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Maximum Likelihood Estimation for the Laplacian Autoregressive Time Series Model [†]

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Abstract

The maximum likelihood estimation is discussed for the NLAR model with Laplacian marginals. Since the explicit form of the estimates cannot be obtained due to the complicated nature of the likelihood function we utilize the automatic computer optimization subroutine using a direct search complex algorithm. The conditional least square estimates are used as initial estimates in maximum likelihood procedures. The results of a simulation study for the maximum likelihood estimates of the NLAR(1) and the NLAR(2) models are presented.

Key Words : Maximum likelihood estimation; New Laplacian autoregressive time series model; Direct search complex algorithm; Conditional least squares estimates.

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1. INTRODUCTION

The New Laplacian AutoRegressive(NLAR) model introduced by Dewald and Lewis(1985) is a non-Gaussian autoregressive time series model with Laplacian(double-exponential) marginals. The NLAR model can be applied to marginally double-exponentially distributed data with a larger kurtosis or longer tails than Gaussian data. Son and Cho(1988) discussed the properties and the forecasting procedures of the NLAR process. As for the estimation of model parameters, Karlsen and Tjøstheim(1988) obtained the conditional least squares(CLS) estimates which are consistent and asymptotically normal for all four parameters of the NLAR(2) model. They pointed out, however, that the standard errors of the CLS estimates may be large, especially for small parameter values, so that considerably large sample may be needed to obtain reasonable estimates.

In this paper the maximum likelihood(ML) estimation is considered for the NLAR model. Since the ML estimate can not explicitly be obtained from the likelihood function, the IMSL optimization routine is used and the CLS estimates are used as initial estimates.

2. THE MODELS

Let $\{X_t\}$ be a stationary sequence of random variables whose marginal distribution is standard Laplacian. It is shown in Son and Cho(1988) that the NLAR(p) process can be constructed analogously to the NLAR(2) process of Dewald and Lewis(1985) as follows : for $t = 0, \pm 1, \pm 2, \ldots$,

$$X_t = \left\{ \begin{array}{lll} \beta_1 X_{t-1} & \text{w.p.} & \alpha_1 \\ \beta_2 X_{t-2} & \text{w.p.} & \alpha_2 \\ \vdots & \vdots & \vdots \\ \beta_p X_{t-p} & \text{w.p.} & \alpha_p \\ 0 & \text{w.p.} & 1-\alpha^* \end{array} \right\} + \varepsilon_t \tag{2.1}$$

with $\alpha^* = \sum_{i=1}^{p} \alpha_i$, where $0 < |\beta_i| < 1$, $0 < \alpha_i < 1$, and $0 < \alpha^* < 1$. In NLAR(p) process, the structure of random error ε_t will be derived so that it should give a stationary standard Laplacian marginal distribution of $\{X_t\}$.

Thus, for example, in the NLAR(2) case of $p = 2$,

$$\varepsilon_t = \left\{ \begin{array}{lll} L_t & \text{w.p.} & 1 - p_2 - p_3 \\ |b_2| L_t & \text{w.p.} & p_2 \\ |b_3| L_t & \text{w.p.} & p_3, \end{array} \right. \tag{2.2}$$

where $\{L_t\}$ is a sequence of i.i.d. standard Laplacian variables and b_2, b_3, p_2, and p_3 are determined from the equations (3.8)-(3.12) of Dewald and Lewis(1985), which are the functions of four parameters α_1, α_2, β_1, and β_2 of the NLAR(2) model. Similarly, in the NLAR(1) case of $p = 1$,

$$\varepsilon_t = \begin{cases} L_t & \text{w.p. } 1-p \\ |\beta|\sqrt{1-\alpha}L_t & \text{w.p. } p, \end{cases} \tag{2.3}$$

where $p = \alpha\beta^2/\{1-(1-\alpha)\beta^2\}$. Note that the NLAR(1) model is the special case of $\alpha_2 = 0$ and $\beta_2 = 0$ in the NLAR(2) model.

