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Nonparametric Estimation of Bivariate Mean Residual Life Function under Univariate Censoring†

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Abstract

We, in this paper, propose a nonparametric estimator of bivariate mean residual life function based on Lin and Ying's(1993) bivariate survival function estimator of paired failure times under univariate censoring and prove the uniform consistency and the weak convergence result of this estimator. Through Monte Carlo simulation, the performances of the proposed estimator are tabulated and are illustrated with the skin grafts data.

Key Words: Mean residual life function; Bivariate distribution function; Paired failure times; Univariate censoring.

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1. INTRODUCTION

Let $\mathbf{T} = (T_1, T_2)$ be a non-negative pair of random variables with joint probability distribution function F on $\mathbf{R}^2$ and let us define the bivariate mean residual life(BVMRL) function or remaining life expectancy function at $\mathbf{t} = (t_1, t_2)$ as

$$\mathbf{e}(\mathbf{t}) = E[\mathbf{T} - \mathbf{t} | \mathbf{T} \geq \mathbf{t}]$$

for all $\mathbf{t} \in \mathbf{R}^2$ such that $\Pr[\mathbf{T} \geq \mathbf{t}] > 0$. Note that the j-th component of $\mathbf{e}(\mathbf{t})$, $e_j(\mathbf{t})$, is the mean of the remaining lifetime given survival up to time $\mathbf{t}$ along the direction of the j-th axis. Hence $e_j(\mathbf{t})$ is obviously different from the univariate j-th marginal mean residual life(MRL) function of the marginal distribution of the joint distribution F. But if $\mathbf{T}$ has independent marginal then $e_j(\mathbf{t})$ is the same as the marginal MRL function.

Arnold and Zahedi(1988) provided some general characterization properties of multivariate mean residual life(MVMRL) function and proved the relationship between the MVMRL function and the hazard gradient. Nair and Nair(1989) introduced a concept of BVMRL function and derived the relationship between the reliability and the mean residual life function.

In the presence of censoring, nonparametric estimation problem of bivariate survival function has been studied by many authors including Campbell(1981), Tsai, Leurgans and Crowley(1986), Dabrowska(1988) and Prentice and Cai(1992).

In bivariate survival studies, if the two failure times are censored by a single censoring variable then this censoring mechanism is called the univariate censoring. Under univariate censoring, Lin and Ying(1993) developed a very simple nonparametric estimator for the bivariate survival curve using the natural representation of the bivariate survival function in terms of the bivariate at-risk probability and the survival function of the censoring time. They also provided the uniform consistency and the weak convergence to a mean zero Gaussian process.

In Section 2, we propose an estimator of BVMRL function based on Lin and Ying's bivariate survival function estimator of paired failure times under univariate censoring and prove the uniform consistency and the weak convergence result of this estimator. In Section 3, we investigate the properties of the proposed estimator via Monte Carlo simulation. Finally, an example is illustrated with the skin grafts data in Section 4.

2. ASYMPTOTIC PROPERTIES

Let $(T_{1i}, T_{2i}), i = 1, 2, \cdots, n$ be independent and identically distributed($i.i.d.$) pairs of random variables(r.v.'s) with joint survival function $S(x, y) = \Pr(T_1 \geq x, T_2 \geq y)$, and let $C_i, i = 1, 2, \cdots, n$ be $i.i.d.$ r.v.'s with survival function $G(t) = \Pr(C \geq t)$. Suppose that the two sequences $\{(T_{1i}, T_{2i})\}_{i=1}^n$ and $\{C_i\}_{i=1}^n$ are independent. We will refer to the (T_{1i}, T_{2i})'s as pairs of lifetimes and to the C_i's as censoring times. In the random univariate censorship model from the right, the (T_1, T_2) may be censored on the right by the single censoring variable C, so that we only observe the random vectors $(X_i, Y_i, \delta_i^x, \delta_i^y), i = 1, 2, \cdots, n$, where $X_i = (T_{1i} \wedge C_i), Y_i = (T_{2i} \wedge C_i), \delta_i^x = I(T_{1i} \leq C_i)$ and $\delta_i^y = I(T_{2i} \leq C_i)$. Here and in the sequel, $I(A)$ denotes the indicator function of the event A, $a \wedge b = \min(a, b)$, and $a \vee b = \max(a, b)$.

