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A Renewal Theorem for Random Walks with Time Stationary Random Distribution Function

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Abstract

Sums of independent random variables $S_n = X_1 + X_2 + \cdots + X_n$ are considered, where the X_n are chosen according to a stationary process of distributions. Given the time $t \geq 0$, let $N(t)$ be the number of indices n for which $0 < S_n \leq t$. In this set up we prove that $N(t)/t$ converges almost surely and in L^1 as $t \to \infty$, which generalizes classical renewal theorem.

Key Words : Renewal theorem; Random walks; Stationarity; Ergodicity; Random distribution function.

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1. INTRODUCTION

Let $\mathcal{F}$ be a set of distributions on R with the topology of weak convergence, and let $\mathcal{A}$ be the σ–field generated by the open sets. We denote by $\mathcal{F}_1^\infty$ the space consisting of all infinite sequence $(F_1, F_2, \cdots), F_n \in \mathcal{F}$ and R_1^∞ the space consisting of all infinite sequences $(x_1, x_2, \cdots)$ of real numbers. Take the σ–field $\mathcal{A}_1^\infty$ to be the smallest σ–field of subsets of $\mathcal{F}_1^\infty$ containing all finite–dimensional rectangles and take $\mathcal{B}_1^\infty$ to be the Borel σ–field of $\mathcal{F}_1^\infty$. Let $\omega = (F_1^\omega, F_2^\omega, \cdots)$ be the coordinate process in $\mathcal{F}$ and ν its distribution on $\mathcal{A}_1^\infty$. Let θ be the coordinate shift: $\theta^k(\omega) = \omega'$ with $F_n^{\omega'} = F_{n+k}^\omega, k = 1, 2, \cdots$. On $(R_1^\infty, \mathcal{B}_1^\infty)$ we also define the shift transformation $\sigma : R_1^\infty \to R_1^\infty$ by $\sigma(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. ν is called stationary if for every $A \in \mathcal{A}_1^\infty, \nu(\theta^{-1}(A)) = \nu(A)$ and we let π be its marginal distribution. Let $\mathcal{I}$ be the σ–field of invariant sets in $\mathcal{A}_1^\infty$, that is, $\mathcal{I} = \{A|\theta^{-1}(A) = A, A \in \mathcal{A}_1^\infty\}$ and let $\mathcal{J}$ be the σ–field of invariant sets in $\mathcal{B}_1^\infty$, that is, $\mathcal{J} = \{B|\sigma^{-1}(B) = B, B \in \mathcal{B}_1^\infty\}$. For each ω, define a probability measure P_ω on $(R_1^\infty, \mathcal{B}_1^\infty)$ so that $P_\omega = \Pi_{i=1}^\infty F_i^\omega$. A monotone class argument shows that $P_\omega(B), B \in \mathcal{B}_1^\infty$, is $\mathcal{A}_1^\infty$–measurable as a function of ω. So we can define a new probability measure such that $P(B) = \int P_\omega(B)\nu(d\omega)$. Define the process $\{X_n\}$ on $(R_1^\infty, \mathcal{B}_1^\infty)$ such that $X_n(x_1, x_2, \cdots) = x_n$ and set $S_n = X_1 + X_2 + \cdots + X_n$. By the definition of $P_\omega, \{X_n\}$ are independent with respect to P_ω and we also note that $\{X_n\}$ is a sequence of independent and identically distributed random variables when $\mathcal{F}$ has just one element. The purpose of this paper is to generalize the classical renewal theorem in this set up.

2. STRONG LAW OF LARGE NUMBERS

In this section we consider some strong law of large numbers.

Lemma 1. Let $\mathcal{F} = \{F| \int |x|dF(x) < \infty, \int xdF(x) = 0\}$, and let ν be stationary with $\int\int |x|dF(x)\pi(dF) < \infty$. Then X_1 with respect to P satisfies

$$E[X_1|\ |\mathcal{J}] = 0 \text{ a.s..}$$

Proof. By the assumption, $E|X_1| < \infty$ and hence $E[X_1|\ \mathcal{J}]$ exists. Now let $A \in \mathcal{J}$ and let $\{(X_1, X_2, \cdots) \in B\} = A$ for some $B \in \mathcal{B}_1^\infty$. Then we have

$$\int_A X_1 dP = \int_{\{(X_1, X_2, \cdots) \in B\}} X_1 dP$$

$$\begin{aligned} &= \int_{\{(X_2,X_3,\cdots)\in B\}} X_1 dP \\ &= \int \left(\int x_1 dF_1^{\omega}(x_1) \int_B \Pi_{i=2}^{\infty} dF_i^{\omega}(x_i) \right) \nu(d\omega) \\ &= 0, \end{aligned}$$

the last equality holding because $\int x dF(x) = 0$ for all $F \in \mathcal{F}$. This proves the lemma.

The following propositions of Hong and Kwon(1993) are needed.

