Journal of the Korean
Statistical Society
Vol. 25, No. 1, 1996

Bayesian Hypothesis Testing in Multivariate Growth Curve Model

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Abstract

This paper suggests a new criterion for testing the general linear hypothesis about coefficients in multivariate growth curve model. It is developed from a Bayesian point of view using the highest posterior density region methodology. Likelihood ratio test criterion(LRTC) by Khatri(1966) results as an approximate special case. It is shown that under the simple case of vague prior distribution for the multivariate normal parameters a LRTC-like criterion results; but the degrees of freedom are lower, so the suggested test criterion yields more conservative test than is warranted by the classical LRTC, a result analogous to that of Berger and Sellke(1987). Moreover, more general(non-vague) prior distributions will generate a richer class of tests than were previously available.

Key Words : Multivariate growth curve model; General linear hypothesis; HPD region methodology; Bayesian test criterion; Box's approximation; p-value.

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1. INTRODUCTION

The model considered here is the generalized growth curve model first proposed by Potthoff and Roy(1964). The model is defined as

$$\mathbf{Y}_{p\times N} = \mathbf{X}_{p\times m}\tau_{m\times q}\mathbf{A}_{q\times N} + \mathcal{E}_{p\times N}, \tag{1.1}$$

where τ is unknown, $\mathbf{X}$ and $\mathbf{A}$ are known matrices of ranks $m < p$ and $q < N$, respectively. Further the columns of $\mathcal{E}$ are independent p-variate normal with mean vector 0 and common covariance matrix Σ, *i.e.* $Vec(\mathcal{E}') \sim N(0, \Sigma \otimes I_N)$. In general, p is the number of time(or spatial) points observed on each of N cases.

Several examples of growth curve applications for the model (1.1) were given by Potthoff and Roy(1964), Zerbe and Jones(1980), Rao(1977, 1984), and Lee(1988) among others. In particular, the polynomial curves in time as models for growth curves are an important example. This model comes about from (1.1) by letting

$$\mathbf{X} = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{m-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{m-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & t_p & t_p^2 & \cdots & t_p^{m-1} \end{bmatrix},$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{e}_1' & 0_2' & \cdots & \cdots & 0_q' \\ 0_1' & \mathbf{e}_2' & 0_3' & \cdots & 0_q' \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0_1' & 0_2' & \cdots & \cdots & \mathbf{e}_q' \end{bmatrix},$$

where $\mathbf{e}_i$ is a $N_i \times 1$ vector all of whose components are unity and 0_i is the null vector of size N_i, $\sum_{i=1}^{q} N_i = N; i = 1, \ldots, q$. Therefore, when $m = 2$, a linear model results from the polynomial growth curve models:

$$E[y_{1j}, y_{2j}, \ldots, y_{pj}] = (\tau_{1u} + \tau_{2u}t_1, \tau_{1u} + \tau_{2u}t_2, \ldots, \tau_{1u} + \tau_{2u}t_p), \tag{1.2}$$

where $\mathbf{Y} = \{y_{ij}\}$, $\tau = \{\tau_{k\ell}\}$, and $\sum_{i=1}^{u} N_{i-1} < j \le \sum_{i=1}^{u} N_i$, $u = 1, \ldots, q$.

The analysis of the model (1.1), which can be regarded as a special case of generalized multivariate analysis of variance model has been subsequently studied by many authors (including Bayesian statisticians). A brief survey of such studies(with references) is in Geisser(1980) and Rao(1987). An important class of studies, where there has been much activity, concerns the test criterion for general linear hypothesis under the growth curve model. Potthoff and Roy(1964) proposed a couple of test criteria by means of the maximum root criterion by Roy(1938) and the trace criterion by Lawley(1938) and Hotelling(1951). On the other hand, Khatri(1966) derived the likelihood ratio test criterion for testing the linear hypothesis. However, a Bayesian criterion for testing the linear hypothesis under the growth curve model has not been seen yet.

