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Some Positive Dependent Orderings †

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Abstract

Let $\underline{X}$ and $\underline{Y}$ be random vectors in R^n. A random vector $\underline{X}$ is 'more associated' than $\underline{Y}$ if and only if $P(\underline{X} \in A \cap B) - P(\underline{X} \in A)P(\underline{X} \in B) \geq P(\underline{Y} \in A \cap B) - P(\underline{Y} \in A)P(\underline{Y} \in B)$ for all open upper sets A and B. By requiring the above inequality to hold for some open upper sets A and B various notions of positive dependence orderings which are weaker than 'more associated' ordering are obtained. First a general theory is given and then the results are specialized to some concepts of a particular interest. Various properties and interrelationships are derived.

Key Words : Positive dependence; More associated; More positive dependent; More positive orthant dependent.

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1. INTRODUCTION

In the recent years there has been growing interest in concepts of positive dependence for families of random variables. Lehmann(1966) introduced a simple and natural definition of positive dependence : A sequence $\{X_j : j \geq 1\}$ of random variables is said to be pairwise positive quadrant dependent(pairwise PQD) if for any real r_i, r_j and $i \neq j$

$$P\{X_i > r_i, X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}.$$

A much stronger concept than PQD was considered by Esary, Proschan, and Walkup(1967) : A random vector $\underline{X} = (X_1, X_2, \cdots, X_n)$ is said to be associated if for every componentwise nondecreasing functions f and g on R^n

$$Cov(f(\underline{X}), g(\underline{X})) \geq 0, \tag{1.1}$$

provided the underlying expectations exist. A random vector $\underline{X}$ is said to be positively upper orthant dependent(PUOD) if for every $\underline{x} = (x_1, x_2, \cdots, x_n)$

$$P(\underline{X} > \underline{x}) \geq \prod_{i=1}^{n} P(X_i > x_i) \tag{1.2}$$

and $\underline{X}$ is called positively lower orthant dependent(PLOD) if for every $\underline{x}$

$$P(\underline{X} \quad \underline{x}) \leq \prod_{i=1}^{n} P(X_i \leq x_i) \tag{1.3}$$

It is well known(see Esary et al.[3]) that (1.1) holds if and only if for all open upper sets A, B

$$P(\underline{X} \in A \cap B) \geq P(\underline{X} \in A)P(\underline{X} \in B) \tag{1.4}$$

(A is an upper set if $\underline{x} \in A$ and $\underline{y} \geq \underline{x}$ imply $\underline{y} \in A$).

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets in R^n. Usually, the sets in $\mathcal{A}$ and $\mathcal{B}$ may be upper sets. A random vector $\underline{X}$ (or its distribution function) is positively dependent relative to $\mathcal{A}$ and $\mathcal{B}$(denoted by $PD(\mathcal{A}, \mathcal{B})$) if for all $A \in \mathcal{A}, B \in \mathcal{B}$

$$P(\underline{X} \in A \cap B) \geq P(\underline{X} \in A)P(\underline{X} \in B). \tag{1.5}$$

Specially, $PD(\mathcal{A}, \mathcal{A})$ is denoted by $PD(\mathcal{A})$. (see Shaked[7]). The inequalities in (1.1)-(1.5) are notions of positive dependence.

In this paper, we generalize these concepts of positive dependence in Shaked(1982) to compare the levels of dependence of two random vectors, that is, we study some positive dependence orderings in multivarite case. The discussion is done under a general framework and illustrated through what we think are the most intersting special cases.

The general framework and some observations are given in Section 2. In Section 3 the illustrative special cases are introduced.

