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Least Squares Approach for Structural Reanalysis†

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Abstract

A study is made of approximate technique for structural reanalysis based on the force method. Perturbation analysis of generalized least squares problem is adopted to reanalyze a damaged structure, and related results are presented.

Key Words : Generalized Least Squares Problem; Structural Reanalysis; Perturbation Theory.

1. INTRODUCTION

Given the external loads on a structure, the object of structural analysis is to determine the resulting internal forces, stresses, and displacements. The solution to this problem is provided by a variational principle (minimization of energy) subject to the linear elastic relationships among the nodes and

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elements of the finite element model of the structure, which can be stated as the quadratic programming problem

$$\text{Min } \frac{1}{2} f^T A f \text{ subject to } Ef = s. \tag{1.1}$$

First-order necessary conditions for a solution to the quadratic programming problem above are given by the 2×2 block system of linear equations

$$\begin{bmatrix} A & E^T \\ E & 0 \end{bmatrix} \begin{bmatrix} f \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix}, \tag{1.2}$$

where λ is a vector of Lagrange multipliers. Here A is the element flexibility matrix (or equivalently the element stiffness matrix is A^{-1}), E is the equilibrium matrix, s is the vector of external loads, $-\lambda$ is the displacement vector, and f is the system force vector. There are two methods generally used to calculate (1.1) or (1.2), the displacement method and the force method.

Displacement Method : Consider (1.2) and assume A is invertible and E has full row rank. Block elimination in (1.2) yields the steps:
(i) Solve $EA^{-1}E^T\lambda = -s$,
(ii) Solve $Af = -E^T\lambda$.

Force Method : Consider (1.2) and assume N^TAN is invertible, where N is a matrix whose columns form a basis of the nullspace of E.
(i) Solve $Ef_p = s$, f_p is any particular solution to $Ef = s$.
(ii) Find a basis of the nullspace N of E, and solve

$$N^TANf_0 = -N^TAf_p, \; f_0 \text{ is a redundant force vector.} \tag{1.3}$$

(iii) Set $f = f_p + Nf_0$.

The purpose of a reanalysis procedure is to analyze a damaged structure using, as much as possible, quantities calculated in the analysis of the original structure. This procedure also can be applied to the statistical problems. Suppose that we have a data and assume some parts of data are contaminated or damaged by external sources during the computation. For example, a study investigating variation of temperature is undergoing but there could be an abrupt change of temperature by external sources such as a storm or unexpected movement of atmospheric pressure, that is, some parts of data are damaged. It is not possible to delete or treat them as outliers in this case. Under this circumstance, it is necessary to analyze the destroyed data as what it is.

Various means to accomplish reanalysis of damaged structures have been investigated by Scott, Westkaemper, Sejal and Stearman(1979), Arora, Haskell and Gavil(1980), and Hemming and Venkayya(1980). This work, for the most part, has been based on the matrix displacement method and iterative schemes. However, a series of papers Heath, Plemmons and Ward(1984), Berry, Heath, Kaneko, Lawo, Plemmons, and Ward(1985), Gilbert and Heath (1987), Coleman and Pothen(1986), and Coleman and Pothen(1987) have done some reanalysis work using the force method, and recent research Batt, Gellin, and Gellatly(1982), Batt and Gellin(1985), and Plemmons and White (1990) indicates that the force method is a viable alternative not only to the solution of problems of dynamics and weight optimization but also to reanalysis.

In this paper we discuss the reanalysis based on the force method in the small scale damage case. There are two different types of problems involved in the small scale damage case. The first problem is a case that either one or two elements have been modified, and the second problem is a case that almost all of the components of the stiffness matrix have been modified. Both cases will be discussed in the next two sections.

2. REANALYSIS WITH QR FACTORIZATION

Given a particular solution f_p, the main task of the force method is the computation of the redundant force vector f_0 which satisfies system (1.3). System (1.3) is simply the normal equation for the weighted least squares problem:

$$\mathrm{Min}_{f_0}\|G^{-1}(Nf_0 + f_p)\|_2, \tag{2.1}$$

where G is Cholesky factor of element stiffness matrix A^{-1}. The traditional method of normal equations consists of the direct application of Cholesky's method to the symmetric positive definite matrix N^TAN. Unfortunately, explicitly forming the matrix N^TAN can lead to loss of sparse structure of N and worsening of the conditioning of the problem. A better approach in this regard is to apply orthogonal transformations to the matrix $G^{-1}N$, leading to an algorithm of the following form:

Orthogonal factorization

$$P_1G^{-1}NP_2^T = Q\begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{2.2}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = -Q^T P_1 G^{-1} f_p,$$

$$f_0 = P_2^T R^{-1} c,$$

where R is an upper triangular matrix of order $n-m$, P_1 and P_2 are permutation matrices of order n and $n-m$, respectively, Q is an orthogonal matrix of order n, and c and d are vectors of length $n-m$ and m, respectively. Several methods for solving problems of the form (2.1) are described in Lawson and Hanson(1974).

