# Multiple-loading condition을 고려한 구조체의 위상학적 최적화

## Topological Structural Optimization under Multiple-Loading Conditions

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요 약

본 연구에서는 구조체의 위상학적 최적화를 위한 비선형 formulation(NLP)가 개발, 검토되었다. 이 NLP는 multiple-loading 하에서 임의의 오브젝티브 함수, 응력, 변위 제약조건들을 쉽게 다룰 수가 있다. 또한 이 NLP는 해석과 최적화 디자인을 동시에 실시함으로써 요소 사이즈가 영으로 접근함에 따른 강성 매트릭스의 singularity를 피할 수 있다. 즉, 평형 방정식을 등제약조건으로 치환함으로써 강성 매트릭스 그자체나 그의 역매트릭스를 구할 필요도 없어진다. 이 NLP는 multiple-loading condition하에서 테스트 되었으며, 이를 통해 이 NLP가 다양한 제약조건하에서 강력하게 작용함이 입증되었다.

Abstract

A simple nonlinear programming(NLP) formulation for the optimal topology problem of structures is developed and examined. The NLP formulation is general, and can handle arbitrary objective functions and arbitrary stress, displacement constraints under multiple loading conditions. The formulation is based on simultaneous analysis and design approach to avoid stiffness matrix singularity resulting from zero sizing variables. By embedding the equilibrium equations as equality constraints in the nonlinear programming problem, we avoid constructing and factoring a system stiffness matrix, and hence avoid its singularity. The examples demonstrate that the formulation is effective for finding an optimal solution, and shown to be robust under a variety of constraints.

**Keywords** : topology, optimization, nonlinear programming formulation(NLP), simultaneous method, SQP, single-loading condition, multiple-loading conditions

### 1. Introduction

The classical paper by Dorn, et. al.[6] proposed an LP formulation of the problem, but could guarantee an optimal topology only for stress constraints and single loading condi-

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• 이 논문에 대한 토의를 1996년 12월 31일까지 본 학회에 보내주시면 1997년 6월호에 그 결과를 게재하겠습니다.

tions. Attempts to generalize the methodology to displacement constraints, multiple loading, etc., lose the guarantee of optimality in the general case(e.g. [5][14][16][18]). Other approaches have employed heuristics to search through the topological space and NLP to solve the fixed-topology problem(e.g. [10]). Motivated by complexity and lose of guarantee of optimality of conventional approaches, we present a general nonlinear programming (NLP) formulation of the topology problems. Auther has presented an NLP formulation for the optimal topology problem of structure.[1] The formulation guarantees at least a local minimum. Stiffness matrix singularity is avoided by embedding the equilibrium equations as equality constraints. However, the formulation is focused only on a single loading condition. Therefore it would be desirable to expand the formulation to multiple-loading conditions. The new formulation follows similar development of the previous NLP formulation. That is, the formulation is based on simultaneous analysis and design, in which behavioral constraints are embedded as equality constraints in optimization model. However, in this study the formulation is modified to generalize the methodology to a variety of constraints, multiple-loading conditions as well as a single loading condition.

Our development addresses weight minimization of structures molded by finite elements, subject to stress, displacement, and linking constraints under multiple-loading conditions.

## 2. NLP Formulation for the Topological Structural Optimization

Our development addresses the minimum weight of structures under multiple loads. It corporates zero sizes: hence, the simultaneous method is used to insure that matrix singularity is avoided. Simultaneous formulation may be expressed as:[1]

$$g_1(x) = C_1 u(x) - u_b \le 0 \quad (1)$$

$$g_2(x) = C_2 u(x) - \sigma_b \le 0 \quad (2)$$

$$g_e(x) = K(x)u(x) - P(x) = 0 \quad (3)$$

where

$g_1(x)$ displacement constraints

$g_1(x)$ stress constraints

$C_1$, $C_2$ matrix of constant coefficients

$K(x)$ stiffness matrix

$P(x)$ a vector of applied loads

$u_b$ displacement limits

$\sigma_b$ stress limits

By embedding the equilibrium equations as equality constraints, one can avoid its singularity. It does not require stiffness matrix inversion. It requires only their evaluation, not their solution, at each optimization iteration.