2. GENERAL FRAMEWORK

Dependence measures have typically been developed to test for independence between two variables or to compare the levels of dependence of two set of variables. The notion of dependence for bivariate random variables is given in the following definition.(See Hollander, Proschan, and Scorning (1990)) : Given four variables X_1, X_2, Y_1, Y_2, we say that (X_1, Y_1) is more PQD than (X_2, Y_2) if for all x, y

$$\begin{aligned} & P(X_1 > x, Y_1 > y) - P(X_1 > x)P(Y_1 > y) \\ \geq\ & P(X_2 > x, Y_2 > y) - P(X_2 > x)P(Y_2 > y) \end{aligned}$$

and that (X_1, Y_1) is more associated than (X_2, Y_2) if for all componentwise nondecreasing functions f, g on R^2

$$Cov(f(X_1, Y_1), g(X_1, Y_1)) \geq Cov(f(X_2, Y_2), g(X_2, Y_2)).$$

Some other positive dependence orderings are also appeared in Hollander et al. (1990). Shriever (1987) has also generalized the ordering of Yanagimoto and Okamoto (1969) by introducing an ordering which he terms " more associated ". He showed that most well known rank measures of positive dependence preserve his ordering " more associated " in populations.

We extend the notion of association ordering to multivariate case :

Definition 1. The random vector $\underline{X} = (X_1, X_2, \cdots, X_n)$ is said to be more associated than $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$ if for all componentwise nondecreasing functions f, g on R^n

$$Cov(f(\underline{X}), g(\underline{X})) \geq Cov(f(\underline{Y}), g(\underline{Y})). \tag{2.1}$$

Proposition 1. For all componentwise nondecreasing functions f, g on R^n (2.1) holds if and only if for all open upper sets A, B

$$P(\underline{X} \in A \cap B) - P(\underline{X} \in A)P(\underline{X} \in B) \geq P(\underline{Y} \in A \cap B) - P(\underline{Y} \in A)P(\underline{Y} \in B). \tag{2.2}$$

Proof. We only show the converse : Let f, g be arbitrary increasing functions on R^n. Then for every real s and t , $A = \{f(\underline{X}) > s\}$ and $B = \{g(\underline{X}) > t\}$ are open upper sets. Thus

$$P(f(\underline{X}) > s, g(\underline{X}) > t) = P(\underline{X} \in A \cap B)$$
$$P(f(\underline{X}) > s) = P(\underline{X} \in A), P(g(\underline{X}) > t) = P(\underline{X} \in B).$$

Define

$$X_f(s) = \begin{cases} 1 & if \quad f(\underline{X}) > s, \\ 0, & otherwise. \end{cases} \qquad X_g(t) = \begin{cases} 1, & if \quad g(\underline{X}) > t, \\ 0, & otherwise. \end{cases}$$

Similarly, define

$$Y_f(s) = \begin{cases} 1, & if \quad f(\underline{Y}) > s, \\ 0, & otherwise, \end{cases} \qquad Y_g(t) = \begin{cases} 1, & if \quad g(\underline{Y}) > t, \\ 0, & otherwise. \end{cases}$$

Then

$$\begin{aligned}
& Cov(f(\underline{X}), g(\underline{X})) \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Cov(X_f(s), X_g(t))\, ds\, dt \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{E(X_f(s)X_g(t)) - EX_f(s)EX_g(t)\}\, ds\, dt \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{P(f(\underline{X}) > s, g(\underline{X}) > t) - P(f(\underline{X}) > s)P(g(\underline{X}) > t)\}\, ds\, dt \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\{(\underline{X} \in A \cap B) - P(\underline{X} \in A)P(\underline{X} \in B)\}\, ds\, dt \\
\geq & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{P(\underline{Y} \in A \cap B) - P(\underline{Y} \in A)P(\underline{Y} \in B)\}\, ds\, dt \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{P(f(\underline{Y}) > s, g(\underline{Y}) > t) - P(f(\underline{Y}) > s)P(g(\underline{Y}) > t)\}\, ds\, dt \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Cov(Y_f(s), Y_g(t))\, ds\, dt \\
= & Cov(f(\underline{Y}), g(\underline{Y})).
\end{aligned}$$

Note the above inequality follows from assumption (2.2). So by (2.1), $\underline{X}$ is more associated than $\underline{Y}$.