From the fact that the observed pairs $\{(X_i, Y_i)\}_{i=1}^n$ have the survival function $S(x, y)G(x \vee y)$, it is natural to estimate the survival function $S(x, y)$ by

$$\widehat{S}_n(x, y) = \frac{n^{-1} \sum_{i=1}^n I(X_i \geq x, Y_i \geq y)}{\widehat{G}_n(x \vee y)} \tag{2.1}$$

where the numerator is the empirical estimator for $S(x, y)G(x \vee y)$ and the denominator is the product-limit estimator for $G(\cdot)$ (see Lin and Ying (1993)). Note that when there is no censoring, the estimator (2.1) is reduced to the usual empirical bivariate survival function. Now let τ be a point such that $S(\tau, \tau)G(\tau) > 0$. Then $\widehat{S}_n(x, y)$ is uniform consistent and has weak convergence result, $i.e.$, for $(x, y) \in [0, \tau]^2$,

$$\sqrt{n}\,\{\widehat{S}_n(x, y) - S(x, y)\} \xrightarrow{d} Z(x, y), \tag{2.2}$$

where $Z(\cdot, \cdot)$ is a mean zero Gaussian process with covariance function

$$Cov\{Z(x_1, y_1), Z(x_2, y_2)\} = \frac{S(x_1 \vee x_2, y_1 \vee y_2)}{G\{(x_1 \vee y_1) \wedge (x_2 \vee y_2)\}}$$
$$-S(x_1, y_1)S(x_2, y_2)\Big(1 - \int_0^{(x_1 \vee y_1) \wedge (x_2 \vee y_2)} \frac{dG(u)}{G^2(u)\Pr\{(T_1 \vee T_2) \geq u\}}\Big)$$

(for details, see Lin and Ying(1993)). Since the BVMRL function may be written as

$$e_i(x, y) = \begin{cases} \{S(x, y)\}^{-1} \int_x^\tau S(u, y)\, du, & \text{if} \quad i = 1 \\ \\ \{S(x, y)\}^{-1} \int_y^\tau S(x, v)\, dv, & \text{if} \quad i = 2\ , \end{cases}$$

we then propose an estimator $\widehat{\mathbf{e}}(x,y)$ by using $\widehat{S}_n(x,y)$

$$\widehat{e}_i(x,y) = \begin{cases} \{\widehat{S}_n(x,y)\}^{-1} \int_x^{X^*} \widehat{S}_n(u,y)\,du, & \text{if} \quad i=1 \\ \{\widehat{S}_n(x,y)\}^{-1} \int_y^{Y^*} \widehat{S}_n(x,v)\,dv, & \text{if} \quad i=2\,, \end{cases} \tag{2.3}$$

where $X^* = \max(X_1, X_2, \cdots, X_n)$ and $Y^* = \max(Y_1, Y_2, \cdots, Y_n)$.

The following two theorems provide the uniform consistency and weak convergence results for the first component of the proposed estimator (2.3).

Theorem 2.1. Suppose that $\sqrt{n}\int_{X^*}^{\tau} S(u,y)\,du \xrightarrow{p} 0$. Then as $n \to \infty$,

$$\sup_{(x,y)\in[0,\tau]^2} |\widehat{e}_1(x,y) - e_1(x,y)| \xrightarrow{p} 0.$$

Proof. For a fixed $(x,y) \in [0,\tau]^2$,

$$\begin{aligned} |\widehat{e}_1(x,y) - e_1(x,y)| &= \left| \frac{1}{\widehat{S}_n(x,y)} \int_x^{X^*} \widehat{S}_n(u,y)\,du - \frac{1}{S(x,y)} \int_x^{\tau} S(u,y)\,du \right| \\ &= \{\widehat{S}_n(x,y)S(x,y)\}^{-1} \Big| S(x,y) \int_x^{\tau} \{\widehat{S}_n(u,y) - S(u,y)\}\,du \\ &\quad - \{\widehat{S}_n(x,y) - S(x,y)\} \int_x^{\tau} S(u,y)\,du - S(x,y) \int_{X^*}^{\tau} \widehat{S}_n(u,y)\,du \Big| \\ &\le \{\widehat{S}_n(x,y)S(x,y)\}^{-1} \Big(S(x,y) \int_x^{\tau} |\widehat{S}_n(u,y) - S(u,y)|\,du \\ &\quad + |\widehat{S}_n(x,y) - S(x,y)| \int_x^{\tau} S(u,y)\,du + S(x,y) \int_{X^*}^{\tau} \widehat{S}_n(u,y)\,du \Big). \end{aligned}$$

By combining the consistency result of $\widehat{S}_n(x,y)$ with partial integration, the first and second terms of the right-hand side of the inequality converge to zero in probability.