3. CONDITIONAL LEAST SQUARE ESTIMATION

Assuming that the observations of $\{X_t\}$ are available for $t = 1, 2, \ldots, n$, the CLS estimates for four parameters, α_1, α_2, β_1, and β_2 in the NLAR(2) model can be directly obtained from the equations (3.4) and (3.12) of Karlsen and Tjøstheim(1988). The CLS estimates are given as follows :

$$\hat{\alpha}_{CLS.i} = \frac{\hat{a}_i^2}{\hat{\sigma}_{ii} + \hat{a}_i^2} \quad \text{and} \quad \hat{\beta}_{CLS.i} = \frac{\hat{\sigma}_{ii} + \hat{a}_i^2}{\hat{a}_i}, \quad i = 1, 2, \tag{3.1}$$

where

$$\hat{a}_i = \frac{\sum_{t=3}^n x_{t-j}^2 \sum_{t=3}^n x_t x_{t-i} - \sum_{t=3}^n x_t x_{t-j} \sum_{t=3}^n x_{t-1} x_{t-2}}{\sum_{t=3}^n x_{t-1}^2 \sum_{t=3}^n x_{t-2}^2 - (\sum_{t=3}^n x_{t-1} x_{t-2})^2}, \tag{3.2}$$

$$\hat{\sigma}_{ii} = \frac{\sum_{t=3}^n (x_{t-j}^2 - 2)^2 \sum_{t=3}^n \hat{G}_t (x_{t-i}^2 - 2) - \sum_{t=3}^n \hat{G}_t (x_{t-j}^2 - 2) H_n}{\sum_{t=3}^n (x_{t-1}^2 - 2)^2 \sum_{t=3}^n (x_{t-2}^2 - 2)^2 - H_n^2} \tag{3.3}$$

with $\hat{G}_t = (x_t - \hat{a}_1 x_{t-1} - \hat{a}_2 x_{t-2})^2 + 2\hat{a}_1\hat{a}_2 x_{t-1}x_{t-2} + 2(\hat{a}_1^2 + \hat{a}_2^2 - 1)$, $H_n = \sum_{t=3}^n (x_{t-1}^2 - 2)(x_{t-2}^2 - 2)$ and $j = 2(1)$ in case of $i = 1(2)$. Also the estimates $\hat{\alpha}_{CLS.i}$ and $\hat{\beta}_{CLS.i}$, $i = 1, 2$, will be strongly and jointly asymptotically normal by Theorem 3.1 of Karlsen and Tjøstheim(1988). The admissible region for α_1, α_2, β_1, and β_2 in the NLAR(2) implies the restrictions on a_1, a_2, σ_{11}, and σ_{22} as follows :

$$\left\{ \begin{array}{c} 0 < |a_1| + |a_2| < 1 \\ \sigma_{11}\sigma_{22} > a_1^2 a_2^2 \\ 0 < \sigma_{ii} < \min\{0.25,\ |a_i|(1 - a_i)\} \end{array} \right\} \tag{3.4}$$

Specially, letting $\alpha_2 = \beta_2 = 0$ in (3.4) and (3.12) of Karlsen and Tjøstheim(1988), the CLS estimates of α and β in the NLAR(1) model are given by

$$\hat{\alpha}_{CLS} = \frac{\hat{a}^2}{\hat{\sigma} + \hat{a}^2} \quad \text{and} \quad \hat{\beta}_{CLS} = \frac{\hat{\sigma} + \hat{a}^2}{\hat{a}}, \tag{3.5}$$

where

$$\hat{a} = \frac{\sum_{t=2}^{n} x_t x_{t-1}}{\sum_{t=2}^{n} x_{t-1}^2} \quad \text{and} \quad \hat{\sigma} = \frac{\sum_{t=2}^{n}(x_{t-1}^2 - 2)\{(x_t - \hat{a}x_{t-1})^2 + 2(\hat{a}^2 - 1)\}}{\sum_{t=2}^{n}(x_{t-1}^2 - 2)^2}. \tag{3.6}$$