A possible way of weaking the condition of more associated ordering is to require that (2.2) holds for all A and B which belong to a subcollection of the collection of all upper sets.

Definition 2. The random vector $\underline{X}$ is said to be more positively dependent relative to $\mathcal{A}$ and $\mathcal{B}$ than $\underline{Y}$ if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\begin{aligned} &P(\underline{X} \in A \cap B) - P(\underline{X} \in A)P(\underline{X} \in B) \\ &\geq P(\underline{Y} \in A \cap B) - P(\underline{Y} \in A)P(\underline{Y} \in B). \end{aligned} \quad (2.3)$$

Remark 1. It is not necessary for random vectors to be positively dependent.

Remark 2. According to Proposition 1 and Definition 2 we observe several positive dependence ordering between association ordering and positive orthant dependence ordering and obtain some implications among them (See Lemma 1).

Proposition 2. Let $\mathcal{A}$ $\tilde{\mathcal{A}}$ and $\mathcal{B} \subset \tilde{\mathcal{B}}$. If $\underline{X}$ is more $PD(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ than $\underline{Y}$ then $\underline{X}$ is more $PD(\mathcal{A}, \mathcal{B})$ than $\underline{Y}$.

Proof. It is easy to prove. Let $\mathcal{A}$ be a collectionn of sets in R^n. Put $\mathcal{A}^c = \{A : A^c \in \mathcal{A}\}$($A^c$ denotes the complement of A in R^n) and $-\mathcal{A} = \{A : -A \in \mathcal{A}\}$ ($-A$ denotes $\{\underline{x} : -\underline{x} \in A\}$). In some instances

$$\mathcal{A}^c = -\mathcal{A} \quad (2.4)$$

Proposition 3. $\underline{X}$ is more $PD(\mathcal{A}, \mathcal{B})$ than $\underline{Y}$ if and only if $\underline{X}$ is more $PD(\mathcal{A}^c, \mathcal{B}^c)$ than $\underline{Y}$.

Proof. Assume that $\underline{X}$ is more $PD(\mathcal{A}, \mathcal{B})$ than $\underline{Y}$. Then

$$\begin{aligned} &P(\underline{X} \in A^c \cap B^c) - P(\underline{X} \in A^c)P(\underline{X} \in B^c) \\ &\geq P(\underline{Y} \in A^c \cap B^c) - P(\underline{Y} \in A^c)P(\underline{Y} \in B^c) \end{aligned} \quad (2.5)$$

since $P(\underline{X} \in A \cap B) - P(\underline{X} \in A)P(\underline{X} \in B) = P(\underline{X} \in A^c \cap B^c) - P(\underline{X} \in A^c)P(\underline{X} \in B^c)$.

Thus $\underline{X}$ is more $PD(\mathcal{A}^c, \mathcal{B}^c)$ than $\underline{Y}$. Similarly the converse is also proved.

Proposition 4. $\underline{X}$ is more $PD(\mathcal{A}, \mathcal{B})$ than $\underline{Y}$ if and only if $-\underline{X}$ is more $PD(-\mathcal{A}, -\mathcal{B})$ than $-\underline{Y}$.

Proof. Assume that $\underline{X}$ is more $PD(\mathcal{A},\mathcal{B})$ than $\underline{Y}$. Then

$$P(-\underline{X}\in(-A)\cap(-B))-P(-\underline{X}\in(-A))P(-\underline{X}\in(-B))$$
$$\geq P(-\underline{Y}\in(-A)\cap(-B))-P(-\underline{Y}\in(-A))P(-\underline{Y}\in(-B))$$

since $P(\underline{X}\in A\cap B)=P(\underline{X}\in A,\underline{X}\in B)=P(-\underline{X}\in(-A),-\underline{X}\in(-B))=P(-\underline{X}\in(-A))\cap(-B))$. Thus $-\underline{X}$ is more $PD(-\mathcal{A},-\mathcal{B})$ than $-\underline{Y}$. Similarly the converse is also obtained.