On the other hand, the main part of third term is rewritten as

$$\sqrt{n}\int_{X^*}^{\tau} \widehat{S}_n(u,y)\,du = \sqrt{n}\int_{X^*}^{\tau} \{\widehat{S}_n(u,y) - S(u,y)\}\,du + \sqrt{n}\int_{X^*}^{\tau} S(u,y)\,du.$$

So by the convergence result of $\widehat{S}_n(x,y)$ and the above assumption of this theorem, the third term converges to zero in probability. Thus the result follows.

Theorem 2.2. Suppose that $\sqrt{n}\int_{X^*}^{\tau} S(u,y)\,du \xrightarrow{p} 0$. Then as $n \to \infty$,

$$\sqrt{n}\{\widehat{e}_1(x,y) - e_1(x,y)\} \xrightarrow{d} W_1(x,y),$$

where $W_1(\cdot,\cdot)$ is a mean zero Gaussian process and is given by

$$W_1(x,y) = \{S(x,y)\}^{-2}\Big(S(x,y)\int_x^{\tau} Z(u,y)\,du - Z(x,y)\int_x^{\tau} S(u,y)\,du\Big).$$

Proof. For a fixed $(x,y) \in [0,\tau]^2$, we have

$$\begin{aligned}\sqrt{n}\{\widehat{e}_1(x,y) - e_1(x,y)\} &= \sqrt{n}\Big(\frac{1}{\widehat{S}_n(x,y)}\int_x^{X^*} \widehat{S}_n(u,y)\,du - \frac{1}{S(x,y)}\int_x^{\tau} S(u,y)\,du\Big)\\ &= \{\widehat{S}_n(x,y)S(x,y)\}^{-1}\Big(S(x,y)\int_x^{\tau} \sqrt{n}\{\widehat{S}_n(u,y) - S(u,y)\}\,du\\ &\quad - \sqrt{n}\{\widehat{S}_n(x,y) - S(x,y)\}\int_x^{\tau} S(u,y)\,du - S(x,y)\sqrt{n}\int_{X^*}^{\tau} \widehat{S}_n(u,y)\,du\Big).\end{aligned}$$

Now let $D([0,\tau]^2)$ be the space of functions on the rectangle $[0,\tau]^2$ that are right continuous and have left-hand limits. Let $\mathbf{d}$ be the Skorohod metric on $D([0,\tau]^2)$ and let us define a map $H : D([0,\tau]^2) \longrightarrow D([0,\tau]^2)$, by having

$$H(Z(x,y)) = S(x,y)\int_x^{\tau} Z(u,y)\,du - Z(x,y)\int_x^{\tau} S(u,y)\,du$$

for $Z \in D([0,\tau]^2)$ where the limiting distribution Z is defined in (2). Then H is a continuous map with respect to $\mathbf{d}$ (see Yang (1977)). Thus by the continuity theorem in Billingsley(1968) and the above assumption of this theorem, the result follows.

Remark 1. The covariance function of the limit distribution $W_1(\cdot,\cdot)$ defined in Theorem 2.2 is given by, for $0 \le x_1 \le x_2 \le \tau$, and $0 \le y_1 \le y_2 \le \tau$,

$$\begin{aligned}Cov\{W_1(x_1,y_1), W_1(x_2,y_2)\} &= \{S(x_1,y_1)S(x_2,y_2)\}^{-2}\\ &\Big(S(x_1,y_1)S(x_2,y_2)\ E\Big[\int_{x_1}^{\tau} Z(u,y_1)\,du\int_{x_2}^{\tau} Z(u,y_2)\,du\Big]\\ &+ E\{Z(x_1,y_1)Z(x_2,y_2)\}\int_{x_1}^{\tau} S(u,y_1)\,du\int_{x_2}^{\tau} S(u,y_2)\,du\\ &- S(x_1,y_1)\int_{x_2}^{\tau} S(u,y_2)\,du\ E\Big[Z(x_2,y_2)\int_{x_1}^{\tau} Z(u,y_1)\,du\Big]\\ &- S(x_2,y_2)\int_{x_1}^{\tau} S(u,y_1)\,du\ E\Big[Z(x_1,y_1)\int_{x_2}^{\tau} Z(u,y_2)\,du\Big]\Big).\end{aligned}$$

Remark 2. For the asymptotic properties of the second component of the proposed estimator (3), it is analogous to the above Theorems 2.1 and 2.2, with X^* replaced by Y^* and $\widehat{S}_n(u,y)$ by $\widehat{S}_n(x,v)$.

