\
& \quad + (\mu \otimes \mu \otimes \mu \otimes \mu)(I_n \otimes I_{n,n} \otimes I_n)(vecI_n \otimes vecI_n)
\end{aligned}
$$

$$= ((V \otimes V) + (\mu \otimes \mu \otimes V) + (V \otimes \mu \otimes \mu) + (\mu \otimes \mu \otimes \mu \otimes \mu))vec(I_n \otimes I_n)$$
$$+(I_p \otimes I_{p,p} \otimes I_p)(V \otimes V)(vecI_n \otimes vecI_n)$$
$$+(I_p \otimes I_{p,p} \otimes I_p)(I_p \otimes I_{p,p^2})(V \otimes V)(I_n \otimes I_{n^2,n})(vecI_n \otimes vecI_n),$$

by (2.8).
We get the following results.

[1] $(V \otimes V)vec(I_n \otimes I_n) = vec(V(I_n \otimes I_n)V') = vec(VV'),$

using (2.9) and (2.1).

[2] $(\mu \otimes \mu \otimes V)vec(I_n \otimes I_n) = vec(V(I_n \otimes I_n)(\mu \otimes \mu)') = vec(V(\mu \otimes \mu)'),$

using (2.9) and (2.1).

[3] $(V \otimes \mu \otimes \mu)vec(I_n \otimes I_n) = vec((\mu \otimes \mu)(I_n \otimes I_n)V') = vec((\mu \otimes \mu)V'),$

using (2.9) and (2.1).

[4] $(\mu\otimes\mu\otimes\mu\otimes\mu)vec(I_n\otimes I_n) = vec((\mu\otimes\mu)(I_n\otimes I_n)(\mu\otimes\mu)') = vec(\mu\mu'\otimes\mu\mu'),$

using (2.9), (2.1) and (2.5).

[5] $(I_p \otimes I_{p,p} \otimes I_p)(V \otimes V)(vecI_n \otimes vecI_n)$

$$= (I_p \otimes I_{p,p} \otimes I_p)(vec\Sigma \otimes vec\Sigma)(vec'I_n \otimes vec'I_n)(vecI_n \otimes vecI_n)$$
$$= vec(\Sigma \otimes \Sigma)vec'I_{n^2}vecI_{n^2} = tr(I_{n^2}I_{n^2})vec(\Sigma \otimes \Sigma)$$
$$= n^2 vec(\Sigma \otimes \Sigma),$$

using (2.10), (2.8), (2.1) and (2.7).

[6]-1 $(I_p \otimes I_{p,p} \otimes I_p)(I_p \otimes I_{p,p^2})(vec\Sigma \otimes vec\Sigma)$

$$= (I_p \otimes I_{p,p} \otimes I_p)(I_p \otimes I_p \otimes I_{p,p})(I_p \otimes I_{p,p} \otimes I_p)(vec\Sigma \otimes vec\Sigma)$$
$$= (I_p \otimes I_p \otimes I_{p,p})(I_p \otimes I_{p,p} \otimes I_p)(I_p \otimes I_p \otimes I_{p,p})(vec\Sigma \otimes vec\Sigma)$$
$$= (I_p \otimes I_p \otimes I_{p,p})(I_p \otimes I_{p,p} \otimes I_p)(vec\Sigma \otimes vec\Sigma)$$
$$= (I_p \otimes I_p \otimes I_{p,p})vec(\Sigma \otimes \Sigma) = vec(I_{p,p}(\Sigma \otimes \Sigma)),$$

using (2.12), (2.10), (2.3), (2.4), (2.8) and (2.9).

[6]-2 $(vec'I_n \otimes vec'I_n)(I_n \otimes I_{n^2,n})(vecI_n \otimes vecI_n)$

$$= (vec'I_n \otimes vec'I_n)(I_n \otimes I_{n,n} \otimes I_n)(I_n \otimes I_n \otimes I_{n,n})(vecI_n \otimes vecI_n)$$
$$= (vec'I_n \otimes vec'I_n)(I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n)$$
$$\times(I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n)(vecI_n \otimes vecI_n)$$

$$\begin{aligned}
&= (vec'I_n \otimes vec'I_n)(I_n \otimes I_{n,n} \otimes I_n) \\
&\times (I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n)(vecI_n \otimes vecI_n) \\
&= vec'(I_n \otimes I_n)(I_{n^2} \otimes I_{n,n})vec(I_n \otimes I_n) \\
&= vec'(I_n \otimes I_n)vec(I_{n,n}I_{n^2}) \\
&= tr(I_{n^2}I_{n,n}I_{n^2}) = trI_{n,n} = n,
\end{aligned}$$

using (2.10), (2.12), (2.3), (2.10), (2.2), (2.4), (2.9) and (2.6).