Proposition 1. If ν is stationary, then $\{X_n\}$ is a stationary process with respect to P.

Proposition 2. If ν is ergodic, then $\{X_n\}$ is ergodic with respect to P.

Proposition 3. Let $A \subset R_1^{\infty}$ be measurable. Then $P_{\omega}(A) = 1$ for ν – a.e. ω if and only if $P(A) = 1$.

Theorem 1. Let $\mathcal{F} = \{F | \int x dF(x) = 0, \int |x| dF(x) < \infty\}$ and ν be stationary with $\int \int |x| dF(x) \pi(dF) < \infty$. Then

$$P_{\omega}\{\frac{S_n}{n} \to 0\} = 1, \quad \nu - \text{a.e. } \omega.$$

Proof. The proof follows directly from Proposition 1 and 3, Lemma 1, and the Birkhoff's ergodic theorem.

In general we then prove the following theorem.

Theorem 2. Let $\mathcal{F} = \{F | \int |x| dF(x) < \infty\}$ and let ν be stationary with $\int \int |x| dF(x) \pi(dF) < \infty$. Then

$$P_{\omega}\{\frac{S_n}{n} \to E[\int x dF_1^{\cdot}(x) | \mathcal{I}](\omega)\} = 1, \qquad \nu - \text{a.e. } \omega.$$

($E[\int x dF_1^{\cdot}(x) | \mathcal{I}](\omega) = E[\int x dF_1^{\omega}(x)] = \int \int x dF(x) \pi(dF)$ in case ν is ergodic.)

Proof. Let $F_i^{\omega\prime}(x) = F_i^{\omega}(x + \int y dF_i^{\omega}(y))$ and note that $\int x dF_i^{\omega\prime}(x) = 0$. Let $\mathcal{F}' = \{F_1^{\omega\prime} | \omega \in \mathcal{F}_1^{\omega}\}$. Define $\phi : \mathcal{F}_1^{\infty} \to (\mathcal{F}')_1^{\infty}$ by $\phi(\omega) = \omega' = (F_1^{\omega\prime}, F_2^{\omega\prime}, \cdots)$. Now let $\nu' = \nu \circ \phi^{-1}$. Then ν' is stationary(ergodic). Applying Theorem 1 to this probability measure, $P_{\omega'}\{\frac{S_n}{n} \to 0\} = P_{\omega}$

$\{\frac{S_n - E_\omega S_n}{n} \to 0\} = 1$, ν - a.e. ω, where $E_\omega S_n = \sum_{k=1}^{n} \int X_k dP_\omega = \sum_{k=1}^{n} \int x dF_k^\omega(x)$. We know $\frac{1}{n} E_\omega S_n \to E[\int x dF_1^{\cdot}(x)|\mathcal{I}](\omega)$, ν - a.e. ω by the ergodic theorem. Hence

$$P_\omega\{\frac{S_n}{n} \to E[\int x dF_1^{\cdot}(x)|\mathcal{I}](\omega)\} = 1, \quad \nu - \text{a.e. } \omega.$$

Theorem 3. If ν is stationary and ergodic with $\int \int_{-\infty}^{0} |x| dF(x)\ \pi(dF) < \infty$ and $\int \int_0^\infty x dF(x) \pi(dF) = \infty$, then

$$P_\omega\{\frac{S_n}{n} \to \infty\} = 1, \qquad \nu - \text{a.e. } \omega.$$

Proof. By Proposition 1 and 2, $\{X_n\}$ is stationary and ergodic process with respect to P such that $\int X_1^+ dP = \infty$ and $\int X_1^- dP < \infty$. Then using truncatoin and the ergodic theorem we have $\frac{S_n}{n} \to \infty$ a.s. with respect to P. Hence by Proposition 3, $P_\omega\{\frac{S_n}{n} \to \infty\} = 1$, ν–a.e. ω.

3. RENEWAL THEORY

An interesting application of the law of large numbers occurs in renewal theory. We shall assume in this section that for every $F \in \mathcal{F}$, $F(0-) = 0$, i.e., $P_\omega\{X_n < 0\} = 0$ for all ω and for all n. We also assume that $\pi\{F|F(0) = 1\} \neq 1$ in order that exclude the trivial case. Let us consider the following question. Given the time $t \geq 0$, let $N(t)$ be the number of renewals up to and including the time t, that is formally,

$$N(t) = \text{the number of indices } n \text{ for which } 0 < S_n \leq t,$$

where $S_n = X_1 + X_2 + \cdots + X_n$, $n \geq 1$. It is clear that we have

$$\{x|N(t,x) = n\} = \{x|S_n(x) \leq t < S_{n+1}(x)\}$$

for $n \geq 0$, $S_0 = 0$ by convention. Summing over $n \leq m - 1$, we obtain

$$\{x|N(t,x) < m\} = \{x|S_m(x) > t\}, \qquad m = 1, 2, \cdots .$$

The family of r.v.'s $\{N(t)\}$ indexed by $t \in [0, \infty)$ may be called a renewal process. Let us first provide the following proposition.