3. SIMULATION STUDIES

Simulation studies were carried out to examine the properties of the proposed estimator with various sample sizes, n=20, 50 and 100.

Three sets of 500 simulations were carried out. In each simulation, n pairs of failure times with unit exponential marginal distributions were generated. These values were then subject to be censored to the right by an independent exponentially distributed random variate with hazard rate of 0.111 and 0.429. Here the values of hazard rates were calculated to make censoring rate to be 10% and 30%, respectively.

In our simulations the pairs of failure times were, respectively, independent and distributed according to the Gumbel(1960)'s bivariate exponential distribution

$$S(x,y) = e^{-(x+y)}\{1+\theta(1-e^{-x})(1-e^{-y})\},$$

with θ=0.25 and 0.50. Here θ stands for the relationship between x and y. That is,

$$\theta = \frac{4\cdot\rho}{(1-e^{-2x})(1-e^{-2y})},$$

where ρ is a correlation coefficient of x and y. Thus the Gumbel model with θ=0 simply means that two components are independent. In Gumbel model, the pairs of failure times (x,y) were generated from uniform (0,1)-variates u_1 and u_2 using the following transformation

$$y = -\log(1-u_2), \qquad x = -\log\left[\frac{(1+a)-\{(1+a)^2-4a(1-u_1)\}^{\frac{1}{2}}}{2a}\right],$$

where $a = \theta(2u_2-1)$.

Simulation results are tabulated in Tables 3.1 to 3.3. In the tables, the estimates and MSE's of the proposed estimator are given at pairs of time points where (x,y) take values (0.0,0.0), (0.2231,0.2231), (0.5108,0.5108), (0.9163,0.9163) corresponding to maginal survival probabilities of 1, 0.8, 0.6 and 0.4.

Table 3.1. Bivariate MRL function estimates and their MSE's in under independent exponential model. (Censoring rate : 10%)

Sample	$y \setminus x$	0.000	0.223	0.510	0.916
20	0.000	(1.000, 1.000)[a] (0.996, 0.990)[b] (0.057, 0.060)[c]	(1.000, 1.000) (0.984, 1.003) (0.069, 0.083)	(1.000, 1.000) (0.987, 1.009) (0.097, 0.124)	(1.000, 1.000) (0.974, 1.000) (0.171, 0.173)
	0.2231	(1.000, 1.000) (1.004, 0.989) (0.074, 0.091)	(1.000, 1.000) (0.992, 1.007) (0.086, 0.129)	(1.000, 1.000) (1.002, 1.012) (0.130, 0.176)	(1.000, 1.000) (1.004, 0.996) (0.324, 0.242)
	0.5108	(1.000, 1.000) (1.020, 0.985) (0.108, 0.174)	(1.000, 1.000) (1.009, 1.003) (0.127, 0.208)	(1.000, 1.000) (1.019, 0.999) (0.207, 0.264)	(1.000, 1.000) (1.004, 0.966) (0.394, 0.357)
	0.9163	(1.000, 1.000) (1.009, 0.987) (0.169, 0.301)	(1.000, 1.000) (0.993, 1.004) (0.225, 0.350)	(1.000, 1.000) (1.004, 1.011) (0.361, 0.468)	(1.000, 1.000) (0.953, 0.952) (0.561, 0.666)
100	0.0000	(1.000, 1.000) (1.001, 1.003) (0.011, 0.011)	(1.000, 1.000) (0.998, 1.004) (0.014, 0.014)	(1.000, 1.000) (0.999, 1.001) (0.019, 0.020)	(1.000, 1.000) (1.006, 0.995) (0.032, 0.031)
	0.2231	(1.000, 1.000) (1.004, 1.010) (0.014, 0.018)	(1.000, 1.000) (1.002, 1.012) (0.018, 0.022)	(1.000, 1.000) (1.002, 1.006) (0.022, 0.028)	(1.000, 1.000) (1.009, 0.995) (0.040, 0.041)
	0.5108	(1.000, 1.000) (0.997, 1.009) (0.018, 0.030)	(1.000, 1.000) (1.009, 1.014) (0.127, 0.036)	(1.000, 1.000) (1.019, 1.009) (0.207, 0.047)	(1.000, 1.000) (1.008, 0.999) (0.053, 0.063)
	0.9163	(1.000, 1.000) (0.994, 1.009) (0.030, 0.055)	(1.000, 1.000) (0.989, 1.015) (0.038, 0.067)	(1.000, 1.000) (0.992, 1.013) (0.050, 0.085)	(1.000, 1.000) (0.999, 0.999) (0.090, 0.118)