Therefore,

$$\begin{aligned}
&vecE(\otimes^2 YY') \\
&\quad = vec(VV') + vec(V(\mu \otimes \mu)') + vec((\mu \otimes \mu)V') \\
&\qquad\qquad + vec(\mu\mu' \otimes \mu\mu') + n^2 vec(\Sigma \otimes \Sigma) + nvec(I_{p,p}(\Sigma \otimes \Sigma)) \\
&\quad = vec(VV' + V(\mu \otimes \mu)' + (\mu \otimes \mu)V' + (\mu\mu' \otimes \mu\mu') \\
&\qquad\qquad + n^2(\Sigma \otimes \Sigma) + nI_{p,p}(\Sigma \otimes \Sigma)).
\end{aligned}$$

Hence, the exact form of the second moment of non-central Wishart distribution is

$$\begin{aligned}
E(\otimes^2 YY') = VV' &+ V(\mu \otimes \mu)' + (\mu \otimes \mu)V' + (\mu\mu' \otimes \mu\mu') \\
&+ n^2(\Sigma \otimes \Sigma) + nI_{p,p}(\Sigma \otimes \Sigma),
\end{aligned}$$

where $V = vec\Sigma vec'I_n$.

Example 2. (The vectorizing of kurtosis of non-central Wishart distribution) When $J = 4$, (4.6) becomes

$$\begin{aligned}
&vecE(\otimes^4 YY') \\
&\quad = \Big(\prod_{d=1}^{3} P_{2d-1;1,4-d;4-d}\Big)\Big[\sum_{i=2}^{8}\sum_{j=2}^{6}\sum_{k=2}^{4} P_{1;1,i-2;8-i}P_{3;1,j-2;6-j}P_{5;1,k-2;4-k}(\otimes^4 V) \\
&\quad \times N_{5;k-2,1;4-k}N_{3;j-2,1;6-j}N_{1;i-2,1;8-i}\Big]\Big(\otimes^4 vecI_n\Big) \\
&\quad + \Big[(\otimes^4\mu) \otimes \Big(\sum_{i=2}^{4} P_{1;1,i-2;4-i}(\otimes^2 V)N_{1;i-2,1;4-i}\Big)\Big]\Big(\prod_{d=1}^{3} N_{2d-1;1,4-d;4-d}\Big)\Big(\otimes^4 vecI_n\Big) \\
&\quad + \Big[\Big(\sum_{i=2}^{4} P_{1;1,i-2;4-i}(\otimes^2 V)N_{1;i-2,1;4-i}\Big) \otimes (\otimes^4\mu)\Big]\Big(\prod_{d=1}^{3} N_{2d-1;1,4-d;4-d}\Big)\Big(\otimes^4 vecI_n\Big) \\
&\quad + (\otimes^8\mu)\Big(\prod_{d=1}^{3} N_{2d-1;1,4-d;4-d}\Big)\Big(\otimes^4 vecI_n\Big),
\end{aligned}$$

using Corollary 2 and (2.14).
Further, when $p = 1$, $M_J(YY')$ is the J-th moment of non-central χ^2– distribution with degrees of freedom n.

5. CONCLUSION

The importance of multivariate analysis in statistics is now widely recognized in the statistical community as its range of application grows rapidly. In studying multivariate analysis, however, we encounter some difficulties because in most cases multivariate version of some statements are not the direct generalizations of their univariate version, and because the calculations involved in multivariate analysis are extremely convoluted and sometimes impossible. For these reasons there are yet many theoretical questions to be answered in multivariate analysis. In this paper we, hoping to shed some light on such questions, get the general form of moments of non-central Wishart distribution whose univariate version is the non-central χ^2– distribution. Furthermore, we establish the complete formula of second moment of non-central Wishart distribution. This result can be applied usefully in studying the characteristics of non-central Wishart distribution such as mean, variance, skewness and kurtosis, just to name a few. Moreover, we think that the above results can be used in testing the covariance matrices in multivariate analysis and in estimation the distribution of parameter in multivariate regression analysis.

REFERENCES

(1) Kang, C. and Kim, B. C.(1996). The N-th Moment of Matrix Quadratic Form, *Statistics and Probability Letters*, **28**, 291-297.

(2) Magnus, J. R. and Neudecker, H. (1979). The Commutation Matrix: Some Properties and Applications, *The Annals of Statistics*, **7**, 381-394.

(3) Neudecker, H. and Wansbeek, T. (1983). Some Results on Commutation Matrices, with Statistical Applications, *The Canadian Journal of Statistics*, **11**, 221-231.

(4) Neudecker, H. and Wansbeek, T. (1987). Fourth-Order Properties of Normally Distributed Random Matrices, *Linear Algebra and its Applications*, **97**, 13-21.

(5) Rosen, D. von (1988). Moments for Matrix Normal Variables, *Statistics*, **19**, 575-583.

(6) Tracy, D. S. and Sultan, S. A. (1993). Third Moment of Matrix Quadratic Form, *Statistics and Probability Letters*, **16**, 71-76.