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On Convergence in p–Mean of Randomly Indexed Partial Sums and Some First Passage Times for Random Variables Which Are Dependent or Non–identically Distributed

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Abstract

Let $S_n, n = 1, 2, \cdots$ denote the partial sums of not necessarily independent random variables. Let $N(c) = min\{n; S_n > c\}$, $c \geq 0$. Theorem 2 states that $N(c)$, (suitably normalized), tends to 0 in p–mean, $1 \leq p < 2$, as $c \to \infty$ under mild conditions, which generalizes earlier result by Gut(1974). The proof follows by applying Theorem 1, which generalizes the known result $E|S_n|^p = o(n), 0 < p < 2$, as $n \to \infty$ to randomly indexed partial sums.

Key Words : First passage time; Stopping time; Martingale; Uniformly conditionally integrable.

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1. INTRODUCTION

Let $X_1, X_2, \cdots$ be a sequence of random variables, let $S_n = X_1 + \cdots + X_n$, and for $c > 0$ set $N = N(c) =$ first $n \geq 1$ such that $S_n > c$. We set $\mathcal{F}_n = \sigma\{X_i, 1 \leq i \leq n\}$, $n \geq 1$, and $\mathcal{F}_0 = \{\phi, \Omega\}$. Pyke and Root (1968) prove the L_p-convergence for $n^{-1/p}S_n$, $0 < p < 2$, for sequences in the i.i.d. case. The result was extended by Chatterji(1969) to martingale differences under some domination condition.

Chow(1971) relaxes the domination condition to uniform integrability. Recently Chandra(1989) and Gut(1992) prove the theorem under the condition of uniform integrability in the Cesáro sense which is weaker than uniform integrability. Gut(1974), inspired by Chow(1971), proves a corresponding result for stopped random walks for a sequence of i.i.d. random variables and by applying this he proves $c^{-1} \cdot E|N - c/\mu|^p \to o$ as $c \to \infty$ if $E|X|^p < \infty$, $1 \leq p < 2$. Chow and Robbins(1963) prove that if $E(X_n|\mathcal{F}_{n-1}) = EX_n = \mu_n$, $\lim_{n\to\infty} (\mu_1 + \cdots \mu_n)/n = \mu$, $0 < \mu < \infty$ and $E(|X_n - \mu_n|^\alpha|\mathcal{F}_{n-1}) \leq K < \infty$ for some $\alpha > 1$(for the independent case, this is replaced by the assumption that $\{X_n - EX_n\}$ are uniformly integrable) then $\lim_{c\to\infty} EN/c = 1/\mu$. The purpose of this paper is to generalize results in Gut(1974) applying Chow and Robbins'(1963) result with the same scheme as Chandra(1989) and Gut(1992).

2. RESULTS

Let $\|X\| = sup\{\alpha : P\{|X| > \alpha\} > 0\}$, where X is a random variable.

Definition 1. A sequence $\{X_n\}$ of random variables is uniformly conditionally integrable(UCI) if

$$\lim_{a\to\infty} sup_n\{\|E(|X_n|I\{|X_n| > a\}|\mathcal{F}_{n-1})\|\} = 0,$$

where $I\{\cdot\}$ denotes the indicator function of the set in braces.

Definition 2. A sequence $\{X_n\}$ of random variables is uniformly conditionally integrable in the Cesàro sence(UCIC) if

$$\lim_{a\to\infty} sup_n\{\frac{1}{n}\sum_{k=1}^{n} \|E(|X_k|I\{|X_k| > a\}|\mathcal{F}_{k-1})\|\} = 0.$$

Clearly, the above condition is implied by UCI. Actually UCIC is strictly weaker than UCI(see Chandra(1989) Example 2). It is also noted that if $\{X_n\}$ is a sequence of independent random variables, then $\{X_n\}$ are UCI(UCIC), if and only if, $\{X_n\}$ are uniformly integrable(uniformly integrable in the Cesàro sense (see Chandra(1989) and Gut(1992)), respectively). We first consider the following theorem which generalizes results by Gut(1974, Theorem 2) and Pyke and Root(1968).

Theorem 1. Let $\{|X_n|^p,\ n \geq 1\}$ be UCIC for some $0 < p < 2$, and let $E(X_n|\mathcal{F}_{n-1}) = 0$ for all $n \geq 1$ if $1 \leq p < 2$. Let $\{\tau(c),\ c \geq 0\}$ be a non–decreasing family of stopping times such that $E\tau(c) < \infty$ and $E\tau(c) \uparrow \infty$, as $c \to \infty$. Then

$$E|S_{\tau(c)}|^p = o(E\tau(c)), \quad as \quad c \to \infty. \tag{2.1}$$

If moreover, $c^{-1}E\tau(c) \to \mu^{-1}$, as $c \to \infty$, where μ is a positive constant, then

$$E|S_{\tau(c)}|^p = o(c), \quad as \quad c \to \infty. \tag{2.2}$$

Proof. The technique is similar to the one used in Chow(1961) and also in Gut(1974). A slight different situation is that we cannot use Wald's lemma. For this the following lemma is stated and proved in Chow, Robbins and Teicher(1965).