This paper concerns hypothesis testing in multivariate growth curve analysis from a Bayesian point of view. Generally, estimation and prediction are of much greater interest to Bayesian statisticians than is hypothesis testing, but there are those situations in which hypothesis testing is desirable and appropriate. Those situations are the ones with which we will be concerned in this paper.

2. BAYESIAN HIGHEST POSTERIOR DENSITY(HPD) REGION TESTING

Under the growth curve model (1.1), a variety of hypotheses concerning the elements of τ are easily formulated as $\mathbf{C}\tau\mathbf{D} = \mathbf{\Phi}_0$ where $\mathbf{D}$ is a $q \times d$ matrix of rank $d \leq q$ and $\mathbf{C}$ is a $c \times m$ matrix of rank $c \leq m$. For example, in the previously discussed linear case (1.2), one may be only interested in testing that all the groups "grew" at an equal rate($H_0 : \tau_{21} = \ldots = \tau_{2q}$). Hence,

$$\mathbf{C}\tau\mathbf{D} = (0,1)\tau\mathbf{D} = \mathbf{0},$$

where

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1q} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{2q} \end{bmatrix},$$

and $\mathbf{D}$ is any $q \times (q-1)$ matrix of rank $q-1$ such that the columns of $\mathbf{D}$ sum to zero.

HPD regions introduced by Box and Tiao(1973, p.122) are ideally suited for testing the hypotheses of interest in Bayesian growth curve analysis. This is because in higher dimensions we are generally interested in the event that some vector or matrix belongs to a particular region, and this event can generally be specified either directly or in terms of some monotonic functions. This can be stated more specifically. From the properties of HPD regions, we see that if $\Re_\alpha$ is an HPD region of probability content $(1-\alpha)$, then the event $\mathbf{\Phi} \equiv \mathbf{C}\tau\mathbf{D} \in \Re_\alpha$ is equivalent to the event that

$$f(\mathbf{\Phi} \mid \mathbf{Y}) > a, \tag{2.1}$$

where a is a suitably chosen positive constant and $f(\mathbf{\Phi} \mid \mathbf{Y})$ denotes a posterior density of $\mathbf{\Phi}$. It follows that a particular matrix $\mathbf{\Phi}_0$ is covered by HPD region of content $(1-\alpha)$ if and only if

$$Pr(f(\mathbf{\Phi} \mid \mathbf{Y}) > f(\mathbf{\Phi}_0 \mid \mathbf{Y}) \mid \mathbf{Y}) \leq 1-\alpha. \tag{2.2}$$

In this expression, the density function is treated as a random variable. Thus, once the posterior distribution of the quantity $f(\mathbf{\Phi} \mid \mathbf{Y})$ or some monotonic functions of it can be determined, our interest in testing $H_0 : \mathbf{\Phi} = \mathbf{\Phi}_0$ can be resolved.

The probability statements defining HPD regions can be derived directly from the posterior distribution; any kind of prior information may be used(vague or not) so that non-vague prior distributions will lead to a rich family of tests; and no multidimensional integrations are involved once we have the posterior distribution for $\mathbf{\Phi}$(we must of course be able to evaluate the integral for the cdf to calculate the probability content of the distribution). In the sequel, we will adopt this HPD approach to develop a Bayesian test for the growth curve model.

2.1. Posterior Density of $\mathbf{\Phi}$

From the generalized growth curve model (1.1), we can express the likelihood of τ and Σ as

$$L(\tau,\Sigma) \propto \mid \Sigma^{-1} \mid^{N/2} \exp\left\{-\frac{1}{2}tr\Sigma^{-1}(\mathbf{Y}-\mathbf{X}\tau\mathbf{A})(\mathbf{Y}-\mathbf{X}\tau\mathbf{A})'\right\}. \tag{2.3}$$

In order to express the notion of "knowing little", and to provide a "reference-type prior" that often produces frequentist-types of results, we adopt a vague, Jeffreys-type of prior density(see Geisser and Cornfield, 1963; Jeffreys, 1961) for τ and the precision matrix, Σ^{-1},

$$g(\tau,\Sigma^{-1}) \propto \mid \Sigma \mid^{(p+1)/2}. \tag{2.4}$$

Results are easily extendible to natural conjugate families of prior distributions with little change in results(except for changes in the numbers of degrees of freedom).

