Two-Dimensional Probability Functions of Morphological Dilation and Erosion of a Memoryless Source

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Abstract

This paper derives the two-dimensional probability distribution and density functions of morphological dilation and erosion of a one-dimensional memoryless source and reports numerical results for a uniform source, thus providing methodology for joint distributions for other morphological operations. The joint density functions expressed in closed forms contain the Dirac delta functions due to the joint discontinuity within the dilation and erosion. They also exhibit symmetry between these two morphological operations. Applications of the result can be found in the computation of the autocorrelation and the power spectral density functions of dilated and/or eroded sources, in the computation of other higher moments thereof, and in multidimensional quantization.

I. Introduction

This paper considers the morphological dilation and erosion of an independent identically-distributed (iid) random sequence $X=\{X_k\}_{k=-\infty}^{\infty}$ by a structuring element $G=\{G_k\}_{k=-\infty}^{\infty}$ of size L, which is assumed to only select, as opposed to adding a bias and selecting, appropriate samples X_k. Then for a given source sequence X and a structuring element sequence G, the dilation $U=\{U_k\}_{k=-\infty}^{\infty}$ and the erosion $V=\{V_k\}_{k=-\infty}^{\infty}$ are expressed as $U_k=(X\oplus G)(k)=\max_{j\in \underline{G}}\{X_{k-j}\}$ and $V_k=(X\ominus G)(k)=\min_{j\in \underline{G}}\{X_{k+j}\}$, where $\underline{G}$ denotes the support of G, i.e., the set of k such that $G_k\neq 0$. The structuring element considered herein has the property that $\underline{G}$ is a set of contiguous integers, i.e., $G_k=0$ for all k, except for $l\leq k\leq h$ for some integers l and h. This type of structuring element will be called a *contiguous structuring element* or a *contiguous window* of size h-l+1. It is also assumed in this paper that $\underline{G}=\{-L+1,\cdots,-1,0\}$. This assumption is made for convenience, but not necessary because the dilation and the erosion of a stationary source are stationary [1]. Note that it does not matter what values G_k assumes on $\underline{G}$.

This paper derives the two-dimensional probability distribution and density functions, and applies them to a source uniformly-distributed over (0,1).

The reason for this study is that two dimensional probability distributions, specifically two dimensional probability density functions, are necessary in order that statistical properties of dilated and/or eroded sources may be considered. For example, to find out the correlation between dilated and/or eroded source output samples that are continuous-valued, their two dimensional density function must be known. Therefore, the result of the paper will be valuable in computing the autocorrelation function and hence the power spectral density function of the dilated and/or eroded sources. The latter will show the power profile as a function of frequency when a source is dilated or eroded as done in [2,3]. Another application of the result of this paper is found in quantization of the dilated or eroded source: the multidimensional probability distribution or density functions are essential for efficient quantization.

The probabilistic aspects of morphological operations on a stationary memoryless source (iid random sequence) have been investigated, e.g.,[4,5,6]. However, they focused on the one-dimensional distribution. In case of the two-dimensional distribution Kuhlmann and Wise [2], in

Manuscript received July 25, 1994; accepted May 9, 1995.
This work was supported by the 1993 Non-Directed Research Fund of the Korea Research Foundation.
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their study on statistical properties of median filters, obtained an expression for the two-dimensional probability distribution function of, what is known in morphology, dilation and numerically computed the autocorrelation and power spectral density functions of iid discrete random sequences, i.e., quantized Gaussian sources.

However, this paper differs from Kuhlmann and Wise's work in that (1) the results in this paper are for continuous sources, and (2) it presents elegant closed form expressions for the two-dimensional density functions as well. This paper shows that the dilation and erosion of a memoryless source by a contiguous structuring element result in jointly discontinuous neighboring random variables and therefore the joint density functions contain singularity functions. These density functions also exhibit symmetry between the dilation and the erosion.

The rest of the paper is organized as follows: in Section 2 the joint distribution and density functions of the dilated and eroded sequences are presented;in Section 3 the numerical results for a uniform source are given; and in Section 4 the conclusions are drawn.