Lemma 1(Chow, Robbins and Teicher). For any stopping time τ and any $r > 0$,

$$E\sum_{k=1}^{\tau}|X_k|^r = E\sum_{k=1}^{\tau}E(|X_k|^r|\mathcal{F}_{k-1}).$$

First, let $1 \leq p < 2$. Since $E(X_n|\mathcal{F}_{n-1}) = 0$ for all $n \geq 1$, $\{\sum_{k=1}^{n} X_k,\ n \geq 1\}$ is a martingale. Define $\tau_n(c) = min\{\tau(c), n\}$, and define $U_k = \sum_{i=1}^{k} X_i \cdot I\{\tau_n(c) \geq i\},\ \ k = 1, 2, \cdots, n$. Then also $\{U_k\}_{k=1}^{n}$ is a martingale, $U_n = S_{\tau_n(c)}$, and $E|U_k|^p \leq E|U_n|^p \leq E|S_n|^p < \infty$, since $\{|U_k|^p\}_{k=1}^{n}$ is a sub-martingale (see Doob(1953), Theorem 2.1). Thus by the Burkholder–Davis inequalities (see Burkholder(1966), Theorem 9 and Davis(1970), Theorem 1), there exists a constant $C_p > 0$, depending only on p, such that

$$E|U_n|^p \leq C_p \cdot E|\sum_{k=1}^{\tau_n(c)} X_k^2|^{p/2}.$$

Since $\lim_{a\to\infty} sup_n\{\frac{1}{n}\sum_{i=1}^n \|E(|X_i|^p I\{|X_i|^p > a\}|\mathcal{F}_{i-1})\|\} = 0$, there exists, for every $\epsilon > 0$, an $M > 0$, such that $\sum_{i=1}^n E(|X_i|^p I\{|X_i|^p > M\}|\mathcal{F}_{i-1}) \leq n\epsilon$ for all $n \geq M$. Put

$$X_n^{'} = X_n \cdot I\{|X_n|^p \leq M\} \; and \; X_n^{''} = X_n - X_n^{'},\ n = 1, 2, \cdots.$$

By the C_r–inequality(Loève (1977, p. 157)) and Lemma 1,

$$\begin{aligned}
E|\sum_{k=1}^{\tau_n(c)} X_k^2|^{p/2} &= E|\sum_{k=1}^{\tau_n(c)}((X_k^{'})^2 + (X_k^{''})^2)|^{p/2} \\
&\leq E|\sum_{k=1}^{\tau_n(c)}(X_k^{'})^2|^{p/2} + E|\sum_{k=1}^{\tau_n(c)}(X_k^{''})^2|^{p/2} \\
&\leq E|\tau_n(c)\cdot M^{2/p}|^{p/2} + E\sum_{k=1}^{\tau_n(c)}|X_k^{''}|^p \\
&= ME(\tau_n(c))^{p/2} + E\sum_{k=1}^{\tau_n(c)} E(|X_k^{''}|^p|\mathcal{F}_{k-1}) \\
&= ME(\tau_n(c))^{p/2} + E\sum_{i=1}^{n}(I\{\tau_n(c) = i\}\sum_{k=1}^{i} E(|X_k^{''}|^p|\mathcal{F}_{k-1})) \\
&\leq ME(\tau_n(c))^{p/2} + E\sum_{i=1}^{n} I\{\tau_n(c) = i\}\cdot i\epsilon \\
&\leq ME(\tau(c))^{p/2} + \epsilon E\tau(c). \qquad (2.3)
\end{aligned}$$

Thus, $E|S_{\tau_n(c)}|^p \leq C_p\cdot ME(\tau(c))^{p/2} + C_p\cdot\epsilon\cdot E\tau(c) < \infty$, and by Fatou's lemma,

$$E|S_{\tau(c)}|^p \leq C_p\cdot M\cdot E(\tau(c))^{p/2} + C_p\cdot\epsilon\cdot E\tau(c) < \infty.$$

Therefore $E|S_{\tau(c)}|^p = o(E\tau(c))$ as $c \to \infty$. Now, let $0 < p < 1$. By applying the C_r–inequality and Lemma 1 as above we obtain

$$\begin{aligned}
E|S_{\tau_n(c)}|^p &= E|\sum_{k=1}^{\tau_n(c)} X_k|^p \\
&\leq E|\sum_{k=1}^{\tau_n(c)} X_k^{'}|^p + E|\sum_{k=1}^{\tau_n(c)}(X_k^{''})^2|^{p/2} \\
&\leq ME(\tau_n(c))^p + E\sum_{k=1}^{\tau_n(c)} E(|X_k^{''}|^p|\mathcal{F}_{k-1})
\end{aligned}$$

$$\leq \quad ME(\tau(c))^p + \epsilon E\tau(c),$$

and hence, by Fatou's lemma,

$$E|S_{\tau(c)}|^p \leq M \cdot E(\tau(c))^p + \epsilon E\tau(c) < \infty.$$

This proves (2.1), from which the proof (2.2) is immediate.