Since posterior inference about τ are made most easily from the marginal posterior density of τ, integrating the joint posterior density of τ and precision matrix with respect to Σ^{-1} yields the marginal posterior density for τ (see Geisser, 1970):

$$\begin{aligned} p(\tau \mid \mathbf{Y}) &= \int_{\Sigma^{-1}>0} L(\tau, \Sigma) g(\tau, \Sigma^{-1}) d\Sigma^{-1} \qquad (2.5) \\ &= \frac{\mid \mathbf{X}'\mathbf{S}^{-1}\mathbf{X} \mid^{-(N-q)/2} \mid \mathbf{R} \mid^{m/2}}{k(N,q,m)} \mid (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1} + (\tau - \hat{\tau})\mathbf{R}(\tau - \hat{\tau})' \mid^{-N/2}, \end{aligned}$$

where

$$\begin{aligned} N &> m + q - 1, \\ \hat{\tau} &= (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\mathbf{Y}\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}, \qquad (2.6) \\ \mathbf{S} &= \mathbf{Y}(I_N - \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A})\mathbf{Y}', \\ \mathbf{R}^{-1} &= (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\mathbf{Y}'[\mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}]\mathbf{Y}\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1} + (\mathbf{A}\mathbf{A}')^{-1}, \end{aligned}$$

$$k(N,q,m) = \pi^{mq/2}\Gamma_q\left(\frac{N-m}{2}\right)/\Gamma_q\left(\frac{N}{2}\right).$$

Here $\Gamma_q(\lambda)$ denotes the multivariate gamma function,

$$\Gamma_q(\lambda) \equiv \pi^{q(q-1)/4}\Gamma(\lambda)\Gamma(\lambda - 1/2)\cdots\Gamma(\lambda - q/2 + 1/2).$$

It is to be noted that the marginal posterior distribution of τ in (2.5) follows a matrix **T**-distribution with N degrees of freedom(cf. Press, 1982; Dickey, 1967); we shall, as the previous authors, say that the $m \times q$ matrix of parameters τ is distributed as $\mathbf{T}_{mq}((\mathbf{X}'\mathbf{S}^{-1}\mathbf{X}), \mathbf{R}^{-1}, \hat{\tau}, N)$.

Lemma 1 (Box and Tiao, 1973). Let the $m \times q$ matrix of parameters θ is distributed as $\mathbf{T}_{mq}(\mathbf{P}, \mathbf{Q}, \mathbf{\Omega}, N)$. Let $\mathbf{C}$ be $c \times m (c \leq m)$ matrix of rank c and $\mathbf{D}$ be $q \times d (d \leq q)$ matrix of rank d. Suppose $\mathbf{\Psi}$ is the $c \times d$ matrix of random variables obtained from the linear transformation $\mathbf{\Psi} = \mathbf{C}\theta\mathbf{D}$. Then $\mathbf{\Psi}$ is distributed as

$$\mathbf{T}_{cd}((\mathbf{C}\mathbf{P}^{-1}\mathbf{C}')^{-1}, \mathbf{D}'\mathbf{Q}\mathbf{D}, \mathbf{C}\mathbf{\Omega}\mathbf{D}, N - q - m + c + d). \qquad (2.7)$$

From Lemma 1 we see that the linear transformation $\mathbf{\Phi} = \mathbf{C}\tau\mathbf{D}$ has the posterior distribution $\mathbf{T}_{cd}((\mathbf{C}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{C}')^{-1}, \mathbf{D}'\mathbf{R}^{-1}\mathbf{D}, \hat{\mathbf{\Phi}}, N-q-m+c+d)$ with density

$$f(\mathbf{\Phi}|\mathbf{Y}) \propto |\mathbf{C}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{C}' + (\mathbf{\Phi} - \hat{\mathbf{\Phi}})(\mathbf{D}'\mathbf{R}^{-1}\mathbf{D})^{-1}(\mathbf{\Phi} - \hat{\mathbf{\Phi}})'|^{-(N-q-m+c+d)/2}, \tag{2.8}$$

where $\hat{\mathbf{\Phi}} = \mathbf{C}\hat{\tau}\mathbf{D}$.