II. Two-Dimensional Probability Functions

Let $\{X_k\}$ be an iid random sequence with a common probability distribution $F(x)$ and density $F(x)$. We wish to find the two-dimensional probability density functions of $\{U_k\}$ and $\{V_k\}$. Toward this goal, the probability distribution function $F_{U_kU_l}(u_1, u_2)$ will be computed and the density function $f_{U_kU_l}(u_1, u_2)$ will be obtained by taking the partial derivative of $F_{U_kU_l}(u_1, u_2)$ with respect to u_1 and u_2. Then the joint density of $\{V_k\}$ will be derived from that of $\{U_k\}$.

Theorem 1 below states the result for the joint distribution function of $\{U_k\}$.

Theorem 1: For an iid source $\{X_k\}$ the dilated sequence $\{U_k\}$ has the two-dimensional probability distribution

$$F_{U_kU_l}(u_1, u_2) = \begin{cases} F^L(\min(u_1, u_2)), & \text{if } l=k, \\ F^{l-k}(u_1)F^{L-l-k}(\min(u_1, u_2))F^{l-k}(u_2), & \text{if } 0<|l-k|\le L-1, \\ F^L(u_1)F^L(u_2), & \text{if } |l-k|>L-1. \end{cases} \quad (1)$$

We note that the first two cases can be merged. Also note that the joint distribution function depends only on *l-k*, as it should because U_k is strict sense stationary [1]. Marginals are found when $u_1=\infty$ or $u_2=\infty$ and agree with the results previously reported, for example, in [5].

Proof From the definition of the distribution function, we have

$$\begin{aligned} F_{U_kU_l}(u_1, u_2) &= P(U_k\le u_1, U_l\le u_2) \\ &= P(\max\{X_k, \cdots, X_{k+L-1}\}\le u_1, \max\{X_l, \cdots, X_{l+L-1}\}\le u_2) \\ &= P(X_k\le u_1, \cdots, X_{k+L-1}\le u_1, X_l\le u_2, \cdots, X_{l+L-1}\le u_2), \end{aligned} \quad (2)$$

where the second equality follows from the definitions of U_k and U_l, and the third from a set equality. Depending on the values of the indices k,l and the window size L, we may have (1) $l=k$, (2) $0<|l-k|\le L-1$, and (3) $|l-k|>L-1$. Each case is separately dealt with in the following.

Case 1: If $l=k$, the joint distribution

$$\begin{aligned} F_{U_kU_l}(u_1, u_2) &= P(X_k\le \min(u_1, u_2), \cdots, X_{k+L-1}\le \min(u_1, u_2)) \\ &= F^L(\min(u_1, u_2)), \end{aligned} \quad (3)$$

where the first equality follows from the set equality $\{X\le a, X\le b\} = \{X\le \min(a, b)\}$ and the second from X_k being iid.

Case 2: If $0<|l-k|\le L-1$, assume that $l>k$ for the moment. Then

$$\begin{aligned} F_{U_kU_l}(u_1, u_2) &= P(X_k\le u_1, X_{k+1}\le u_1, \cdots, X_{l-1}\le u_1, \\ &\quad X_l\le \min(u_1, u_2), \cdots, X_{k+L-1}\le \min(u_1, u_2), \\ &\quad X_{k+L}\le u_2, \cdots, X_{l+L-1}\le u_2) \\ &= P(X_k\le u_1, X_{k+1}\le u_1, \cdots, X_{l-1}\le u_1) \\ &\quad \cdot P(X_l\le \min(u_1, u_2), \cdots, X_{k+L-1}\le \min(u_1, u_2)) \\ &\quad \cdot P(X_{k+L}\le u_2, \cdots, X_{l+L-1}\le u_2), \end{aligned} \quad (4)$$

where the first equality follows from combining constraints on the common X_k and the second from the independence of X_k. Since the roles of l and k can be interchanged, we obtain

$$F_{U_kU_l}(u_1, u_2) = F^{|l-k|}(u_1)F^{L-|l-k|}(\min(u_1, u_2))F^{|l-k|}(u_2). \quad (5)$$

Case 3: If $|l-k|>L-1$, we have

$$\begin{aligned} F_{U_kU_l}(u_1, u_2) &= P(X_k\le u_1, X_{k+1}\le u_1, \cdots, X_{k+L-1}\le u_1, \\ &\quad X_l\le u_2, X_{l+1}\le u_2, \cdots, X_{l+L-1}\le u_2) \\ &= F^L(u_1)F^L(u_2), \end{aligned} \quad (6)$$

where the second equality follows from the independence and stationarity of X_k.