The following result generalizes Theorem 1 of Gut(1974).

Theorem 2. Let $\{|X_n|^p, n \geq 1\}$ be UCI, $1 < p < 2$ and let $E(|X_n - E(X_n|\mathcal{F}_{n-1})|^\alpha|\mathcal{F}_{n-1}) \leq K < \infty$ for some $\alpha > 1$ if $p = 1$ (if the $\{X_n\}$ are independent, we assume $\{X_n - EX_n\}$ are uniformly integrable for $p = 1$). If $E(X_n|\mathcal{F}_{n-1}) = EX_n = \mu_n$, and $\sum_{i=1}^n(\mu_i - \mu) = o(n^{1/p}), 0 < \mu < \infty$ then

$$c^{-1} \cdot E|N - c/\mu|^p \to 0, \quad as \quad c \to \infty.$$

To prove Theorem 2 we begin with the following lemma.

Lemma 2. Let $\{|X_n|^p, \ n \geq 1\}$ be UCI, $p > 1$, and let $E(|X_n - E(X_n|\mathcal{F}_{n-1})|^\alpha$ $|\mathcal{F}_{n-1}) \leq K < \infty$ for some $\alpha > 1$ if $p = 1$ (if the $\{X_n\}$ are independent, we assume $\{X_n - EX_n\}$ are uniformly integrable for $p = 1$). If $E(X_n|\mathcal{F}_{n-1}) = EX_n = \mu_n$ and $\sum_{i=1}^n(\mu_i - \mu) = o(n), \ \ 0 < \mu < \infty$, then

$$c^{-1} \cdot E(S_N - c)^p \to 0 \quad as \quad c \to \infty.$$

Proof. First note that if $\{|X_n|^p\}$ is UCI, then $sup_n\{\|max\{E(|X_n|^p|\mathcal{F}_{n-1}), E(|X_n - \mu_n|^p|\mathcal{F}_{n-1})\}\|\} \leq K$ for some $K < \infty$ and hence $EN(c) < \infty$ for all $c > 0$(see proof of Theorem 1, Chow and Robbins(1963)). Let $\epsilon > 0$ be an arbitrary small given number and choose M so large that $sup_k\{\|E(|X_k|^p I\{|X_k|^p > \epsilon^2 n\}|\mathcal{F}_{k-1})\|\} < \epsilon$ if $n \geq M$. Then we have

$$\begin{aligned} EX_N^p &= E(X_N^+)^p = E((X_N^+)^p \cdot I\{(X_N^+)^p \leq \epsilon N\}) + E((X_N^+)^p \cdot I\{(X_N^+)^p > \epsilon N\}) \\ &\leq \epsilon EN + E(\sum_{k=1}^N (X_k^+)^p \cdot I\{(X_k^+)^p > \epsilon k\}) \\ &= \epsilon EN + E((\sum_{k=1}^N (X_k^+)^p \cdot I\{(X_k^+)^p > \epsilon k\}) \cdot I\{N \leq \epsilon M\}) \\ &\quad + E((\sum_{k=1}^N (X_k^+)^p \cdot I\{(X_k^+)^p > \epsilon k\}) \cdot I\{N > \epsilon M\}) \end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon EN + E((\sum_{k=1}^{[\epsilon M]}(X_k^+)^p \cdot I\{N \leq \epsilon M\}) \\
&+ E((\sum_{k=1}^{[\epsilon M]}(X_k^+)^p) \cdot I\{N > \epsilon M\}) \\
&+ E((\sum_{k=[\epsilon M]+1}^{N}(X_k^+)^p \cdot I\{(X_k^+)^p > \epsilon^2 M\}) \cdot I\{N > \epsilon M\}) \\
&\leq \epsilon EN + E(\sum_{k=1}^{[\epsilon M]}(X_k^+)^p) + E(\sum_{k=1}^{N}(X_k^+)^p \cdot I\{(X_k^+)^p > \epsilon^2 M\}) \\
&= \epsilon EN + E(\sum_{k=1}^{[\epsilon M]} E(|X_n|^p|\mathcal{F}_{n-1})) + E(\sum_{k=1}^{N} E(|X_k|^p I\{|X_k|^p > \epsilon^2 M\}|\mathcal{F}_{n-1})) \\
&\leq \epsilon EN + K\epsilon M + EN \cdot \epsilon \\
&= \epsilon(2EN + KM). \qquad (2.4)
\end{aligned}
$$

The last inequality holds because of the way ϵ and M are chosen and preceding equality is from Lemma 1.