a : the value of $(e_1(x,y), e_2(x,y))$
b : the value of $(\widehat{e}_1(x,y), \widehat{e}_2(x,y))$
c : the MSE of $(\widehat{e}_1(x,y), \widehat{e}_2(x,y))$

In our simulation studies, we may see the following results: (1) In the case of independent model, the true value of each component is always one at all pairs of time points (x, y) because of the properties of exponential distribution. The MSE's of the proposed estimator are reduced as the sample size increases and censoring rate decreases at each component. (2) For the Gumbel model, the estimators of both component are over-estimated. As the time point increases, the estimates of first component are decreased while those of the second increased. The estimates of both component are increased as the censoring rate decreases and θ increases. The MSE's of the proposed estimator are reduced as the sample size increases, θ decreases and the amount of censoring decreases at each component.

Table 3.2 Bivariate MRL function estimates and their MSE's in under Gumbel bivariate exponential model with $\theta = 0.25$. (Censoring rate : 10%)

Sample	$y \setminus x$	0.000	0.223	0.510	0.916
20	0.0000	(1.000, 1.000)[a] (1.108, 1.026)[b] (0.069, 0.060)[c]	(1.000, 1.025) (1.088, 1.110) (0.078, 0.086)	(1.000, 1.050) (1.060, 1.201) (0.100, 0.140)	(1.000, 1.075) (1.031, 1.292) (0.166, 0.247)
	0.2231	(1.025, 1.000) (1.163, 1.059) (0.093, 0.118)	(1.019, 1.019) (1.127, 1.136) (0.100, 0.146)	(1.014, 1.039) (1.087, 1.223) (0.121, 0.203)	(1.009, 1.058) (1.042, 1.301) (0.222, 0.309)
	0.5108	(1.050, 1.000) (1.240, 1.078) (0.141, 0.198)	(1.039, 1.014) (1.184, 1.149) (0.146, 0.238)	(1.028, 1.022) (1.127, 1.230) (0.168, 0.302)	(1.018, 1.042) (1.083, 1.304) (0.298, 0.427)
	0.9163	(1.075, 1.000) (1.346, 1.127) (0.266, 0.443)	(1.058, 1.009) (1.248, 1.174) (0.242, 0.469)	(1.042, 1.018) (1.163, 1.251) (0.271, 0.563)	(1.027, 1.027) (1.085, 1.327) (0.396, 0.784)
50	0.0000	(1.000, 1.000) (1.116, 1.011) (0.039, 0.023)	(1.000, 1.025) (1.097, 1.091) (0.040, 0.034)	(1.000, 1.050) (1.070, 1.174) (0.045, 0.059)	(1.000, 1.075) (1.043, 1.262) (0.067, 0.109)
	0.2231	(1.025, 1.000) (1.173, 1.030) (0.058, 0.044)	(1.019, 1.019) (1.142, 1.107) (0.057, 0.059)	(1.014, 1.039) (1.103, 1.184) (0.060, 0.088)	(1.009, 1.058) (1.064, 1.263) (0.085, 0.145)
	0.5108	(1.050, 1.000) (1.242, 1.093) (0.087, 0.086)	(1.039, 1.014) (1.187, 1.162) (0.078, 0.109)	(1.028, 1.022) (1.133, 1.237) (0.080, 0.151)	(1.018, 1.042) (1.074, 1.304) (0.107, 0.214)
	0.9163	(1.075, 1.000) (1.333, 1.126) (0.142, 0.135)	(1.058, 1.009) (1.241, 1.181) (0.116, 0.157)	(1.042, 1.018) (1.167, 1.252) (0.119, 0.209)	(1.027, 1.027) (1.097, 1.325) (0.144, 0.305)

a : the value of $(e_1(x,y), e_2(x,y))$
b : the value of $(\widehat{e}_1(x,y), \widehat{e}_2(x,y))$
c : the MSE of $(\widehat{e}_1(x,y), \widehat{e}_2(x,y))$