Reanalysis by the force method work based on QR factorization has been done by Plemmons and White(1990) for the case that only one element has been modified. The element flexibility matrix A=diag(A_k), where each A_k is an $n_k \times n_k$ symmetric positive definite, is symmetric positive definite. If we assume only one block of the matrix A has been modified, that is, one element changed, then A will be modified by changing one A_k to $A_k + \delta_k \delta_k^T$ where each δ_k is $n_k \times n_k$. Suppose we use an orthogonal factorization to solve (2.1), then the advantage of the force method is that one can use the QR factorization of the unperturbed problem of (2.2) to solve the perturbed problem

$$N^T(A + e_k \delta_k \delta_k^T e_k^T) N (f_0 + \Delta f_0) = -N^T (A + e_k \delta_k \delta_k^T e_k^T) f_p, \qquad (2.3)$$

where each e_k is an $n \times n_k$ matrix having all zero components except kth block which is the $n_k \times n_k$ identity. Based on the assumption that only one block of the matrix A has been modified, Plemmons and White(1990) established following theorem.

Theorem 1. Let f_0 be the solution of (1.3). Let $G^{-1}N = QR$, where G is a Cholesky factor of symmetric positive definite matrix A^{-1} and N has full column rank. Then the solution of (2.3) is given by $f_0 + \Delta f_0$, where

$$\begin{aligned} \Delta f_0 = & \left[R^{-1}R^{-T} - R^{-1}R^{-T}U_k(I + U_k^T R^{-1} R^{-T} U_k)^{-1} U_k^T R^{-1} R^{-T}\right] q_k, \\ U_k = & \ N^T e_k \delta_k, \ \ (n-m) \times n_k \ \text{ matrix}, \\ I = & \ n_k \times n_k, \ \text{ identity matrix}, \\ q_k = & -N^T e_k \delta_k \delta_k^T e_k^T (N f_0 + f_p). \end{aligned}$$

However, poor results of (2.1) may be obtained with either QR factorization or normal equation when the matrix A is ill-conditioned, and this gives us motivation to use Paige's formulation in Paige(1979), which can considerably reduce this difficuly, to solve (2.1). Furthermore we apply perturbation analysis discussed in Paige(1979) to the more general case that almost entire components of the stiffness matrix have been modified(e.g. perturbation occurs due to excitation of frequency in forced response of an elastic structure to a time-harmonic load).

3. REANALYSIS WITH LEAST SQUARES SCHEME

In this section we present formulation of Paige's linearly constrained sum-of-squares scheme, then apply the perturbation analysis with these schemes to our discussion about reanalysis.

3.1. Paige's Formulation

Following Paige(1979), if we define the weighted residual vector

$$v = G^{-1}(Nf_0 + f_p),$$

then problem (2.1) can be written in the equivalent form

$$\mathrm{Min}_{v,f_0} v^T v \text{ subject to } Gv = Nf_0 + f_p. \tag{3.1}$$

Because of its special form, problem (3.1) is sometimes referred to as the linearly constrained sum-of-squares problem. In addition to leading to a better numerical method, (3.1) also has important theoretical advantages over (2.1) in that it requires no restrictive assumptions regarding the ranks of the matrices involved. In particular, it is possible to compute a G which is suitable for use in (3.1) even if the element stiffness matrix is only semidefinite. Furthermore (3.1) could be expressed as

$$\mathrm{Min}_{v,f_0} \| \begin{bmatrix} 0, & I \end{bmatrix} \begin{bmatrix} f_0 \\ v \end{bmatrix} \|_2 \text{ subject to } \begin{bmatrix} -N, & G \end{bmatrix} \begin{bmatrix} f_0 \\ v \end{bmatrix} = f_p,$$

and the methods in Lawson and Hanson(1974) could then be applied. The method in Lawson and Hanson(1974) appears to be the most numerically reliable of these, although no rounding error analysis is given. However, such a method does not treat f_0, v, N, G separately. In reanalysis of the damaged structure case, the nullspace basis matrix N and particular solution f_p remains unchanged while G has been modified. So, it is important in the analysis to treat them separately. Here we will give a numerically stable algorithm that takes advantage of the special form of (3.1), and maintains f_0, v, N, G as separate throughout. This will allow us to carry out a reanalysis based on the resulting decomposition.