The admissible region for α and β in the NLAR(1) implies the restrictions on a and σ as follows :

$$\left\{ \begin{array}{c} 0 < |a| < 1 \\ 0 < \sigma < \min\{0.25,\ |a|(1-a)\} \end{array} \right\} \tag{3.7}$$

4. MAXIMUM LIKELIHOOD ESTIMATION

Since the NLAR(2) process is the second-order Markovian process, the conditional probability density of $(X_3, X_4, \cdots, X_n)$ given X_1 and X_2 is of the form,

$$f_{X_3,X_4,\ldots,X_n|X_1,X_2}(x_3, x_4, \ldots, x_n|x_1, x_2) = \prod_{t=3}^{n} f_{X_t|X_{t-1},X_{t-2}}(x_t|x_{t-1}, x_{t-2}). \tag{4.1}$$

Also, the conditional distribution function of X_t given X_{t-1} and X_{t-2} is given by

$$\begin{array}{l} \Pr(X_t \leq x_t | X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}) \\ = \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i p_j \Pr(L_t \leq (x_t - \beta_i x_{t-i})/b_j) \end{array} \tag{4.2}$$

with $\alpha_3 = 1 - \alpha_1 - \alpha_2$, $p_1 = 1 - p_2 - p_3$, $\beta_3 = 0$, and $b_1 = 1$. Now, after differentiating the function (4.2) with respect to x_t, we have the monotone function of the log likelihood conditional on x_1 and x_2 as follows :

$$L^*(\alpha_1, \alpha_2, \beta_1, \beta_2) = \sum_{t=3}^{n} \ln \left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i p_j |b_j|^{-1} \exp\{-|x_t - \beta_i x_{t-i}|/|b_j|\} \right\}. \tag{4.3}$$

The ML estimates $\hat{\alpha}_{ML.1}$, $\hat{\alpha}_{ML.2}$, $\hat{\beta}_{ML.1}$, and $\hat{\beta}_{ML.2}$ are the values of α_1, α_2, β_1, and β_2 which maximize $L^*(\alpha_1, \alpha_2, \beta_1, \beta_2)$ of (4.3). Here, we can not but point out the fact that the function (4.3) becomes infinite on the boundaries of the parameter space. But, this problem can be solved by looking for a maximum in the interior of the parameter space.

Letting $\alpha_2 = \beta_2 = 0$ in (4.3), the monotone function of the log likelihood conditional on x_1 in the NLAR(1) process is given by

$$L^*(\alpha, \beta) = \sum_{t=2}^{n} \ln \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \alpha_i p_j d_j^{-1} \exp\{-|x_t - \beta_i x_{t-i}|/d_j\} \right\}, \tag{4.4}$$

where $\alpha_1 = 1 - \alpha$, $\alpha_2 = \alpha$, $\beta_1 = \beta$, $\beta_2 = 0$, $p_2 = p = \alpha\beta^2/\{1 - (1 - \alpha)\beta^2\}$, $p_1 = 1 - p$, $d_1 = 1$, and $d_2 = |\beta|\sqrt{1 - \alpha}$. The ML estimates $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ of parameters α and β in the NLAR(1) model are the values of α and β which maximize $L^*(\alpha, \beta)$ of (4.4). Since these monotone likelihood functions, which are to be maximized, are nonlinear and complex in parameters and have the absolute component, not only there are no closed form expressions for the estimates which maximize L^*, but also standard numerical optimization will not work.