Formulation

The NLP for the optimal topology is stated as follows :

objective function:

$$\text{minimize } F = \text{total weight} = \sum_{i=1}^{k} A_i \rho_i t_i \quad (4)$$

constraints:

subject to:

Equilibrium equation:

$$K\, u_j - P_j = 0 \qquad j = 1, 2, \cdots, p \quad (5)$$

Stress constraints:

$$\sigma_{ij}^L \le \sigma_{ij} \le \sigma_{ij}^U \qquad i = 1, 2, \cdots, k \quad (6)$$

Displacement constraints:

$$u_j^L \le u_j \le u_j^U \quad (7)$$

Thickness constraints:

$$t_i^L \leq t_i \leq t_i^U \qquad i = 1, 2, \cdots, k \tag{8}$$

Parameters are defined as:

- $k$ number of total elements
- $p$ number of loading conditions
- n number of degree of freedom after applying boundary condition
- K n×n-stiffness matrix
- $P_j$ n-vector of applied nodal loads for loading condition j
- $\sigma_{ij}^L$, $\sigma_{ij}^U$ stress lower (upper) bounds of element i for loading condition j
- $u_j^L$, $u_j^U$ n-vector of nodal displacement lower (upper) bounds for loading condition j
- $t_i^L$, $t_i^U$ thickness lower (upper) bounds of element i
- $A_i$ area of element i
- $\rho_i$ density of element i

and the variables are defined as:

- $t_i$ thickness of element i
- $u_j$ n-vector of nodal displacement for loading condition j

Remarks

- All functions (4)-(8) are assumed to be continuously differentiable.
- The nonlinearity in this formulation is found in the equilibrium equations (5) and stress constraints (6), which include bilinear product of displacement and thickness. The objective function (4) and all other constraints are linear.
- If none of the tLi is zero, then the NLP (4)-(8) is no longer a topological design problem and topology is fixed by the thickness lower bounds.
- There is no guarantee that a unique minimum exist, or that a local minimizer coincides with a global minimizer.
- A single stress constraints (6) or displacement constraints (7) can be chosen, if needed.

2.4 Sequential Quadratic Programming Algorithm

The sequential quadratic programming (SQP) method is used to solve the NLP. This method is based on the iterative formulation and solution of quadratic programming subproblems, obtains subproblems by using a quadratic approximation of Lagrangian. That is:

$$\text{minimize} \quad \frac{1}{2} p_k^T B(x_k, \lambda_k) p_k + \nabla F(x_k)^T p_k$$

subject to:

$$g_i(x_k)^T p_k + g_i(x_k) = 0 \qquad i = 1, 2, \cdots, m_e$$

$$g_i(x_k)^T p_k + g_i(x_k) \geq 0 \qquad i = m_{e+1}, \cdots, m$$

$$x^L - x_k \leq p_k \leq x^U - x_k$$

where $B_k$ is a positive definite approximation of the Hessian of the Lagrangian function. $x_k$ represents the current iterate points. Let $p_k$ be the solution of the subproblem. A line search is used to find a new point $x_{k+1}$, where

$$x_{k+1} = x_k + \alpha p_k \qquad \alpha \in (0, 1]$$

such that a merit function will have a lower function value at the new point. The augmented Lagrange function is used here as the merit function. When optimality is not achieved, $B_k$ is updated according to the BFGS formula. This algorithm may generate infeasible points during the solution process. Therefore, if feasibility must be maintained for intermediate points, then this routine may not be suitable.

A summary of SQP algorithm

1. Update $c_k$ and $B_k$

The vector $c_k$ represents the equality constraints in addition to any active inequality constraints.