Various theorems by Dickey(1967) allow us to make conditional(or marginal) inferences about a specific columns or rows of $\mathbf{\Phi}$.

2.2. HPD Region Test Criterion

The density function $f(\mathbf{\Phi}|\mathbf{Y})$ is a monotonic decreasing function of

$$M = -\nu \log \mathbf{B}(\mathbf{\Phi}), \tag{2.9}$$

where, from (2.8),

$$\mathbf{B}(\mathbf{\Phi}) = \frac{|\mathbf{C}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{C}'|}{|\mathbf{C}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{C}' + (\mathbf{\Phi} - \hat{\mathbf{\Phi}})(\mathbf{D}'\mathbf{R}^{-1}\mathbf{D})^{-1}(\mathbf{\Phi} - \hat{\mathbf{\Phi}})'|}, \tag{2.10}$$

and $\nu = N - m - q + 1$. Thus, the event $f(\mathbf{\Phi}|\mathbf{Y}) > f(\mathbf{\Phi}_0|\mathbf{Y})$ is equivalent to the event $M < -\nu \log \mathbf{B}(\mathbf{\Phi}_0)$, where $\mathbf{B}(\mathbf{\Phi}_0)$ is obtained by substituting $\mathbf{\Phi}_0$ for $\mathbf{\Phi}$ in (2.10). This leads to the HPD region criterion (2.2) for testing the general linear hypothesis as accepting $H_0 : \mathbf{\Phi} = \mathbf{\Phi}_0$ if

$$Pr(M < -\nu \log \mathbf{B}(\mathbf{\Phi}_0)) \leq 1 - \alpha \tag{2.11}$$

for some preassigned α. Since the exact distribution of M is complicated(see e.g. Schatzoff, 1966; Pillai and Gupta, 1969), to carry out the test, we need an asymptotic result for M. Such a result is given in the next section.

3. APPROXIMATE TEST CRITERION

The asymptotic distribution of $M = -\nu \log \mathbf{B}(\mathbf{\Phi})$ is obtained from the Box approximation (Box, 1949; Anderson, 1984). For our context the result is given in the theorems below.

Lemma 2. The h-th posterior moment of $\mathbf{B}(\mathbf{\Phi})$ is

$$E(\mathbf{B}(\mathbf{\Phi})^h|Y) = \frac{\Gamma_c[\frac{1}{2}(N-q-m+c)+h]\Gamma_c[\frac{1}{2}(N-q-m+d+c)]}{\Gamma_c[\frac{1}{2}(N-q-m+c)]\Gamma_c[\frac{1}{2}(N-q-m+d+c)+h]}, \tag{3.1}$$

and this holds for all h for which the gamma function exist including purely imaginary h.

Proof. Using the matrix T density integral(see equation (2.5)) :

$$\int |\mathbf{P}|^{-(N-q)/2}|\mathbf{Q}|^{-m/2}|\mathbf{P}^{-1}+(\mathbf{T}-\mathbf{\Omega})\mathbf{Q}^{-1}(\mathbf{T}-\mathbf{\Omega})'|^{-N/2}d\mathbf{T} = \frac{\pi^{mq/2}\Gamma_q\left(\frac{N-m}{2}\right)}{\Gamma_q\left(\frac{N}{2}\right)}, \tag{3.2}$$

we obtain, from the posterior distribution of $\mathbf{\Phi}$ in (2.8), the characteristic function of $W = -2\log \mathbf{B}(\mathbf{\Phi})$:

$$\begin{aligned} E(e^{itW}) &= E(\mathbf{B}(\mathbf{\Phi})^{-2it}) \\ &= \Delta\int \frac{|\mathbf{C}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{C}'|^{(N-q-m+c-2it)/2}|\mathbf{D}'\mathbf{R}^{-1}\mathbf{D}|^{-c/2}}{|\mathbf{C}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{C}' + (\mathbf{\Phi}-\hat{\mathbf{\Phi}})(\mathbf{D}'\mathbf{R}^{-1}\mathbf{D})^{-1}(\mathbf{\Phi}-\hat{\mathbf{\Phi}})'|^{\delta/2}}d\mathbf{T} \\ &= \Delta\frac{\Gamma_c[\frac{1}{2}(N-q-m+c)-2it]}{\Gamma_c[\frac{1}{2}(N-q-m+d+c)-2it]} \end{aligned} \tag{3.3}$$

where $\delta = N-q-m+c+d+2it$ and

$$\Delta = \frac{\Gamma_c(\frac{1}{2}(N-q-m+c+d))}{\Gamma_c(\frac{1}{2}(N-q-m+c))}.$$

Letting $h = -2it$ gives the result.

Let us define $\mathbf{U}(\mathbf{\Phi}) = \mathbf{B}(\mathbf{\Phi})^{\nu/2}$, $\nu = N-m-q+1$, so that $M \equiv -2\log \mathbf{U}(\mathbf{\Phi})$. Using Lemma 2, we obtain the h-th posterior moment of $\mathbf{U}(\mathbf{\Phi})$ as

$$\begin{aligned} E(\mathbf{U}(\mathbf{\Phi})^h \mid \mathbf{Y}) &= E(\mathbf{B}(\mathbf{\Phi})^{\nu h/2} \mid \mathbf{Y}) \\ &= \frac{\Gamma_c(\frac{1}{2}(\nu+c+d-1))\Gamma_c(\frac{1}{2}(\nu+c-1+\nu h))}{\Gamma_c(\frac{1}{2}(\nu+c-1))\Gamma_c(\frac{1}{2}(\nu+c+d-1+\nu h))}, \end{aligned} \tag{3.4}$$

and this holds for all h for which the gamma functions exist including purely imaginary h. Thus we have the following theorem.

Theorem 3. The asymptotic cumulative posterior distribution of $M = -\nu\log \mathbf{B}(\mathbf{\Phi}) = -2\log \mathbf{U}(\mathbf{\Phi})$ is given by

$$Pr(M \le M_0 \mid \mathbf{Y}) = Pr(\chi_f^2 \le \rho M_0) \tag{3.5}$$

$$
\begin{aligned}
&+ \frac{\gamma_2}{\beta^2}\{Pr(\chi^2_{f+4} \le \rho M_0) - Pr(\chi^2_f \le \rho M_0)\} \\
&+ \frac{1}{\beta^4}[\gamma_4\{Pr(\chi^2_{f+8} \le \rho M_0) - Pr(\chi^2_f \le \rho M_0)\} \\
&- \gamma_2^2\{Pr(\chi^2_{f+4} \le \rho M_0) - Pr(\chi^2_f \le \rho M_0)\}] \\
&+ \mathbf{O}(\beta^{-6}),
\end{aligned}
$$

where $f = cd$, $\rho = 1 + \frac{1}{2\nu}(c+d-3)$, $\beta = \nu\rho$, $\gamma_2 = \frac{cd}{48}(c^2+d^2-5)$, and

$$\gamma_4 = \frac{\gamma_2^2}{2} + \frac{cd}{1920}[3c^4 + 3d^4 + 10c^2d^2 - 50(c^2+d^2) + 159].$$

Proof. Set

$$a = b = c, \quad x_k = \frac{\nu}{2}, \quad \xi_k = \frac{1}{2}(k-1), \quad y_j = \frac{\nu}{2}, \text{ and } \eta_j = \frac{1}{2}(d+j-1). \tag{3.6}$$