We note that the joint distribution function in Theorem 1 is symmetric about $u_1=u_2$, i.e., the change of variables u_1 for u_2 and vice versa will result in the same distribution function.

Once the joint distribution function is obtained, the joint density function can be found by taking its partial derivative with respect to u_1 and u_2. However, care must be taken in so doing because, for $|l-k|\le L-1$, U_k and U_l are not jointly continuous and hence the usual type of joint density function does not exist. Nevertheless, using the singularity functions, such as the Dirac delta function $\delta(\cdot)$ (e.g.,[7, pp. 25-26] and the indicator function $I(\cdot)$, Lemma 1 is obtained. The indicator function $I(u_1 < u_2)$ is defined to take on 1 if $u_1 < u_2$, and 0 otherwise. Then $I(u_1 < u_2)$ can be related to the delta function as

$$I(u_1 < u_2) = \int_{-\infty}^{u_2} \delta(\alpha - u_1), d\alpha. \tag{7}$$

Lemma 1: For a continuous $f(x)$ the joint distribution function $F(\min(u_1, u_2))$ can be expressed as follows:

$$\begin{aligned} F(\min(u_1, u_2)) &= \int_{-\infty}^{u_1} f(\alpha) I(\alpha < u_2) d\alpha \\ &= \int_{-\infty}^{u_2} f(\beta) I(\beta < u_1) d\beta \\ &= \int_{\alpha=-\infty}^{u_1} \int_{\beta=-\infty}^{u_2} f(\alpha)\delta(\beta-\alpha) d\beta \, d\alpha. \end{aligned} \tag{8}$$

Corollary 1 follows from Lemma 1.

Corollary 1: A continuous density function $f(\cdot)$ and the distribution $F(\min(u_1, u_2))$ are related as follows:

$$\begin{aligned} &\frac{\partial^2}{\partial u_1 \partial u_2} F(\min(u_1, u_2)) = f(u_1)\delta(u_2 - u_1), \\ &\frac{\partial}{\partial u_1} F(\min(u_1, u_2)) = f(u_1) I(u_1 < u_2), \\ &\frac{\partial}{\partial u_2} F(\min(u_1, u_2)) = f(u_2) I(u_2 < u_1). \end{aligned} \tag{9}$$

Using(9) and the fact that $\delta(u_1 - u_2) = \delta(u_2 - u_1)$ and $h(u_1, u_2)\delta(u_2 - u_1) = h(u_1, u_1)\delta(u_2 - u_1)$ for a continuous function $h(u_1, u_2)$, the joint density function, stated as Theorem 2, follows from Lemma 1.

Theorem 2 For an iid source $\{X_k\}$ with a continuous density function $f(x)$, the dilated sequence $\{U_k\}$ has the two-dimensional probability density function $f_{U_k U_l}(u_1, u_2)$ given as follows:

Case 1: *If $l = k$,*

$$f_{U_k U_l}(u_1, u_2) = L F^{L-1}(u_1) f(u_1) \delta(u_2 - u_1). \tag{10}$$

Case 2: If $0 < |l-k| \le L-1$,

$$\begin{aligned} f_{U_k U_l}(u_1, u_2) = \; & (L-|l-k|)|l-k| F^{|l-k|-1}(u_1) f(u_1) F^{L-|l-k|-1} \\ & (\min(u_1, u_2)) I(u_2 < u_1) F^{|l-k|}(u_2) f(u_2) + \\ & (L-|l-k|)|l-k| F^{|l-k|}(u_1) f(u_1) F^{L-|l-k|-1} \\ & (\min(u_1, u_2)) I(u_1 < u_2) F^{|l-k|-1}(u_2) f(u_2) + \\ & |l-k|^2 F^{|l-k|-1}(u_1) f(u_1) F^{L-|l-k|}(\min(u_1, u_2)) \\ & F^{|l-k|-1}(u_2) f(u_2) + \\ & (L-|l-k|) F^{L+|l-k|-1}(u_1) f(u_1) \delta(u_2 - u_1), \end{aligned} \tag{11}$$

where the following identity is used

$$\frac{\partial}{\partial u_1} F(\min(u_1, u_2)) \frac{\partial}{\partial u_2} F(\min(u_1, u_2)) = 0. \tag{12}$$