Remark 3. From the last two propositions it follows that if $\mathcal{A}$ and $\mathcal{B}$ satisfy (2.4) then $\underline{X}$ is more $PD(\mathcal{A},\mathcal{B})$ than $\underline{Y}$ if and only if $-\underline{X}$ is more $PD(\mathcal{A},\mathcal{B})$ than $-\underline{Y}$.

In some applications the collection $\mathcal{A}$ is closed under intersections, that is,

$$A_k\in\mathcal{A},k=1,2,\cdots,K\Rightarrow\cap_{k=1}^{K}A_k\in\mathcal{A},K=2,3,\cdots \tag{2.6}$$

In that case we have the following proposition :

Proposition 5. Assume that $\mathcal{A}$ satisfies (2.6) and that $P(\underline{X}\in A_k)\geq P(\underline{Y}\in A_k)$ $k=1,2,3,\cdots K$. If $\underline{X}$ is more $PD(\mathcal{A})$ than $\underline{Y}$ then, for all $A_k\in\mathcal{A},k=1,2,\cdots,K:K=2,\cdots,$

$$P(\underline{X}\in\cap_{k=1}^{K}A_k)-\prod_{k=1}^{K}P(\underline{X}\in A_k)\geq P(\underline{Y}\in\cap_{k=1}^{K}A_k)-\prod_{k=1}^{K}P(\underline{Y}\in A_k) \tag{2.7}$$

In many instances, if $\underline{X}$ is more $PD(\mathcal{A},\mathcal{B})$ than $\underline{Y}$, then there exist families of n-variate functions $\mathcal{F}$ and $\mathcal{G}$ such that

$$Cov(f(\underline{X}),g(\underline{X})\geq Cov(f(\underline{Y}),g(\underline{Y}))\ whenever\ f\in\mathcal{F},g\in\mathcal{G} \tag{2.8}$$

provided the expectations exist. When $\underline{X}$ satisfies (2.8) we will say that $\underline{X}$ is more functionally positively dependent $FPD(\mathcal{F},\mathcal{G})$ than $\underline{Y}$.

Proposition 6. Let $\mathcal{F}\subset\tilde{\mathcal{F}}$ and $\mathcal{G}\subset\tilde{\mathcal{G}}$. If $\underline{X}$ is more $FPD(\tilde{\mathcal{F}},\tilde{\mathcal{G}})$ than $\underline{Y}$ then $\underline{X}$ is more $FPD(\mathcal{F},\mathcal{G})$ than $\underline{Y}$, and if $\underline{X}$ is more $FPD(\tilde{\mathcal{F}})$ than $\underline{Y}$ then $\underline{X}$ is more $FPD(\mathcal{F})$ than $\underline{Y}$.

3. CONCEPTS OF POSITIVE DEPENDENCE ORDERINGS

The following collections of upper sets in R^n were already discussed by Shaked(1982).

(1) Let $\mathcal{A}_1^{(n)}$ be the collection of all upper orthants in R^n, that is, $A \in \mathcal{A}_1^{(n)}$ if and only if

$$A = \{\underline{X} : x_i > a_i, i = 1, 2, \cdots, n\} \\ \text{for some } a_i \in [-\infty, \infty], i = 1, 2, \cdots, n. \tag{3.1}$$