2. Check optimality (if YES, then terminate: if NO, then go to step 3)

3. Solve QP subproblem to find a search direction $p_k$

$$\begin{bmatrix} B_k & -A_k^T \\ -A_k & 0 \end{bmatrix} \begin{bmatrix} p_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} -\nabla f_k \\ c_k \end{bmatrix}$$

where $A_k$ is the jacobian of the active constraints, $\lambda_k$ the vector of Lagrange multipliers, $\nabla f_k$ the gradient of the objective function and ck the value of the constraints at the current iteration.

4. Perform a line search on some suitable defined merit function which involves the objective function and constraints to determine a step length $\alpha$.

5. Put:

$$x_{k+1} \leftarrow x_k + \alpha p_k \qquad \alpha \in (0, 1]$$

$$k \leftarrow k+1$$

6. Go to step 1

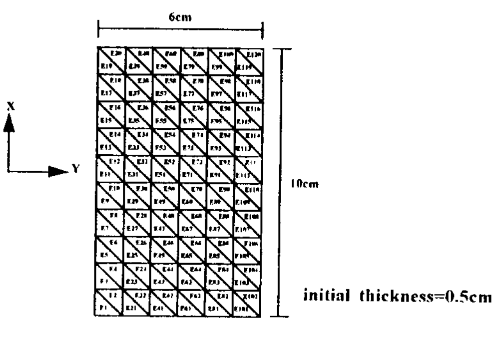


Fig. 1 Model for Examples

### 3. Examples of NLP for Topology Optimization

The NLP formulation of the topology problem is tested with 12×20 element rectangular model fixed at bottom depicted in Figure 1. The initial guesses for the state variables(displacements) are computed from the equilibrium equations and an initial guess for thickness of 0.5cm for an initial design to initiate the SQP method. Exact derivatives are used to construct the Jacobian matrix.

Common data for problems

- Aluminum (Al 6061-T6) is the material, i. e.

$$E=70\text{ GPa} \qquad \sigma_y=240\text{MPa} \qquad \tau_y=140\text{MPa}$$

$$\rho=0.002710\text{kg}/\text{cm}^3 \qquad \nu=0.34615$$

- Triangular finite elements are used.
- The structure is in plane stress.
- For stress constraints, 2 principle stresses ($\sigma_1$, $\sigma_2$) and maximum shear stress ($\tau_{max}$) are calculated for each element, and these stresses should be less than (or equal to) the maximum tension (compression, shear) stresses. That is,

$$\sigma_1 \leq \sigma_y$$

$$\sigma_2 \geq -\sigma_y$$

$$\tau_{max} \leq \tau_y$$

- For thickness constraints, following is used:

$$0 \leq \mathbf{t} \leq 10^{7}\text{cm}$$

- Density and areas of all elements are equal in each example, hence, the objective function is set to $F=\sum_{i=1}^{k} t_i$. The minimum volume is multiplied by density and area to obtain weight.

- If thickness of any element reaches zero, stress in that element is defined as zero.
- SQP terminates when the optimality condition is less than $10^{-7}$

3.1 Single loading condition

At first, the NLP formulation is tested with 3 cases under a single loading condition. The NLP formulation for this example can be expressed as:

$$\text{minimize} \quad F=\sum_{i=1}^{k} t_i \tag{9}$$

subject to:

Equilibrium equation: one of the equations (10)-(12)

$$\mathbf{K}\,\mathbf{u} - \mathbf{P} = 0 \qquad \text{(for case 1)} \tag{10}$$

$$\mathbf{K}\,\mathbf{u} - \mathbf{W} = 0 \qquad \text{(for case 2)} \tag{11}$$

$$\mathbf{K}\,\mathbf{u} - (\mathbf{P}+\mathbf{W}) = 0 \qquad \text{(for case 3)} \tag{12}$$

Stress constraints:

$$-24\text{MPa} \le \sigma \le 240\text{MPa} \tag{13}$$

Displacement constraints:

$$-0.1\text{cm} \le \mathbf{u} \le 0.1\text{cm} \tag{14}$$

Thickness constraints:

$$0.0\text{cm} \le \mathbf{t} \le 10^7\text{cm} \tag{15}$$

This model contains 140 D.O.F. and 120 elements under a single load. This problem has one equality constraints and five inequality constraints(three stress, one displacement and one thickness constraints). Hence, it has total 260 (D.O.F + total elements) variables and 760 constraints. Therefore, its Jacobian size is 760×260.