5. SIMULATION EXPERIMENTS

To observe the sample behaviors of the ML estimates which we introduce in Section 4 and also compare the performances of the CLS and the ML estimates, we conducted some Monte Carlo experiments for the NLAR(1) process. The simulation experiments were performed on samples of size $n = 15, 30, 60, 100, 200, 500, 1000, 10000$ and 30000 for each of four models with $(\alpha, \beta) = (0.1, -0.2), (0.3, 0.4), (0.5, -0.6)$ and $(0.9, 0.8)$. 1000 repetitions were made on samples of size $n = 15,\ 30,\ 60$, 100 repetitions on $n = 100,\ 200,\ 500,\ 1000$, 20 repetitions on $n = 10000$, and 10 repetitions on $n = 30000$. The optimization subroutine BCPOL, PC/IMSL subroutine, which minimizes a function of parameters subject to bounds on the parameters using a direct search Complex algorithm which is based on only function-value information, not on derivative information at each iteration, was used to find the ML estimates. Since the BCPOL subroutine requires starting values for α and β, the CLS estimates $\hat{\alpha}_{CLS}$ and $\hat{\beta}_{CLS}$ of (3.5) were used as initial estimates. When the CLS estimates of α and β which are outside the admissible region were obtained, they were corrected by boundary values of the restriction (3.7). Table 5.1 – 5.4 display the simulation results which show the average bias, the average mean square error(MSE) of the CLS and the ML estimates, and the proportions of repetitions producing corrections in CLS estimation procedures.

From four tables, we can observe some features as follows :

(i) The proportion of repetitions producing the CLS estimates of α and β which are outside the admissible region generally decreases as n increases for each model. Also this proportion for each n becomes large when $|\alpha\beta|$ approaches 0 or 1. The reasons for these results are explained by the fact that the data of the stochastic process when $|\alpha\beta| \approx 0$ or 1, or small samples are in the close vincinty to the boundary of parameter values.

(ii) In the model with $\alpha = 0.1$, $\beta = -0.2$(i.e.$|\alpha\beta| = 0.02$) the bias and the MSE are very large even for the considerably large samples. The reason for this result is that the nature of the stochastic process with $|\alpha\beta| \approx 0$ is mainly determined by the random error term ε_t.

(iii) For $n \leq 200$ in the model with $\alpha = 0.3$, $\beta = 0.4$(i.e. $|\alpha\beta| = 0.12$) and $n \leq 30$ with $\alpha = 0.5$, $\beta = -0.6$(i.e. $|\alpha\beta| = 0.30$) including the model with $\alpha = 0.1$, $\beta = -0.2$, the bias or the MSE are so large that the estimates are practically useless.

(iv) The MSE decreases as $|\alpha\beta|$ approaches 1.

(v) The decrease of MSE with the increase in n becomes apparent as $|\alpha\beta|$ approaches 1.

(vi) Except a few cases besides the estimation of β in the model with $\alpha = 0.1$ and $\beta = -0.2$ the ML estimates are generally better than the CLS estimates in the sense of the bias and the MSE. The accuracy of the ML estimates depends on that of the CLS estimates used as initial estimates in ML estimation procedures.

The simulation experiments for the NLAR(2) process were also performed. To compare the performance of the ML estimates with the CLS estimates of Karlsen and Tjøstheim(1988), data were generated from the NLAR(2) models with $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (0.4, 0.5, 0.6, 0.7)$ and $(0.25, 0.25, 0.5, 0.5)$, respectively. The numbers of repetitions made on each sample of size n are as same as the cases of the NLAR(1). As in the case of NLAR(1) the CLS estimates of $\alpha_1, \alpha_2, \beta_1$, and β_2 outside the admissible range were corrected by boundary values of the restriction (3.4). Thus, the ML estimates $\hat{\alpha}_{ML.1}, \hat{\alpha}_{ML.2}, \hat{\beta}_{ML.1}$ and $\hat{\beta}_{ML.2}$ which maximize L^* of (4.3) were obtained by using the CLS estimates $\hat{\alpha}_{CLS.i}$ and $\hat{\beta}_{CLS.i}$, $i = 1, 2$, of (3.1) as initial estimates. The features of results shown in Table 5.5 – 5.6 are similar to those of the NLAR(1) model.