Formulation : First, decompose N in (3.1) as

$$Q^T N = \begin{bmatrix} Q_1^T N \\ Q_2^T N \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{3.2}$$

where $Q = (Q_1, Q_2)$ is an orthogonal matrix order n, R is a nonsingular upper triangular matrix of order $n - m$. The constraints in (3.1) then split into

$$Q_1^T Gv = Rf_0 + Q_1^T f_p \tag{3.3}$$

$$Q_2^T Gv = Q_2^T f_p. \tag{3.4}$$

Since R has full row rank, (3.3) can always be solved for f_0 once v is given, and so (3.4) gives the constraints on v, and (3.1) becomes

$$\mathrm{Min}_v v^T v \text{ subject to } Q_2^T Gv = Q_2^T f_p. \tag{3.5}$$

Next, apply the QR factorization to $(Q_2^T G)^T$ starting from the lower right components to decompose $Q_2^T G$ so that

$$Q_2^T GP = (0, L_2), \quad P = (P_1, P_2) \text{ orthogonal} \tag{3.6}$$

and L_2 has full column rank.
That is, decompose $Q^T G$ as

$$Q^T GP = \begin{bmatrix} Q_1^T GP_1 & Q_1^T GP_2 \\ 0 & L_2 \end{bmatrix} = \begin{bmatrix} L_1 & L_{12} \\ 0 & L_2 \end{bmatrix}. \tag{3.7}$$

Assuming L_2 is nonsingular we now obtain

$$v = P_2 L_2^{-1} Q_2^T f_p, \tag{3.8}$$

since $Q_2^T GP_2 = L_2$. Finally, f_0 is recovered from the triangular system (3.3).

3.2. Reanalysis with Paige's Formulation

Applying Paige's formulation to reanalysis can be stated as follows: Since each block A_k of A is symmetric positive definite matrix, we can compute $G + \delta G_k$ of perturbed block from $A + \delta A_k$. Now let our perturbed data result in $\overline{G} = G + \delta G$ leading to the solution $v + \delta v, f_0 + \delta f_0$ of the perturbed problem (3.1). Note that N and f_p remains unchanged in our approach to reanalyze the structures based on the force method. Considering (3.1) for both the original and perturbed problems, we see that δv and δf_0 give

$$\mathrm{Min}_{\delta v, \delta f_0} 2v^T(\delta v) + (\delta v)^T(\delta v) \tag{3.9}$$

$$\text{subject to } \overline{G}(\delta v) = N(\delta f_0) - (\delta G)v. \tag{3.10}$$

The constraints (3.10) have the same form as in (3.1), so we can proceed as in (3.7)

$$Q^T\overline{GP} = \begin{bmatrix} \overline{L}_1 & \overline{L}_{12} \\ 0 & \overline{L}_2 \end{bmatrix}, \tag{3.11}$$

where $\overline{P} = (\overline{P}_1, \overline{P}_2)$ is orthogonal, and $\overline{L}_2$ has full column rank. For solving a sequence of problems with fixed N but varing G, in theory it is necessary to compute the orthogonal transformation Q only once. Applying the perturbation analysis discussed in Paige(1979) to our case, we get the following results which could be viewed as an important special case of Paige's work.

Lemma 1. Let v be the solution of (3.5). Let G be the Cholesky factor of the symmetric positive definite matrix A^{-1}, and N has full column rank. Assume L_2 and $\overline{L}_2$ are nonsingular. If δv satisfy (3.9) and (3.10), then

$$\delta v = \left[\overline{P}_1\overline{P}_1^T(\delta G)^T Q_2(L_2^{-1})^T P_2^T + \overline{P}_2\overline{L}_2^{-1}Q_2^T(\delta G)\right] v \ \textit{and}, \tag{3.12}$$

$$\|\delta v\|_2 \leq \left[\frac{1}{\sigma(L_2)} + \frac{1}{\sigma(\overline{L}_2)}\right] \epsilon_G \|v\|_2, \tag{3.13}$$

where $\epsilon_G = \|\delta G\|_2$ and Q_2 is as in (3.2), and, $\sigma(L_2)$ and $\sigma(\overline{L}_2)$ are the smallest nonzero singular values of L_2 and $\overline{L}_2$, respectively.