Thus $0 \leq EX_N^p/EN \leq 2\epsilon + \epsilon KM/EN$, from which it follows that $0 \leq \limsup_{c\to\infty} EX_N^p/EN \leq 2\epsilon$. Since ϵ was arbitrary, we have $\lim_{c\to\infty} EX_N^p/EN = 0$. And since the assumptions satisfy those of Theorem 1 in Chow and Robbins' paper(Theorem 2(Chow and Robbins) when $\{X_n\}$ are independent), $EN/c \to \mu^{-1}$ as $c \to \infty$, from which $\lim_{c\to\infty} EX_N^p/c = 0$. Now from $c < S_N \leq c + X_N$, it follows that

$$0 \leq \frac{E(S_N - c)^p}{c} < \frac{EX_N^p}{c} \to 0 \quad as \quad c \to \infty.$$

Lemma 3. If $\|E(|X_n|^\alpha|\mathcal{F}_{n-1})\| \leq K < \infty$ for some $\alpha > 1$, then $\{|X_n|, n \geq 1\}$ is UCI.

Proof. The elementary inequality

$$a^{\alpha-1}E(|X_n|I\{|X_n|^\alpha > a\}|\mathcal{F}_{n-1}) \leq E(|X_n|^\alpha I\{|X_n|^\alpha > a\}|\mathcal{F}_{n-1})$$

implies that

$$sup_n\{\|E(|X_n|I\{X_n| > a^{1/\alpha}|\mathcal{F}_{n-1}\})\|\} \leq \frac{K}{a^{\alpha-1}}$$

and since $\alpha > 1$, the last expression decreases to 0 as $a \uparrow \infty$.

Proof of Theorem 2. Now, let $1 \le p < 2$. As in the proof of Lemma 2, by Theorem 1 and 2(Chow and Robbins), $c^{-1} \cdot EN \to \mu^{-1}$ as $c \to \infty$. Set $Y_n = X_n - \mu_n$. Then we can easily check that $\{|Y_n|^p, n \ge 1\}$ are UCI by elementary computation for $1 < p < 2$ and by Lemma 3 for $p = 1$. Applying Theorem 1, $E|S_N - \sum_{i=1}^N \mu_i|^p = E|\sum_{i=1}^N Y_n|^p = o(EN)$, hence, combining these, we have

$$c^{-1} \cdot E|S_N - \sum_{i=1}^{N} \mu_i|^p \to 0, \quad as \quad c \to \infty. \tag{3}$$

Now, by Minkowski's inequality,

$$\begin{aligned}(E|\mu N - c|^p)^{1/p} &\le (E|S_N - \mu N|^p)^{1/p} + (E|S_N - c|^p)^{1/p} \\ &\le (E|S_N - \sum_{i=1}^{N} \mu_i|^p)^{1/p} + (E|\sum_{i=1}^{N} \mu_i - \mu N|^p)^{1/p} + (E|S_N - c|^p)^{1/p}\end{aligned} \tag{2.5}$$

By (3) and Lemma 2, it suffices to show

$$c^{-1} \cdot E|\sum_{i=1}^{N} (\mu_i - \mu)|^p \to 0 \quad as \quad c \to \infty.$$

Since $\sum_{i=1}^n (\mu_i - \mu) = o(n^{1/p})$, for given $\epsilon > 0$, choose n_0 so that $|\sum_{i=1}^n (\mu_i - \mu)|^p \le n\epsilon$ for all $n \ge n_0$. Then we have

$$\begin{aligned}E|\sum_{i=1}^{N}(\mu_i - \mu)|^p &= E|(\sum_{i=1}^{N}(\mu_i - \mu))I\{N \le n_0\}|^p + E|(\sum_{i=1}^{N}(\mu_i - \mu))I\{N > n_0\}|^p \\ &\le |\sum_{i=1}^{n_0}(\mu_i - \mu)|^p + \sum_{i=1}^{N} P\{N = n_0 + i\}|\sum_{k=1}^{n_0+i}(\mu_i - \mu)|^p \\ &\le C + \epsilon EN,\end{aligned} \tag{2.6}$$

where C is an unimportant constant.

Therefore $E|\sum_{i=1}^N (\mu_i - \mu)|^p = o(c)$ since $\lim_{c\to\infty} EN/c = \mu^{-1}$ and ϵ is arbitrary.

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