Table 5.1 Simulation results for the NLAR(1) model with $\alpha = 0.1$ and $\beta = -0.2$ ($|\alpha\beta| = 0.02$).

n	Proportion of corrections	Bias(α)		MSE(α)		Bias(β)		MSE(β)	
		CLS	ML	CLS	ML	CLS	ML	CLS	ML
15	0.813	0.535	0.627	0.438	0.481	0.195	0.208	0.444	0.375
30	0.790	0.554	0.581	0.463	0.446	0.141	0.178	0.339	0.301
60	0.788	0.541	0.536	0.457	0.420	0.159	0.174	0.312	0.282
100	0.840	0.557	0.540	0.481	0.419	0.127	0.182	0.248	0.190
200	0.780	0.555	0.506	0.484	0.395	0.112	0.152	0.224	0.208
500	0.760	0.419	0.382	0.380	0.317	0.182	0.166	0.371	0.303
1000	0.680	0.393	0.331	0.336	0.262	0.068	0.050	0.238	0.215
10000	0.450	0.245	0.184	0.226	0.148	-0.162	-0.133	0.148	0.140
30000	0.500	0.336	0.275	0.320	0.228	-0.092	-0.003	0.089	0.048

Table 5.2 Simulation results for the NLAR(1) model with $\alpha = 0.3$ and $\beta = 0.4$ ($|\alpha\beta| = 0.12$).

n	Proportion of corrections	Bias(α)		MSE(α)		Bias(β)		MSE(β)	
		CLS	ML	CLS	ML	CLS	ML	CLS	ML
15	0.793	0.357	0.400	0.274	0.248	-0.248	-0.206	0.444	0.376
30	0.743	0.345	0.346	0.267	0.219	-0.207	-0.153	0.356	0.284
60	0.710	0.310	0.284	0.254	0.192	-0.157	-0.137	0.301	0.254
100	0.600	0.303	0.208	0.237	0.133	-0.106	-0.066	0.225	0.184
200	0.610	0.303	0.245	0.246	0.172	-0.030	-0.052	0.149	0.095
500	0.470	0.192	0.121	0.177	0.091	0.025	-0.027	0.129	0.075
1000	0.270	0.202	0.114	0.145	0.063	-0.019	-0.045	0.072	0.029
10000	0.000	0.033	0.004	0.007	0.001	-0.019	-0.003	0.007	0.001
30000	0.000	0.004	0.000	0.004	0.000	0.009	0.000	0.009	0.000

Table 5.3 Simulation results for the NLAR(1) model with $\alpha = 0.5$ and $\beta = -0.6$ ($|\alpha\beta| = 0.30$).

n	Proportion of corrections	Bias(α)		MSE(α)		Bias(β)		MSE(β)	
		CLS	ML	CLS	ML	CLS	ML	CLS	ML
15	0.813	0.149	0.227	0.163	0.121	0.214	0.225	0.399	0.314
30	0.697	0.165	0.165	0.152	0.088	0.121	0.124	0.229	0.184
60	0.494	0.142	0.111	0.123	0.058	0.056	0.055	0.117	0.072
100	0.410	0.157	0.059	0.110	0.034	0.041	-0.004	0.063	0.032
200	0.170	0.093	0.034	0.067	0.013	0.027	0.006	0.047	0.014
500	0.020	0.095	0.008	0.040	0.004	0.048	0.001	0.027	0.003
1000	0.010	0.026	0.001	0.016	0.002	0.014	0.003	0.014	0.001
10000	0.000	0.009	0.002	0.003	0.000	0.007	0.002	0.003	0.000
30000	0.000	0.007	0.002	0.001	0.000	0.006	0.002	0.001	0.000

Table 5.4 Simulation results for the NLAR(1) model with $\alpha = 0.9$ and $\beta = 0.8$ ($|\alpha\beta| = 0.72$).