Proof. From combining (3.10) and (3.11), that is,

$$\begin{bmatrix} \overline{L}_1 & \overline{L}_{12} \\ 0 & \overline{L}_2 \end{bmatrix} \begin{bmatrix} \overline{P}_1^T \\ \overline{P}_2^T \end{bmatrix} (\delta v) = \begin{bmatrix} R \\ 0 \end{bmatrix} (\delta f_0) - \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} (\delta G)v,$$

we get

$$\overline{L}_2\overline{P}_2^T(\delta v) = -Q_2^T(\delta G)v, \tag{3.14}$$

and this must be a consistent system for the perturbation to be meaningful for this problem, since (3.14) is underdetermined system. We can then express

$$\delta v = \overline{P}_1 z_1 + \overline{P}_2 z_2,\ z_2 = \overline{L}_2^{-1}Q_2^T(\delta G)v \ \text{ for all } \ z_1, \tag{3.15}$$

since $\overline{L}_2\overline{P}_2^T = Q_2^T\overline{G}$ and $Q_2^T\overline{GP}_1 = 0$. Substituting (3.15) in (3.9) and taking the derivative with respect to z_1 gives

$$z_1 = -\overline{P}_1^T v,\ \delta v = -\overline{P}_1\overline{P}_1^T v + \overline{P}_2\overline{L}_2^{-1}Q_2^T(\delta G)v. \tag{3.16}$$

The second term of (3.16) can easily be bounded, but the first is difficult to bound. From (3.8),

$$\overline{P}_1^T v = \overline{P}_1^T P_2 L_2^{-1} Q_2^T f_p = \overline{P}_1^T P_2 P_2^T v, \tag{3.17}$$

and we will seek an expression for $\overline{P}_1^T P_2$. To do this we first consider the following expression

$$Q^T (G + \delta G)\overline{P} = \begin{bmatrix} \overline{L}_1 & \overline{L}_{12} \\ 0 & \overline{L}_2 \end{bmatrix},$$

from (3.11). And then comparing the first set of columns of both sides, it will give

$$Q_2^T (\delta G)\overline{P}_1 = -Q_2^T G\overline{P}_1 = -L_2 P_2^T \overline{P}_1,$$

since $Q_2^T G P_2 = L_2$. This can be used with (3.16) and (3.17) to give an expression for δv as follows:

$$\begin{aligned} \delta v &= -\overline{P}_1\overline{P}_1^T v + \overline{P}_2\overline{L}_2^{-1}(\delta G)v \\ &= -\overline{P}_1(\overline{P}_1^T P_2 P_2^T v) + \overline{P}_2\overline{L}_2^{-1} Q_2^T(\delta G)v \\ &= -\overline{P}_1\left[L_2^{-1}(-Q_2^T(\delta G)\overline{P}_1)\right]^T P_2^T v + \overline{P}_2\overline{L}_2^{-1} Q_2^T(\delta G)v \\ &= \overline{P}_1\overline{P}_1^T(\delta G)^T Q_2 L_2^{-T} P_2^T v + \overline{P}_2\overline{L}_2^{-1} Q_2^T(\delta G)v. \end{aligned}$$

By taking the 2-norm we obtain

$$\begin{aligned} \|\delta v\| &= \left[\|\overline{P}_1\overline{P}_1^T(\delta G)^T Q_2 L_2^{-T} P_2^T\| + \|\overline{P}_2\overline{L}_2^{-1} Q_2^T(\delta G)\|\right]\|v\| \\ &\le \left[\|(\delta G)^T\|\|L_2^{-T}\| + \|\overline{L}_2^{-1}\|\|\delta G\|\right]\|v\|. \end{aligned}$$

Let $\epsilon_G = \|\delta G\|$, and let $\sigma(L_2)$ and $\sigma(\overline{L}_2)$ be the smallest nonzero singular value of L_2 and $\overline{L}_2$ respectively, then

$$\|\delta v\| \le \left[\frac{\epsilon_G}{\sigma(L_2)} + \frac{\epsilon_G}{\sigma(\overline{L}_2)}\right]\|v\|. \qquad \square$$