(2) Let $\mathcal{A}_2^{(n)}$ be the collection of all open upper half spaces, that is, $A \in \mathcal{A}_2^{(n)}$ if and only if for some $a_0 \in [-\infty, \infty]$ and $a_i \in [0, \infty), i = 1, 2, \cdots, n$,

$$A = \{\underline{x} : \sum_{i=1}^{n} a_i x_i > a_0\} \tag{3.2}$$

(3) Let $\mathcal{A}_3^{(n)}$ be the collection of all sets of the form

$$A = \cap_{1 \le \beta \le \gamma} \cup_{\alpha \in C_\beta} \{\underline{x} : x_\alpha > a_\alpha\} \\ \text{for some } a_i \in [-\infty, \infty], i = 1, 2, \cdots, n, \tag{3.3}$$

or of the form

$$A = \cup_{1 \le \beta \le \delta} \cap_{\alpha \in P_\beta} \{\underline{x} : x_\alpha > a_\alpha\} \\ \text{for some } a_i \in [-\infty, \infty], i = 1, 2, \cdots, n, \tag{3.4}$$

where, some positive integers γ and $\delta, C_\beta \subset \{1, 2, \cdots, n\}, \beta = 1, 2, \cdots, \gamma$ and $P_\beta \subset \{1, 2, \cdots, n\}, \beta = 1, 2, \cdots, \delta$.

(4) Let $\mathcal{A}_4^{(n)}$ be the collection of all convex open upper sets in R^n.

(5) Let $\mathcal{A}_5^{(n)}$ be the collection of all open upper sets in R^n.

In the following the superscript n on the $\mathcal{A}'s$ will be omited when there is no danger of a confusion.

Definition 3. (i) $\underline{X} = (X_1, X_2, \cdots, X_n)$ is called more positively upper orthant dependent(PUOD) than $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$ if

$$P(X_1 > a_1, \cdots, X_n > a_n) - \prod_{i=1}^{n} P(X_i > a_i) \\ \ge P(Y_1 > a_1, \cdots, Y_n > a_n) - \prod_{i=1}^{n} P(Y_i > a_i).$$

(ii) $\underline{X}$ is called more positively lower orthant dependent(PLOD) than $\underline{Y}$ if

$$P(X_1 \le a_1, \cdots, X_n \le a_n) - \prod_{i=1}^{n} P(X_i \le a_i)$$
$$\ge P(Y_1 \le a_1, \cdots, Y_n \le a_n) - \prod_{i=1}^{n} P(Y_i \le a_i).$$

Since $\mathcal{A}_5 \subset \mathcal{A}_4 \subset \mathcal{A}_2, \mathcal{A}_5 \subset \mathcal{A}_3 \subset \mathcal{A}_1$, and $\mathcal{A}_4 \subset \mathcal{A}_1$ from Proposition 2.3 and Definition 3.1 we have following implications :

Lemma 1.

$$\begin{array}{ccccc} \text{more } PD(\mathcal{A}_5) & \Rightarrow & \text{more } PD(\mathcal{A}_4) & \Rightarrow & \text{more } PD(\mathcal{A}_2) \\ \Downarrow & & \Downarrow & & \\ \text{more } PD(\mathcal{A}_3) & \Rightarrow & \text{more } PD(\mathcal{A}_1) & & \end{array}$$

Thus, for $j = 1, 3, 4, 5$ the more $PD(\mathcal{A}_j)$ ordering is weaker than the associated ordering and stronger than the orthant dependent ordering. Some of results of Section 2 can be specialized now to the notions of this section as follows. Since $\mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_5$ satisfy (2.4) from Remark 3 and Lemma 1 we obtain the following theorem :

Theorem 1. For $j = 2, 3, 5,$ $\underline{X}$ is more $PD(\mathcal{A}_j)$ than $\underline{Y}$ if and only if $-X$ is more $PD(\mathcal{A}_j)$ than $-\underline{Y}$. Since $\mathcal{A}_1, \mathcal{A}_4$ and $\mathcal{A}_5$ satisfy (2.6) we obtain from Proposition 2 :