n	Proportion of corrections	Bias(α)		MSE(α)		Bias(β)		MSE(β)	
		CLS	ML	CLS	ML	CLS	ML	CLS	ML
15	0.799	-0.118	-0.013	0.076	0.009	0.037	0.015	0.066	0.037
30	0.695	-0.084	-0.016	0.052	0.006	0.043	0.027	0.021	0.010
60	0.535	-0.050	-0.010	0.032	0.003	0.027	0.015	0.013	0.005
100	0.460	-0.033	-0.009	0.021	0.002	0.025	0.006	0.010	0.003
200	0.390	-0.003	-0.001	0.011	0.001	0.009	0.005	0.007	0.001
500	0.150	-0.016	-0.007	0.008	0.000	0.004	0.006	0.005	0.001
1000	0.060	0.000	-0.003	0.005	0.000	0.000	0.001	0.003	0.000
10000	0.000	0.011	0.000	0.000	0.000	-0.009	0.000	0.000	0.000
30000	0.000	-0.008	0.000	0.000	0.000	0.005	0.000	0.000	0.000

Table 5.5 Mean values, standard deviations(in parentheses) and proportions of corrections for the conditional least square estimates and the maximum likelihood estimates in the NLAR(2) model with $\alpha_1 = 0.4$, $\alpha_2 = 0.5$, $\beta_1 = 0.6$, and $\beta_2 = 0.7$ ($|\alpha_1\beta_1| = 0.24$, $|\alpha_2\beta_2| = 0.35$).

n	Proportion of corrections	CLS	CLS	CLS	CLS	ML	ML	ML	ML
		α_1	α_2	β_1	β_2	α_1	α_2	β_1	β_2
15	0.973	0.43	0.41	0.38	0.39	0.42	0.53	0.53	0.50
		(0.24)	(0.24)	(0.61)	(0.63)	(0.22)	(0.23)	(0.47)	(0.49)
30	0.950	0.43	0.43	0.48	0.53	0.41	0.52	0.58	0.62
		(0.25)	(0.23)	(0.50)	(0.48)	(0.19)	(0.18)	(0.34)	(0.31)
60	0.870	0.42	0.44	0.54	0.63	0.41	0.50	0.62	0.67
		(0.23)	(0.21)	(0.42)	(0.34)	(0.14)	(0.13)	(0.19)	(0.21)
100	0.840	0.43	0.44	0.55	0.70	0.40	0.50	0.60	0.70
		(0.23)	(0.22)	(0.34)	(0.24)	(0.10)	(0.09)	(0.12)	(0.08)
200	0.690	0.44	0.45	0.55	0.68	0.40	0.49	0.60	0.70
		(0.18)	(0.15)	(0.27)	(0.18)	(0.09)	(0.08)	(0.08)	(0.06)
500	0.490	0.43	0.49	0.57	0.68	0.41	0.49	0.60	0.70
		(0.13)	(0.12)	(0.18)	(0.12)	(0.05)	(0.05)	(0.05)	(0.04)
1000	0.310	0.42	0.49	0.59	0.71	0.40	0.49	0.60	0.71
		(0.10)	(0.07)	(0.14)	(0.09)	(0.05)	(0.05)	(0.04)	(0.04)
10000	0.000	0.40	0.50	0.61	0.70	0.40	0.50	0.60	0.70
		(0.03)	(0.03)	(0.04)	(0.04)	(0.01)	(0.01)	(0.01)	(0.01)
30000	0.000	0.40	0.50	0.60	0.70	0.40	0.50	0.60	0.70
		(0.03)	(0.02)	(0.04)	(0.02)	(0.01)	(0.01)	(0.00)	(0.00)

Table 5.6 Mean values, standard deviations(in parentheses) and proportions of corrections for the conditional least square estimates and the maximum likelihood estimates in the NLAR(2) model with $\alpha_1 = 0.25$, $\alpha_2 = 0.25$, $\beta_1 = 0.5$, and $\beta_2 = 0.5$ ($|\alpha_1\beta_1| = 0.06$, $|\alpha_2\beta_2| = 0.25$).