Table 3.3 Bivariate MRL function estimates and their MSE's in under Gumbel bivariate exponential model with $\theta = 0.50$. (Censoring rate : 30%)

Sample	$y \setminus x$	0.000	0.223	0.510	0.916
50	0.000	(1.000, 1.000)[a] (1.187, 1.008)[b] (0.079, 0.032)[c]	(1.000, 1.050) (1.182, 1.118) (0.091, 0.050)	(1.000, 1.100) (1.141, 1.237) (0.095, 0.086)	(1.000, 1.150) (1.078, 1.376) (0.118, 0.173)
	0.223	(1.050, 1.000) (1.288, 1.024) (0.121, 0.055)	(1.039, 1.039) (1.263, 1.125) (0.130, 0.073)	(1.028, 1.076) (1.201, 1.230) (0.126, 0.113)	(1.018, 1.113) (1.113, 1.342) (0.142, 0.194)
	0.510	(1.100, 1.000) (1.417, 1.101) (0.193, 0.095)	(1.076, 1.028) (1.344, 1.177) (0.177, 0.115)	(1.055, 1.055) (1.262, 1.279) (0.172, 0.169)	(1.035, 1.080) (1.145, 1.367) (0.181, 0.259)
	0.916	(1.150, 1.000) (1.572, 1.154) (0.327, 0.188)	(1.113, 1.018) (1.427, 1.194) (0.257, 0.210)	(1.080, 1.035) (1.295, 1.273) (0.232, 0.259)	(1.050, 1.050) (1.178, 1.384) (0.265, 0.398)
100	0.000	(1.000, 1.000) (1.186, 1.003) (0.057, 0.019)	(1.000, 1.050) (1.180, 1.111) (0.061, 0.030)	(1.000, 1.100) (1.145, 1.230) (0.060, 0.056)	(1.000, 1.150) (1.097, 1.376) (0.075, 0.120)
	0.223	(1.050, 1.000) (1.285, 1.042) (0.088, 0.032)	(1.039, 1.039) (1.256, 1.139) (0.085, 0.047)	(1.028, 1.076) (1.199, 1.246) (0.078, 0.080)	(1.018, 1.113) (1.129, 1.370) (0.089, 0.148)
	0.510	(1.100, 1.000) (1.404, 1.091) (0.138, 0.058)	(1.076, 1.028) (1.333, 1.168) (0.119, 0.076)	(1.055, 1.055) (1.261, 1.274) (0.111, 0.122)	(1.035, 1.080) (1.168, 1.385) (0.122, 0.206)
	0.916	(1.150, 1.000) (1.572, 1.157) (0.259, 0.131)	(1.113, 1.018) (1.437, 1.202) (0.195, 0.147)	(1.080, 1.035) (1.312, 1.285) (0.165, 0.195)	(1.050, 1.050) (1.211, 1.412) (0.182, 0.317)

a : the value of $(e_1(x,y), e_2(x,y))$
b : the value of $(\widehat{e}_1(x,y), \widehat{e}_2(x,y))$
c : the MSE of $(\widehat{e}_1(x,y), \widehat{e}_2(x,y))$

4. AN ILLUSTRATION

As an example, let us consider the well-known matched pairs data of Holt and Prentice(1974), which consist of survival times, in days, of closely and poorly matched skin grafts on the same burned patient. With the minor modifications made by Woolson and Lachenbruch(1980) these data are reproduced in Table 4.1. There were only 11 patients, the survival times of two closely matched grafts being censored.