Figure 2-4 show the optimal topology. Interestingly, element 18, 38, 58 and 119 in case 1, and element 85 and 115 in case 2 have nonzero thickness. However, these elements have zero thickness in case 3(which is the combination of case 1 and case 2). Symbolically,

Applied loads: $\mathbf{P}_{case1} + \mathbf{P}_{case2} = \mathbf{P}_{case3}$

however,

Resulting Topology: $\mathbf{T}_{case1} + \mathbf{T}_{case2} \neq \mathbf{T}_{case3}$

3.1 Multiple-loading condition

To demonstrate the capability of the NLP formulation to solve multiple load problems, we solve the design problem of the previous section. In previous section, two independent loading conditions (W in case 1, P in case 2) are applied, and the structural topology is optimized under each separately. In this section, optimization is carried put as a multiple loading condition problem. The NLP formulation under multiple loading conditions of this example can be expressed as follows:

$$\text{minimize } F = \sum_{i=1}^{k} t_i \tag{16}$$

subject to:

Equilibrium equation:

$$\mathbf{K}\,\mathbf{u}_1 - \mathbf{P} = 0 \tag{17}$$

$$\mathbf{K}\,\mathbf{u}_2 - \mathbf{W} = 0$$

Stress constraints:

$$-24\text{MPa} \le \sigma_1 \le 240\text{MPa} \tag{18}$$

$$-24\text{MPa} \le \sigma_2 \le 240\text{MPa}$$

Displacement constraints:

$$-0.1\text{cm} \le \mathbf{u}_1 \le 0.1\text{cm} \tag{19}$$

$$-0.1\text{cm} \le \mathbf{u}_2 \le 0.1\text{cm}$$

Thickness constraints:

$$0.0\text{cm} \le \mathbf{t} \le 10^7\text{cm} \tag{20}$$

In previous section case 3 implied that two loads were applied at the same time(i.e. case $1 \cup$ case 2). However, multiple loading means "case $1 \cap$ case 2" satisfying the same optimal thickness. It has twice D.O.F. and stress and displacement constraints of the single loading condition case.

As a result, the optimal topology under multiple loading is not the intersection of the optimal topologies of each individual loading condition, i.e., element 61 and 119 in case 1, and elements 38 and 58 in case 2 have nonzero thickness. However those elements under multiple loading conditions have zero thickness. Furthermore, although element 96 in both case 1 and case 2 has nonzero thickness, that element under multiple loading conditions has zero thickness. Number of iterations to converge to the optimality and minimum weights are shown in Table 1.