The equation (3.4) can be expressed as

$$E(\mathbf{U}(\mathbf{\Phi})^h \mid \mathbf{Y}) = \mathbf{K}\left(\frac{\prod_{j=1}^b y_j^{y_j}}{\prod_{k=1}^a x_k^{x_k}}\right)^h \frac{\prod_{k=1}^a \Gamma(x_k(1+h)+\xi_k)}{\prod_{j=1}^b \Gamma(y_j(1+h)+\eta_j)}, \quad h = 0, 1, \ldots, \tag{3.7}$$

where $\mathbf{K} = \Gamma_c(\frac{1}{2}(\nu+c+d-1))/\Gamma_c(\frac{1}{2}(\nu+c-1))$ which does not depend on h. Since $\sum_{k=1}^a x_k = \sum_{j=1}^b y_j = c\nu/2$ and $E(\mathbf{U}(\mathbf{\Phi})^0 \mid \mathbf{Y}) = 1$, the random variable $\mathbf{U}(\mathbf{\Phi})$, whose moments are certain functions of gamma functions, satisfies the conditions for Box's(1949) theorem of a general asymptotic expansion of the random variable: such that if we take a fifth order approximation to the distribution of M, Box's theorem (see Anderson, 1984, Theorem 8.5.1) gives the result.

If the first term of (3.5) is used, the error is of order β^{-2}; if the second term is used, the error is of order β^{-4}. Thus the asymptotic HPD region of probability content of the event in (2.11), *i.e.*

$$f(\mathbf{\Phi} \mid \mathbf{Y}) > f(\mathbf{\Phi}_0 \mid \mathbf{Y}) \equiv M < -\nu \log \mathbf{B}(\mathbf{\Phi}_0),$$

may be given by (3.5). Taking just the first two terms of the Box approximation, for illustration, gives following asymptotic result.

Corollary 1. Under the growth curve model (1.1), the asymptotic HPD region criterion for testing $H_0 : \mathbf{\Phi} = \mathbf{\Phi}_0; \mathbf{\Phi} \equiv \mathbf{C}\tau\mathbf{D}$ with significance level α

is to accept H_0 if

$$Pr(\chi^2_{cd} \le \rho M_0) + \frac{\gamma_2}{\beta^2}\{Pr(\chi^2_{f+4} \le \rho M_0) - Pr(\chi^2_f \le \rho M_0)\} \le 1-\alpha, \quad (3.8)$$

where $\rho M_0 = -\beta \log \mathbf{B}(\mathbf{\Phi}_0)$ and $\beta = N - m - q + 1 + \frac{1}{2}(c + d - 3)$.

Proof. Applying the first two terms of (3.5) with remainder term $\mathbf{O}(\beta^{-4})$ to (2.11) gives the result.

If we add the more term(third term) of (3.5) to the left hand side of (3.7), we would expect the better accuracy of the test(see Table I). In particular case where $\mathbf{\Phi}_0 = \mathbf{0}$, the test criterion in Corollary 1 is similar to Box approximation of the likelihood ratio criterion by Khatri(1966), except that $N - q - p + m - (c - d + 1)/2$ in Khatri's criterion is here replaced by $\beta = N - m - q + 1 - (c + d - 3)/2$(see Table II for the effect of this difference). So the classical test by Khatri is different from the suggested Bayesian test in that (i) the classical test depends on the dimension of the response vector of the model (1.1), while the suggested test does not; (ii) the suggested test can be used for any value of $\mathbf{\Phi}_0$, but the classical test can only be applicable for $\mathbf{\Phi}_0 = \mathbf{0}$.

4. NUMERICAL STUDIES

The goal of this section consists of two numerical studies: (i) a study pertaining to the overall performance of the Box's approximation for the distribution of M in (3.5); (ii) a relative comparison between the classical likelihood ratio test by Khatri(1966) and the suggested Bayesian test(see Corollary 1) in terms of $p-$value. For the latter test, we use $p-$value(Bayesian analogy to the classical exact significance level) for a Bayesian measure of evidence against the null hypothesis, so that, for given M_0, the test in Corollary 1 may yield

$$p\text{-value} = 1 - [Pr(\chi^2_{cd} \le \rho M_0) + \frac{\gamma_2}{\beta^2}\{Pr(\chi^2_{f+4} \le \rho M_0) - Pr(\chi^2_f \le \rho M_0)\}].$$