Table 4.1. Days of survival of skin grafts on burn patients

Patient, i	1	2	3	4	5	6	7	8	9	10	11
Survival of close match, X_i	37	19	57*	93	16	22	20	18	63	29	60*
Survival of poor match, Y_i	29	13	15	26	11	17	26	21	43	15	40

$*$ indicates censored

Table 4.2 displays the estimates of each component at all observed failure time points for the skin graft data. For example, $\hat{\mathbf{e}}(29,29) = (26.0, 8.3)$. This means that the mean residual life time of the first component given both skin grafts survive beyond 29 days is 26.0 and those of the second component is 8.3. Thus the mean residual life time of closely matched skin graft is longer than those of poorly matched skin graft in this case. On the other hand, the univariate marginal MRL function estimates of each component are $\widehat{e}_X(29) = 40.8$ and $\widehat{e}_Y(29) = 12.5$, respectively. Thus the BVMRL function estimates are smaller than the univariate marginal MRL function estimates. In general, if two components are dependent then it is more useful using the BVMRL function rather than the univariate marginal MRL function.

Table 4.2. Estimates of the bivariate MRL functions for skin grafts data

$y \setminus x$	16	18	19	20	22	29	37	57	60	63
11	27.0[a]	27.7	29.8	32.5	35.1	34.0	32.8	21.0	18.0	15.0
	12.3[b]	13.5	13.9	15.4	15.4	17.0	19.6	20.0	25.3	23.5
13	29.7	27.7	29.8	32.5	35.1	34.0	32.8	21.0	18.0	15.0
	11.5	11.5	11.9	13.4	13.4	15.0	17.6	18.0	23.3	21.5
15	32.7	30.7	33.5	32.5	35.1	34.0	32.8	21.0	18.0	15.0
	10.8	10.8	11.4	11.4	11.4	13.0	15.6	16.0	21.3	19.5
17	34.3	32.3	36.7	35.7	40.8	44.0	36.0	28.0	18.0	15.0
	11.9	11.9	13.2	13.2	14.0	17.5	17.5	19.3	19.3	17.5
21	39.0	37.0	43.4	42.4	51.0	44.0	36.0	28.0	18.0	15.0
	9.8	9.8	11.8	11.8	13.5	13.5	13.5	15.3	15.3	13.5
26	46.4	44.4	43.4	42.4	51.0	44.0	36.0	28.0	18.0	15.0
	6.8	6.8	6.8	6.8	8.5	8.5	8.5	10.3	10.3	8.5
29	39.0	37.0	36.0	35.0	33.0	26.0	18.0	7.0	2.3	.0
	8.3	8.3	8.3	8.3	8.3	8.3	8.3	12.5	12.5	14.0
40	48.0	46.0	45.0	44.0	42.0	35.0	27.0	7.0	2.3	.0
	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	3.0

a : estimate of the first component
b : estimate of the second component

REFERENCES

(1) Arnold, B. C. and Zahedi, H. (1988). On multivariate mean remaining life functions. *Journal of multivariate analysis*, **25**, 1-9.

(2) Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley & and Sons, New York.

(3) Campbell, G. (1981). Nonparametric bivariate estimation with randomly censored data. *Biometrika*, **68**, 417-422.

(4) Dabrowska, D. M. (1988). Kaplan-Meier estimate on the plane. *Annals of Statistics*, **16**, 1475-1489.

(5) Gumbel, E. J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association*, **55**, 698-707.

(6) Holt, J. D. and Prentice, R. L. (1974). Survival analysis in twin studies and matched pairs experiments. *Biometrika*, **61**, 17-30.

(7) Lin, D. Y. and Ying, Z. (1993). A simple nonparametric estimator of the bivariate survival function under univariate censoring. *Biometrika*, **80**, 573-581.

(8) Nair, K. R. M. and Nair, N. U. (1989). Bivariate mean residual life. *IEEE Transactions on Reliability*, **R-38**, 362-364.

(9) Prentice, R. L. and Cai, J. (1992). Covariance and survival function estimation using censored multivariate failure time data. *Biometrika*, **79**, 495-512.

(10) Tsai, W. Y., Leurgans, S., and Crowley, J. (1986). Nonparametric estimation of a bivariate survival function in the presence of censoring. *Annals of Statistics*, **14**, 1351-1365.

(11) Woolson, R. T. and Lachenbruch, P. A. (1980). Rank tests for censored matched pairs. *Biometrika*, **67**, 597-606.

(12) Yang, G. L. (1977). Life expectancy under random censorship. *Stochastic processes and their applications*, **6**, 33-39.