For further simplification we note that for the first term $F(\min(u_1, u_2)) I(u_2 < u_1) = F(u_2) I(u_2 < u_1)$, similarly for the second term $F(\min(u_1, u_2)) I(u_1 < u_2) = F(u_1) I(u_1 < u_2)$, and for the third term $F(\min(u_1, u_2)) = F(u_2) I(u_2 < u_1) + F(u_1) I(u_1 < u_2)$ except for negligible $u_1 = u_2$.

Then we have

$$\begin{aligned} f_{U_k U_l}(u_1, u_2) = \; & L|l-k| F^{|l-k|-1}(u_1) f(u_1) I(u_2 < u_1) F^{L-1}(u_2) f(u_2) \\ & + L|l-k| F^{L-1}(u_1) f(u_1) I(u_1 < u_2) F^{|l-k|-1}(u_2) f(u_2) \\ & + (L-|l-k|) F^{L+|l-k|-1}(u_1) f(u_1) \delta(u_2 - u_1). \end{aligned} \tag{13}$$

Case 3: *If $|l-k| > L-1$,*

$$f_{U_k U_l}(u_1, u_2) = L^2 F^{L-1}(u_1) f(u_1) F^{L-1}(u_2) f(u_2). \tag{14}$$

Note that $f_{U_k U_l}(u_1, u_2)$ is symmetric about the line $u_1 = u_2$, as it should due to the symmetry of the distribution function. The marginal densities are $f_{U_k}(u) = f_{U_l}(u) = L F^{L-1}(u) f(u)$ and agree with the previously reported results.

The joint density of eroded sequence V_k can be found from that of the dilation U_k through transformations. Basically, $f_{V_k V_l}(v_1, v_2)$ is obtained from $f_{U_k U_l}(u_1, u_2)$ by replacing $F(u_1), F(u_2), I(u_1 < u_2)$ and $I(u_1 > u_2)$ with $1-F(v_1), 1-F(v_2), I(v_1 > v_2)$ and $I(v_1 < v_2)$, respectively. The result is stated as Theorem, in which the joint distribution function is also presented without proof due to virtual triviality. We also note the symmetry of the distribution and density functions just as with dilation. A further symmetry property must be noticed between dilation and erosion. This symmetry comes from the fact that erosion is essentially a shift of the related dilation, that is $X \ominus G =$ a shift of $\{-((-X) \oplus G)\}$.

Theorem 3: For an iid source $\{X_k\}$ with a continuous density function $f(x)$, the eroded sequence $\{V_k\}$ has the two-dimensional probability distribution function $F_{V_k V_l}(v_1, v_2)$ and density function $f_{V_k V_l}(v_1, v_2)$ given as follows:

Case 1: for $|l-k| \le L-1$,

$$\begin{aligned} F_{V_k V_l}(v_1, v_2) = \; & 1 - (1-F(v_1))^L - (1-F(v_2))^L \\ & + (1-F(v_1))^{|l-k|}(1-F(v_2))^{|l-k|}(1-F(\max(v_1, v_2)))^{L-|l-k|}. \end{aligned} \tag{15}$$

$$\begin{aligned} f_{V_k V_l}(v_1, v_2) = \; & L|l-k|(1-F(v_1))^{|l-k|-1} f(v_1) I(v_1 < v_2)(1-F(v_2))^{L-1} f(v_2) \\ & + L|l-k|(1-F(v_1))^{L-1} f(v_1) I(v_2 < v_1)(1-F(v_2))^{|l-k|-1} f(v_2) \\ & + (L-|l-k|)(1-F(v_1))^{L+|l-k|-1} f(v_1) \delta(v_2 - v_1). \end{aligned} \tag{16}$$

Case 2: for $|l-k| > L-1$,

$$F_{V_k V_l}(v_1, v_2) = (1-(1-F(v_1))^L)(1-(1-F(v_2))^L), \tag{17}$$

$$f_{V_k V_l}(v_1, v_2) = L^2(1-F(v_1))^{L-1} f(v_1)(1-F(v_2))^{L-1} f(v_2). \tag{18}$$