4.1. Performance of the Box's approximation

The probability of $Pr(M < M_0|\mathbf{Y})$ in (3.5) were computed using each of the three types of the Box's approximation: (i) Box_1 which uses only the first term on the right hand side of (3.5) with the remainder $\mathbf{O}(\beta^{-2})$; (ii) Box_2

which uses the first and second terms of (3.5) with remainder $\mathbf{O}(\beta^{-4})$; (iii) Box_3 that uses all three terms of (3.5) with remainder $\mathbf{O}(\beta^{-6})$. The performances are made by calculating the error of approximation for significance levels of 0.01 and 0.05 for ν from 2 to 60, $c = 3, 4$, and $d = 2, 4, 6$. Exact critical values of M_0 of M for each ν, c, and d were obtained by using the correction factors in Table 1 of Anderson(1984, p.609) and the probabilities $Pr(M < M_0|\mathbf{Y})$ were computed using each type of the three approximations. Errors of the three approximations for selected values from computations are given in Table I.

As would be expected, Box's approximation with the more terms in (3.5) yields the better accuracy for approximating the distribution of M. In general, the error of the asymptotic distribution for M becomes larger as the value of ν decreases to less than 10. Box_1 approximation highlights this tendency. The maximum errors due to Box_2 and Box_3 are less than 0.019 and 0.008, respectively; in most cases the errors are considerably less. Thus, from the table, we may safely expect that, regardless of dimension p, if $\nu \geq 10$ the suggested test(see Corollary 1) using Box_2(or Box_3) gives almost exact Bayesian test result for testing the linear hypothesis of the growth curve model.

4.2. Comparison with Likelihood Ratio Test Criterion(LRTC)

In this subsection, we compare the suggested Bayesian test criterion(BTC) with the classical LRTC by Khatri(1966). The two criteria were compared in terms of $p-$value obtained from testing $H_0 : \mathbf{\Phi} = \mathbf{0}$. This is to study the overall effectiveness of BTC and to identify some situations where one would(and would not) expect good test result. As mentioned in the previous section, when H_0 is true, Box_2 approximation to LRTC is similar to Corollary 1 except for the constant $\beta = N - m - q + 1 + (c + d - 3)/2$ in (3.7); LRTC replaces the constant by $\beta^* = N - q - p + m - (c - d + 1)/2$. If the hypothesis is not true, LRTC involves complex distribution so that the power function of the criterion still remains uninvestigated(see, Kabe, 1986). Thus, the comparison between the two test criteria were made by $p-$value. This was done by following set up : For a given significance level α, critical value of LRTC was found by use of modified version of Corollary 1(using constant β^* instead of β). This enabled us to get the value of $\rho M_0 = -\beta^* \log \mathbf{B}(\mathbf{\Phi}_0 = \mathbf{0})$ for the given significance level($p-$value of LRTC). Then, from Corollary 1, we evaluated $p-$value of BTC corresponding to the value ρM_0.

Table I. Errors of the Three Approximations for α = .05 and .01: Error = $\{(1 - \alpha) - \text{approx. prob.}\} \times 10^5$

				$d = 2$			$d = 4$			$d = 6$	
α	c	ν	Box_1	Box_2	Box_3	Box_1	Box_2	Box_3	Box_1	Box_2	Box_3
.05	3	3	1180	130	15	2317	636	110	3357	1496	470
		6	399	15	2	972	107	14	1702	329	52
		10	159	-1	-4	440	23	4	838	65	-4
		15	69	-8	-8	239	26	21	453	25	5
		20	46	1	1	121	-9	-10	283	14	7
		30	23	3	2	61	0	-1	144	10	9
		60	0	-5	-5	30	14	14	36	0	-1
.05	4	3	1870	357	42	2797	1009	260	3643	1882	726
		6	72	57	13	1276	191	27	1967	460	84
		10	302	2	-7	588	25	-12	1009	100	-3
		15	153	4	2	302	3	-7	548	26	-5
		20	77	-11	-12	204	20	16	359	26	14
		30	52	10	10	103	13	13	163	-6	-9
		60	0	-11	-11	34	10	10	41	-7	-7
.01	3	3	374	68	9	639	266	75	828	519	245
		6	136	10	3	293	46	3	476	135	26
		10	58	4	3	136	9	0	247	30	-1
		15	26	0	0	74	7	5	132	5	-4
		20	13	-2	-2	42	0	0	87	6	3
		30	7	0	0	17	-3	-3	40	-1	-2
		60	0	-2	-2	9	3	3	10	-2	-2
.01	4	3	548	170	34	731	380	146	867	599	327
		6	228	25	2	373	84	16	534	182	46
		10	98	1	-3	183	19	2	290	45	2
		15	50	1	0	92	0	-4	166	18	4
		20	29	0	0	56	0	-2	107	10	5
		30	15	0	0	28	0	0	55	5	4
		60	0	-4	-4	10	2	2	11	-3	-3