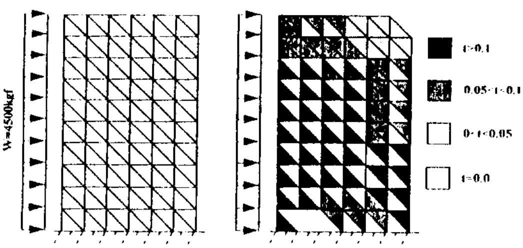


Fig. 2 Optimal topology of case 1

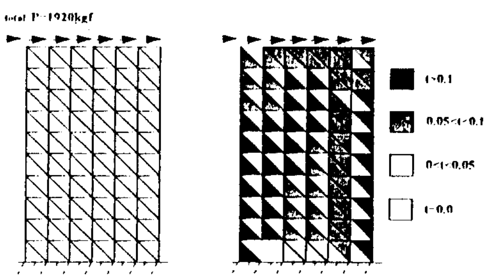


Fig. 3 Optimal topology of case 2

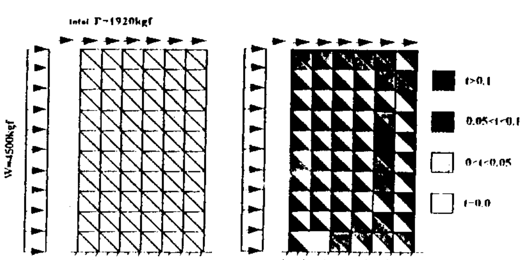


Fig. 4 Optimal topology of case 3

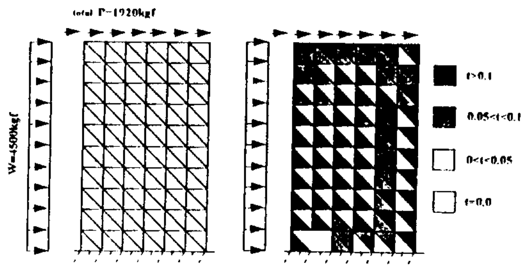


Fig. 5 Optimal topology of multiple loading condition case

Table 1 Optimal results

| case number | Initial weight (kg) | No. of interation | Optimal weight (kg) |
|---|---|---|---|
| case 1 | 0.0813 | 40 | 0.02885943 |
| case 2 | 0.0813 | 24 | 0.02879123 |
| case 3 | 0.0813 | 50 | 0.05677134 |
| multiple loads | 0.0813 | 51 | 0.03427712 |

## 4. Some other examples

Several problems were tested to study the NLP formulation. In this section, these optimal topologies are shown in Figures 6-8.

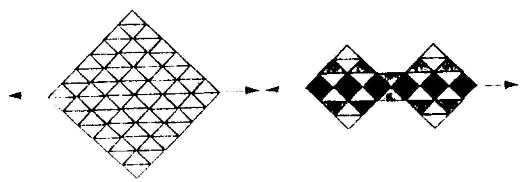

Fig. 6 Original and Optimal topology

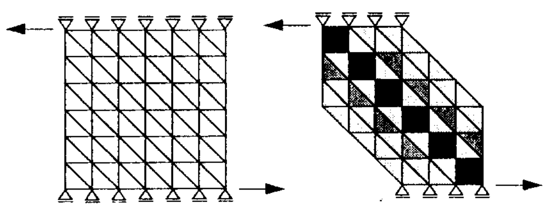

Fig. 7 Original and Optimal topology

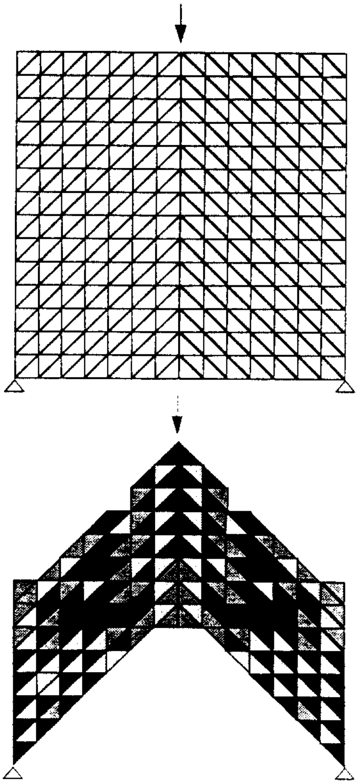

Fig. 8 Original and Optimal topology

## 5. Conclusion

We have presented an NLP formulation for the optimal topology problem of structure. The formulation is based on simultaneous analysis and design, in which behavioral constraints are embedded as equality constraints in optimization model. It insures that stiffness matrix singularity is avoided. Arbitrary objective functions, stress and displacement constraints under multiple-loading conditions, and upper and lower bounds on and linking of the design variables can be easily handled. The formulation is demonstrated on a number of examples of topology optimization of plate structures loaded in plane under multiple loading conditions, and shown to be robust under a variety of constraints.

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