Proposition 4. For all ω, we have

$$P_\omega\{\lim_{t\to\infty} N(t) = +\infty\} = 1.$$

Proof. Since $N(t,x)$ is non-decreasing with t, the limit in (3.3) certainly exists for all t and for all ω. Since $\{x| \lim_{t\to\infty} N(t,x) \leq k \text{ for some } k\} = \{x|S_k(x) > t, \text{ for all } t\} = \phi$, the proposition follows.

Theorem 4. Suppose that ν is stationary and ergodic, with $\int\int |x| dF(x) \pi(dF) < \infty$ and $\int\int x dF(x) \pi(dF) = m$, then

$$P_\omega\{\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{m}\} = 1 \qquad \nu - \text{a.e. } \omega$$

and

$$\lim_{t\to\infty} \frac{E_\omega N(t)}{t} = \frac{1}{m} \qquad \nu - \text{a.e. } \omega;$$

both being true even if $m = +\infty$, provided we take $\frac{1}{m}$ to be 0 in that case.

Proof. First we can easily check that $P_\omega\{N(t,x) = \infty \text{ for some } t < \infty\} = 0 \quad \nu$–a.e. ω by Theorm 2 noting that $m > 0$. It follows from (3.1) that for $x \in R_1^\infty$:

$$S_{N(t,x)}(x) \leq t < S_{N(t,x)+1}(x)$$

and consequently, as soon as t is large enough to make $N(t,x) > 0$,

$$\frac{S_{N(t,x)}(x)}{N(t,x)} \leq \frac{t}{N(t,x)} < \frac{S_{N(t,x)+1}(x)}{N(t,x)+1} \frac{N(t,x)+1}{N(t,x)}.$$

Here we need the following lemma.

Lemma 2. Under the conditions of Theorem 4, we have

$$P_\omega\{\lim_{t\to\infty} \frac{S_{N(t,x)}(x)}{N(t,x)} = m\} = 1 \qquad \nu - \text{a.e. } \omega.$$

Proof. According to Theorem 2,

$$P_\omega\{\lim_{n\to\infty}\frac{S_n}{n}=m\}=1 \qquad \nu-\text{a.e. } \omega.$$

(together with Theorem 3 in case $m=+\infty$). Using (3.3), we conclude that

$$P_\omega\{\lim_{t\to\infty}\frac{S_{N(t,x)}(x)}{N(t,x)}=m\}=1 \qquad \nu-\text{a.e. } \omega.$$

Now back to the proof of Theorem 4. Letting $t\to\infty$ and using (3.3) and (3.6), we conclude that

$$P\{\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{m}\}=1.$$

hence (3.4) is true. The deduction of (3.5) from (3.4) is more tricky than might have been thought. It is not clear that the Lebesgue dominated convergence theorem is applicable. To get around this difficulty we use an idea of Chung(1974), but some refinements, due to the fact that we do not have constant distribution functions, are needed. Since $\pi\{F|F(0)=1\}<1$, there exist $\delta>0$ and $p>0$ such that

$$\pi\{F|F(\delta)\le 1-p\}=q>0.$$

Denote $A=\{F|\int_{x\ge\delta}dF(x)\ge p\}$. Define for each ω

$$P_{\omega'}=F_1^{\omega'}\times F_2^{\omega'}\times\cdots,$$

where

$$F_n^{\omega'}=\begin{cases} p\delta_\delta+(1-p)\delta_0 & \text{if } F_n^\omega\in A,\\ \delta_0 & \text{if } F_n^\omega\notin A.\end{cases}$$

Then it is obvious that $P_\omega(S_n\le r)\le P_{\omega'}(S_n\le r)$ for all r and $P_\omega(N(t)\le s)\ge P_{\omega'}(N(t)\le s)$ for all $s,t\in R$. Here we need another lemma.

Lemma 3. $E_\omega\{(\frac{N(t)}{t})^2\}\le E_{\omega'}\{(\frac{N(t)}{t})^2\}=O(1) \quad \nu-\text{a.e. } \omega.$

Proof. By the ergodic theorem

$$\frac{1}{n}\sum_{i=1}^{n}1_A(F_i^\omega)\to\pi(A), \qquad \nu-\text{a.e. } \omega.$$

Take some $\epsilon > 0$. On the set of ω's above a fraction $(\pi(A) - \epsilon)n$ of the first n $F_k^{\omega'}$'s are in A for all $n \geq n(\omega)$, so S_n/δ is larger than a sum of $[(\pi(A) - \epsilon)n]$ Bernoulli random variables, where $[\cdot]$ stands for integer part of $\cdot$. Hence by elementary computations, $E_{\omega'}(N(t))^2 = O(\frac{t^2}{\delta^2})$ as $t \to \infty$, which completes the lemma.

So we have a uniformly integrable family of random variables which converges a.s., hence they converge in L^1.

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