Proof: Note that V is obtained by shifting $-((-X) \oplus G)$

to the left by $L-1$, where L is the size of the contiguous structuring element G. Since the strict sense stationarity of U implies the strict sense stationarity of V, the shifting does not affect the distribution. Therefore, letting $S=-X$ and $T=(S\oplus G)$, we conclude that V has the same joint probability functions as $-T$. Since X_k is iid, S_k is iid. Then, the joint density $f_{T_kT_l}(t_1,t_2)$ is expressed in terms of the density and distribution functions of S_k and S_l by Theorem 1. Since the transformation $(V_k,V_l)=(-T_k,-T_l)$ is one-to-one, $f_{V_kV_l}(v_1,v_2)=f_{T_kT_l}(-v_1,-v_2)$, where it is used that the Jacobian involved in the transformation is 1. The use of $f_S(s)=f(-s)$ and $F_S(s)=1-F(-s)$ yields the conclusion, where S stands for any of S_k and S_l.

III. Numerical Results

Consider an iid X_k uniformly distributed over the unit interval (0,1). The joint density functions $f_{U_kU_l}(u_1,u_2)$ and $f_{V_kV_l}(v_1,v_2)$ from Theorems 2 and 3 assume 0 outside the unit square $(0,1)\times(0,1)$, on which they assume

$$f_{U_kU_l}(u_1,u_2)=\begin{cases} L|l-k|u_1^{|l-k|-1}u_2^{L-1}I(u_2<u_1) \\ +L|l-k|u_1^{L-1}u_2^{|l-k|-1}I(u_1<u_2) \\ +(L-|l-k|)u_1^{L+|l-k|-1}\delta(u_2-u_1), & \text{if } |l-k|\le L-1, \\ L^2u_1^{L-1}u_2^{L-1}, & \text{if } |l-k|>L-1, \end{cases} \tag{19}$$

$$f_{V_kV_l}(v_1,v_2)=\begin{cases} L|l-k|(1-v_1)^{|l-k|-1}(1-v_2)^{L-1}I(v_1<v_2) \\ +L|l-k|(1-v_1)^{L-1}(1-v_2)^{|l-k|-1}I(v_2<v_1) \\ +(L-|l-k|)(1-v_1)^{L-|l-k|-1}\delta(v_2-v_1), & \text{if } |l-k|\le L-1, \\ L^2(1-v_1)^{L-1}(1-v_2)^{L-1}, & \text{if } |l-k|>L-1. \end{cases} \tag{20}$$

As implied by Theorems 1 and 2, notice that these functions are symmetric about the line $u_1=u_2$ or $v_1=v_2$, i.e., $f_{U_kU_l}(u_1,u_2)=f_{U_{kl}}(u_2,u_1)$ and $f_{V_kV_l}(v_1,v_2)=f_{V_kV_l}(v_2,v_1)$, and that they are point-symmetric to each other about the point $(\frac{1}{2},\frac{1}{2})$, i.e., $f_{U_kU_l}(u_1,u_2)=f_{V_kV_l}(1-u_1,1-u_2)$. It follows from these two kinds of symmetry that $f_{U_kU_l}(u_1,u_2)=f_{V_kV_l}(1-u_2,1-u_1)$ and vice versa. Fig. 1 and 2 show the joint density functions for $L=5$ and various values for $|l-k|$. For $|l-k|=0$ the joint densities take on the value 0, depicted as a mesh in the figures, except $u_1=u_2$ or $v_1=v_2$, on which they consist only of the Dirac delta functions, depicted as a transparent wall whose height curves according to the marginal density function $f_{U_l}(u_1)=Lu_1^{L-1}$ or $f_{V_l}(v_1)=L(1-v_1)^{L-1}$. As $|l-k|$ increases to $L-1$, the maximum height of the delta functions decreases and a corner of the mesh (the point (1,1) for the dilation and the point (0,0) for the erosion) is drawn up higher and higher. This phenomenon reflects the fact that, as the window size L gets bigger, the maximum of the L contiguous source samples— dilation— will get larger too and this in turn causes the shift of the density away from 0 toward 1. Fig. 1 vividly shows the phenomenon described. When $|l-k|=2$, the maximum height of the wall of the delta functions is 3, which occurs at $u_1=u_2=1$, while the jointly continuous part, represented by the mesh, now is drawn up higher to 10 at $u_1=u_2=1$.