Theorem 2. Let $P(\underline{X} \in A) \ge P(\underline{Y} \in A)$ for $A \in \mathcal{A}_j$, $j = 1, 4, 5$. If for $j = 1, 4, 5,$ $\underline{X}$ is more $PD(\mathcal{A}_j)$ than $\underline{Y}$ then

$$P(\underline{X} \in \cap_{k=1}^{K} A_k) - \prod_{k=1}^{K} P(\underline{X} \in A_k) \ge P(\underline{Y} \in \cap_{k=1}^{K} A_k) - \prod_{k=1}^{K} P(\underline{Y} \in A_k)$$

whenever $A_k \in \mathcal{A}_j, k = 1, 2, \cdots, K; K = 1, 2, \cdots$.

Theorem 3. (a) If $\underline{X}$ is more $PD(\mathcal{A}_j)$ than $\underline{Y}$ for $j = 1, 3, 4, 5$, then $\underline{X}$ is more PUOD than $\underline{Y}$.
(b) If $\underline{X}$ is more $PD(-\mathcal{A}_j)$ than $\underline{Y}$ for $j = 1, 3, 4, 5$, then $\underline{X}$ is more PLOD than $\underline{Y}$.
(c) If $\underline{X}$ is more $PD(\mathcal{A}_j)$ than $\underline{Y}$ for $j = 3, 4, 5$, then $\underline{X}$ is more PLOD than $\underline{Y}$.
(d) If $\underline{X}$ is more $PD(-\mathcal{A}_j)$ for $j = 3, 4, 5$, then $\underline{X}$ is more PUOD than $\underline{Y}$.

Proof. By Lemma 1 it is enough to prove (a) for $j = 1$. If $\underline{X}$ is more $PD(\mathcal{A}_1)$ than $\underline{Y}$ then by Theorem 2., for any vector $\underline{a}^{(1)}, \cdots, \underline{a}^{(n)}$,

$$P(\underline{X} > \underline{a}^{(1)}, \cdots, \underline{X} > \underline{a}^{(n)}) - \prod_{i=1}^{n} P(\underline{X} > \underline{a}^{(i)})$$
$$\geq P((\underline{Y} > \underline{a}^{(1)}, \cdots, \underline{Y} > \underline{a}^{(n)}) - \prod_{i=1}^{n} P(\underline{Y} > \underline{a}^{(i)}).$$

Take $\underline{a}^{(1)} = (a_1, -\infty, \cdots, -\infty), \cdots, \underline{a}^{(n)} = (-\infty, \cdots, -\infty, a_n)$ to obtain

$$P(X_1 > a_1, \cdots, X_n > a_n) - \prod_{i=1}^{n} P(X_i > a_i)$$
$$\geq P(Y_1 > a_1, \cdots, Y_n > a_n) - \prod_{i=1}^{n} P(Y_i > a_i),$$

that is $\underline{X}$ is more PUOD than $\underline{Y}$.
(b) For $j = 3, 4, 5$, it is clear that $-\mathcal{A}_j \quad -\mathcal{A}_1$. The proof of (b) for $j = 1$ is similar to the proof of (a) with $j = 1$.
(c) By Proposition 2, if $\underline{X}$ is more $PD(\mathcal{A})$ than $\underline{Y}$ then $\underline{X}$ is more $PD(\mathcal{A}_j^c)$ than $\underline{Y}, j = 3, 4, 5$. It is easy to see that for $j = 3, 4, 5, \mathcal{A}_j^c \supset -\mathcal{A}_1$, thus, if $\underline{X}$ is more $PD(\mathcal{A}_j)$ than $\underline{Y}$ then $\underline{X}$ is more $PD(-\mathcal{A}_1)$ than $\underline{Y}$. Hence by part (b), $\underline{X}$ is more PLOD than $\underline{Y}$.
(d) The proof is similar to the proof of (c).