n	Proportion of corrections	CLS α_1	CLS α_2	CLS β_1	CLS β_2	ML α_1	ML α_2	ML β_1	ML β_2
15	0.975	0.42	0.44	0.15	0.08	0.43	0.48	0.30	0.15
		(0.24)	(0.24)	(0.65)	(0.61)	(0.21)	(0.22)	(0.60)	(0.60)
30	0.945	0.41	0.43	0.23	0.15	0.41	0.44	0.35	0.22
		(0.26)	(0.26)	(0.57)	(0.56)	(0.21)	(0.21)	(0.48)	(0.53)
60	0.933	0.43	0.43	0.28	0.25	0.38	0.42	0.39	0.32
		(0.28)	(0.28)	(0.51)	(0.51)	(0.20)	(0.21)	(0.41)	(0.44)
100	0.910	0.41	0.40	0.31	0.34	0.35	0.38	0.42	0.37
		(0.28)	(0.29)	(0.48)	(0.50)	(0.18)	(0.21)	(0.35)	(0.38)
200	0.770	0.39	0.46	0.37	0.34	0.32	0.38	0.48	0.41
		(0.26)	(0.27)	(0.41)	(0.39)	(0.17)	(0.19)	(0.30)	(0.25)
500	0.580	0.39	0.40	0.44	0.43	0.29	0.30	0.48	0.46
		(0.26)	(0.26)	(0.33)	(0.29)	(0.14)	(0.11)	(0.16)	(0.15)
1000	0.390	0.35	0.33	0.47	0.48	0.25	0.27	0.52	0.48
		(0.20)	(0.20)	(0.26)	(0.26)	(0.06)	(0.07)	(0.10)	(0.11)
10000	0.000	0.25	0.27	0.52	0.52	0.25	0.25	0.50	0.51
		(0.05)	(0.08)	(0.11)	(0.15)	(0.02)	(0.02)	(0.02)	(0.02)
30000	0.000	0.24	0.26	0.53	0.51	0.25	0.25	0.51	0.50
		(0.03)	(0.06)	(0.06)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)

6. CONCLUSIONS

We have shown that the maximum likelihood estimates for the NLAR(1) and NLAR(2) models can be obtained by using the conditional least square estimates as initial estimates of the optimization program. As Karlsen and Tjøstheim(1988) had pointed out, the estimates are virtually useless because of large bias and the MSE if $|\alpha_i\beta_i|$, $i = 1, 2$, in the NLAR(2) and $|\alpha\beta|$ in the NLAR(1) is smaller than 0.1. It is proved in a number of experiments for other models in addition to the models considered in this paper that for moderate sample size in the model with $|\alpha_i\beta_i|$ much above 0.1 the ML estimates obtained by using the CLS estimates as initial estimates in the optimization program are better than the CLS estimates in the sense of bias and MSE. The

ML estimation considered in this paper can be applied to the New Exponential AutoRegressive(NEAR) model of Lawrance and Lewis(1985) in maximum likelihood procedure proposed by Smith(1986). A research on this subject is under study and will be reported in the near future.

REFERENCES

(1) Box, G.E.P. and Jenkins, G.M.(1976). *Time series Analysis Forecasting and Control*, San Francisco : Holden-Day, Inc.

(2) Dewald, L.S. and Lewis, P.A.W.(1985). A new Laplace second-order autoregressive time series model-NLAR(2). *IEEE Transactions Information Theory*, IT-31, 645-651.

(3) Karlsen, H. and Tjøstheim,D.(1988). Consistent estimates for the NEAR(2) and NLAR(2) time series models. *Journal of Royal Statistical Society*, Series B, **50**, 313-320.

(4) Klimko, L.A. and Nelson, P.I.(1978) On conditional least squares estimation for stochastic processes. *The Annals of Statistics*, **6**, 629-642.

(5) Lawrance, A.J. and Lewis, P.A.W.(1985). Modelling and residual analysis of nonlinear autoregressive time series in exponential variables. *Journal of Royal Statistical Society*, Series B, **47**, 165-202.

(6) Smith, R.L.(1986). Maximum likelihood estimation for the NEAR(2) model. *Journal of Royal Statistical Society*, Series B, **48**, 251-257.

(7) Son, Y.S. and Cho, S.(1988). A Laplacian autoregressive time series model. *Journal of the Korean Statistical Society*, **17**, 101-120.