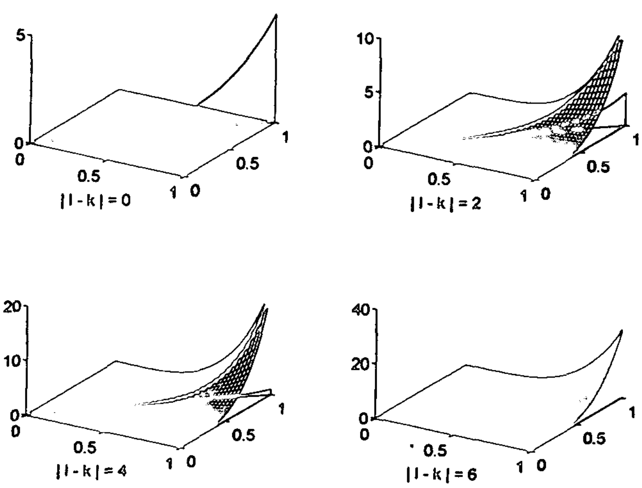

Fig. 1. The joint probability density $f_{U_kU_l}(u_1,u_2)$for L=5 and various values for $|l-k|$.

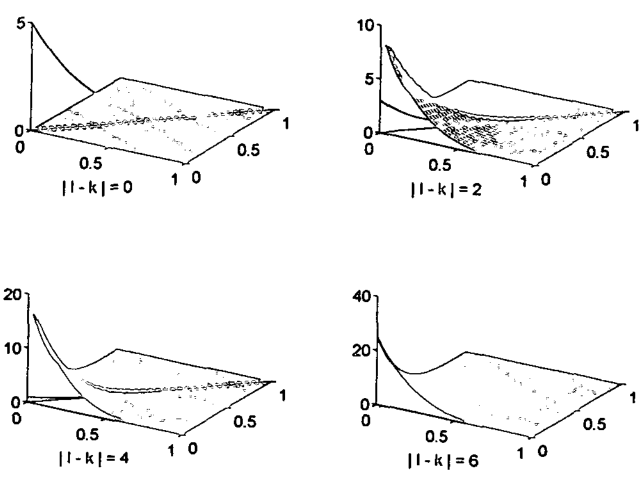

Fig. 2. The joint probability density $f_{V_kV_l}(v_1,v_2)$ for L=5 and various values for $|l-k|$.

When $|l-k|=4$, the maximum height of the wall of the delta functions gets lowered to 1 at $u_1=u_2=1$, while the mesh is drawn up higher to 20 at $u_1 = u_2 = 1$. All these represent well the fact that the maximum height of the delta functions is $L-|l-k|$ and hence must decreases as $|l-k|$ grows and the fact that the jointly continuous part of the density function is drawn up according to $L|l-k|$ and hence increase as $|l-k|$ increases. This mutually complementary relationship between the maximum heights of the delta functions and the mesh continues until $|l-k|$ reaches L. Once $|l-k|$ reaches L, the contiguous samples of the dilated source become independent, which causes the delta functions to disappear and only the smooth mesh surface to remain, and consequently the density function stays the same for all values of $|l-k| \geq L$ with one corner drawn up to a constant height L^2. One such example is shown in Fig. 1 for the case of $|l-k| = 6$.

The exact opposite phenomenon is observed in the erosion as can be seen in Fig. 2. This qualitative argument is clearly substantiated by the figures and the closed form expressions, which provides the quantitative aspect of the shift.

IV. Conclusions

A pair of random variables from a dilated or eroded iid source may not be jointly continuous. However, the closed form joint density functions can be expressed in terms of the distribution and density functions of the given source using the delta and indicator functions. These joint probability functions depend on the difference of the indices, but not on both, as a consequence of the strict sense stationarity of the dilated and eroded iid sequence and show symmetry between the dilation and the erosion. The method herein can be used for the joint probability functions for other morphological operations such as closing and opening. The result of this paper will play an essential role in quantitative study of dilated and/or eroded sources. For example, the computation of the autocorrelation function and the power spectral density function, the computation of other higher moments, and multidimensional quantization are a few applications.

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