Table II. $p-$Values of BTC for $\alpha = .05, .01$ and $q = 6$.

					$p=6$			$p=7$		
α	m	c	N	d	2	4	6	2	4	6
.05	3	3	20		0.01171	0.00802	0.00636	0.00561	0.00314	0.00221
			30		0.02466	0.02003	0.01740	0.01840	0.01364	0.01112
			40		0.03133	0.02716	0.02457	0.02615	0.02140	0.01857
			50		0.03526	0.03163	0.02927	0.03096	0.02662	0.02774
			60		0.03784	0.03466	0.03253	0.03096	0.03029	0.02774
.05	4	3	20		0.03259	0.02936	0.02756	0.01921	0.01513	0.01310
			30		0.04008	0.03766	0.03613	0.03131	0.02738	0.02502
			40		0.04306	0.04121	0.03996	0.03667	0.03341	0.03131
			50		0.04467	0.04317	0.04214	0.03965	0.03693	0.03510
			60		0.04567	0.04442	0.04354	0.04154	0.03922	0.03762
.05	4	4	20		0.01767	0.01347	0.01141	0.00855	0.00534	0.00401
			30		0.02982	0.02559	0.02306	0.02194	0.01710	0.01442
			40		0.03543	0.03186	0.02957	0.02920	0.02467	0.02189
			50		0.03862	0.03560	0.03359	0.03354	0.02952	0.02693
			60		0.04067	0.03807	0.03630	0.03640	0.03285	0.03049
.01	3	3	20		0.00125	0.00080	0.00061	0.00044	0.00022	0.00015
			30		0.00362	0.00280	0.00236	0.00239	0.00165	0.00129
			40		0.00511	0.00427	0.00377	0.00394	0.00307	0.00258
			50		0.00605	0.00528	0.00479	0.00502	0.00416	0.00363
			60		0.00670	0.00600	0.00553	0.00579	0.00497	0.00445
.01	4	3	20		0.00540	0.00475	0.00441	0.00253	0.00190	0.00161
			30		0.00727	0.00673	0.00638	0.00510	0.00432	0.00386
			40		0.00806	0.00763	0.00734	0.00640	0.00570	0.00525
			50		0.00850	0.00814	0.00790	0.00716	0.00655	0.00614
			60		0.00878	0.00847	0.00826	0.00766	0.00712	0.00676
.01	4	4	20		0.00229	0.00165	0.00136	0.00083	0.00048	0.00034
			30		0.00480	0.00397	0.00349	0.00312	0.00229	0.00186
			40		0.00613	0.00537	0.00489	0.00467	0.00378	0.00326
			50		0.00693	0.00625	0.00581	0.00567	0.00483	0.00431
			60		0.00746	0.00686	0.00646	0.00637	0.00560	0.00510

$p-$values of BTC for selected values of N, p, α, c, d, and $m(c \leq m < p$ and $d \leq q < N)$ from the computations(fixing $q = 6$) are given in Table II.

As shown in Table II, $p-$value of BTC is uniformly smaller than that of LRTC($p-$value of LRTC is equal to α). This indicates that LRTC weights the evidence against H_0 more heavily than is warranted by the posterior probability distribution for BTC, a result analogous to Berger and Sellke(1987). In this sense, we can say that BTC yields more conservative test than LRTC does.

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