Remark 4. Theorem 3 does not say anything but about $PD(\mathcal{A}_2)$ ordering. However, it is easy to see that if $(X_1, \cdots, X_n)$ is more $PD(\mathcal{A}_2)$ than $(Y_1, \cdots, Y_n)$[or, equivalently, more $PD(-\mathcal{A}_2)$ han $(Y_1, \cdots, Y_n)$] then (X_i, X_k) is more PQD than $\underline{Y}$ for every $i, k \in \{1, 2, \cdots, n\}$.

The following collections of increasing functions in R^n or in $R_+^n = \{\underline{x} : \underline{x} \geq \underline{0}\}$ were considered by Shaked(1982). Below, for every $x \geq 0, x/0$ is interpreted as ∞, and for every $x, x/\infty$ is interpreted as 0.
(i) Let $\mathcal{F}_1^{(n)}$ be the collection of all functions, defined on R_+^n, of the form

$$f(\underline{x}) = min_{1 \leq i \leq n}\{b_i x_i\} \;\; for\ some \;\; b_i \in [0, \infty], \tag{3.5}$$

(ii) Let $\mathcal{F}_2^{(n)}$ be the collection of all functions, defined on R^n, of the form

$$f(\underline{x}) = \sum_{i=1}^{n} a_i x_i \;\; for\ some \;\; a_i \in [0, \infty], i = 1, 2, \cdots, n. \tag{3.6}$$

(iii) Let $\mathcal{F}_3^{(n)}$ be the collection of all scaled n−dimensional coherent life functions(see, for example, [1]). By definition, life functions defined on R_+^n. In general the scaled coherent life functions are of the form

$$f(\underline{x}) = min_{1\leq\beta\leq\gamma} max_{\alpha\in C_\beta} b_\alpha x_\alpha \quad for\ some\ \ b_i \in [0,\infty), i = 1,2,\cdots,n \quad (3.7)$$

or of the form

$$f(\underline{x}) = max_{1\leq\beta\leq\gamma} min_{\alpha\in P_\beta} b_\alpha x_\alpha \quad for\ some\ \ b_i \in [0,\infty), i = 1,2,\cdots,n \quad (3.8)$$

where , for some positive integers γ and $\delta, C_\beta \subset \{1,2,\cdots,n\}, \beta = 1,2,\cdots,\gamma$ and $P_\beta \subset \{1,2,\cdots,n\}, \beta = 1,2,\cdots,\delta$.

(iv) Let $\mathcal{F}_4^{(n)}$ be the collection of all concave increasing functions on R^n(or on R_+^n when we deal with nonnegative random vectors). Let $\mathcal{F}_5^{(n)}$ be the collection of all measurable increasing functions on R^n(or on R_+^n when we deal with nonnegative random vectors). From Proposition 5 we have the following theorem:

Theorem 4.

$$\begin{array}{ccccc} more\ \ FPD(\mathcal{F}_5) & \Rightarrow & more\ \ FPD(\mathcal{F}_4) & \Rightarrow & moreFPD(\mathcal{F}_2) \\ \Downarrow & & \Downarrow & & \\ more\ \ FPD(\mathcal{F}_3) & \Rightarrow & more\ \ FPD(\mathcal{F}_1). & & \end{array}$$

Remark 5. Esary and Marshall(1974) considered random vectors $\underline{X} = (X_1, \cdots, X_n)$ such that $min_{i\in I} b_i X_i$ has an exponential distribution for all $b_i > 0$ $i = 1,\cdots,n$ and all nonempty sets $I \subset \{1,\cdots,n\}$. Let $\underline{X} = (X_1,\cdots,X_n)$ and $\underline{Y} = (Y_1,\cdots,Y_n)$ be as above and $\underline{X}$ be more positive orthant dependent than $\underline{Y}$ then we may derive an example that $\underline{X}$ is $PD(\mathcal{A}_3)$ than $\underline{Y}$ in the following paper. We may also observe some positive dependent ordering examples among the random vectors with multivariate normal distributions in that paper.

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