Theorem 2. Let f_0 be the solution of (3.3), that is, the solution of (1.3). Let G be the Cholesky factor of the symmetric positive definite matrix A^{-1}, and N has full column rank. Then the solution of (3.9) is given by $f_0 + \delta f_0$ where

$$\delta f_0 = R^{-1}\left[(\overline{L}_{12}\overline{L}_2^{-1} Q_2^T + Q_1^T)(\delta G) + \overline{L}_1\overline{P}_1^T(\delta G)^T Q_2 L_2^{-T} P_2^T\right] v, \tag{3.18}$$

and

$$\|\delta f_0\|_2 \le \left[\left(1+\frac{\|\overline{L}_{12}\|_2}{\sigma(\overline{L}_2)}\right)\epsilon_G + \left(\frac{\|\overline{L}_1\|_2\epsilon_G}{\sigma(L_2)}\right)\right]\frac{\|v\|_2}{\sigma(N)}, \tag{3.19}$$

$\epsilon_G = \|\delta G\|_2$ and Q is as in (3.2), and, $\sigma(L_2)$, $\sigma(\overline{L}_2)$, and $\sigma(N)$ are the smallest nonzero singular values of L_2, $\overline{L}_2$, and N, respectively.

Proof. From combining (3.2) and (3.10),

$$\begin{aligned}
& Q^T N(\delta f_0) - Q^T\overline{G}(\delta v) = Q^T(\delta G)v \\
\Rightarrow\ & \begin{bmatrix} R \\ 0 \end{bmatrix}(\delta f_0) - \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}\overline{G}(\delta v) = \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}(\delta G)v \\
\Rightarrow\ & R(\delta f_0) - Q_1^T\overline{G}(\delta v) = Q_1^T(\delta G)v \\
\Rightarrow\ & \delta f_0 = R^{-1}\left[Q_1^T(\delta G)v + Q_1^T\overline{G}(\delta v)\right], \text{ since } N \text{ has full column rank.}
\end{aligned}$$

By using (3.12) in previous Lemma, and since $Q_1^T\overline{GP}_2 = \overline{L}_{12}$ and $Q_1^T\overline{GP}_1 = \overline{L}_1$,

$$\begin{aligned}
\delta f_0 &= R^{-1}\left[Q_1^T(\delta G)v + Q_1^T\overline{G}\left(+\overline{P}_2\overline{L}_2^{-1}Q_2^T(\delta G)v + \overline{P}_1\overline{P}_1^T(\delta G)^T Q_2 L_2^{-T}P_2^T v\right)\right] \\
&= R^{-1}\left[Q_1^T(\delta G)v + \overline{L}_{12}\overline{L}_2^{-1}Q_2^T(\delta G)v + \overline{L}_1\overline{P}_1^T(\delta G)^T Q_2 L_2^{-T}P_2^T v\right] \\
&= R^{-1}\left[\left(\overline{L}_{12}\overline{L}_2^{-1}Q_2^T + Q_1^T\right)(\delta G) + \overline{L}_1\overline{P}_1^T(\delta G)^T Q_2 L_2^{-T}P_2^T\right]v.
\end{aligned}$$

Since $\sigma(R) = \sigma(N)$, we can get (3.19) from (3.18) by taking the 2-norm. □

Note that assuming L_2 and $\overline{L}_2$ are nonsingular, that is, $Q_2^T G$ and $Q_2^T\overline{G}$ have full row rank, respectively, then the computation for δf_0 is simple. Based on various assumptions in Paige(1979), we can get a tighter bound for δf_0 compared to (3.19).

4. CONCLUSION

It can be concluded that an effective method for reanalysis can be employed for structure initially analyzed by the force method, and that this method utilizes a portion of earlier computations in order to solve such modified problem without starting the computation over from the beginning. We have suggested an implementation of the reanalysis based on the force method which uses Paige's linearly constrained sum-of-squares.

The formulation (3.1) of the problem contributes greatly to its solution and analysis, as well as generalizing the problem. For example, since $Gv = Nf_0 + f_p$ is now just a set of constraints, it is clear that the transformation in (3.2) can be carried out by any well-conditioned nonsingular matrix Q. Then it is clear that the algorithm can often be speeded up by using stabilized nonunitary transformation in Peters and Wilkinson(1970).

The formulation (3.1), however, does not take advantage of any special structure the matrix G have(G will be triangular if it is computed by the Cholesky factorization, and in our case G has block diagonal structures as well); indeed, that structure is in general destroyed by the orthogonal transformation Q. Retaining the triangular structure of G throughout the computations and related numerical stability can be our future work that we hope to soon obtain more results.

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