On Lagrangian Approach to Mixed H_2/H_∞ Control Problem: The State Feedback Case

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Abstract

To improve the reliability of control systems, certain robustness to plant uncertainties and disturbance inputs is required in terms of well founded mathematical basis. Robust control theory was set up and developed until now from this motivation. In this field, H_2 or H_∞-norm performance measures are frequently used nowadays. Moreover a mixed H_2/H_∞ control problem is introduced to combine the merits of each measure since H_2 control usually makes more sense for performance while H_∞ control is better for robustness to plant perturbations. However only some partial analytic solutions are developed to this problem under certain special cases at this time.

In this paper, the mixed H_2/H_∞ control problem is considered. The analytic (or semi-analytic) solutions of (sub)optimal mixed H_2/H_∞ state-feedback controller are derived for the scalar plant case and the multivariable plant case, respectively. An illustrative example is given to compare the proposed analytic solution with the existing numerical one.

I. Introduction

The H_∞ design methodology has become very popular in recent years. The primary significance of H_∞ theory is that it can be combined with certain analysis methods, for example, structured singular value or μ analysis, to give a robust controller synthesis technique for systems with structured uncertainty. There is no comparable method yet for robust H_2 synthesis. Moreover, in addition to the fact that H_∞ design embodies many classidesign objectives, it also presents a natural tool for modeling plant uncertainty in terms of normed H_∞ plant neighborhoods. In contrast, the H_2 topology has been shown to be too weak for a practi robustness theory, while the H_∞-norm is not only suitable for robust stabilization but is also conveniently submultiplicative. The weakness of H_2 theory in robustness is complemented by performance improvement over large frequencies.

Typically an H_∞ controller design gives a lower, flatter closed-loop frequency response than that of the H_2 controller when comparing a pure H_2 controller design on the same problem with the generalized plant G is fixed. This is shown in the following example [1] in Fig. 1.

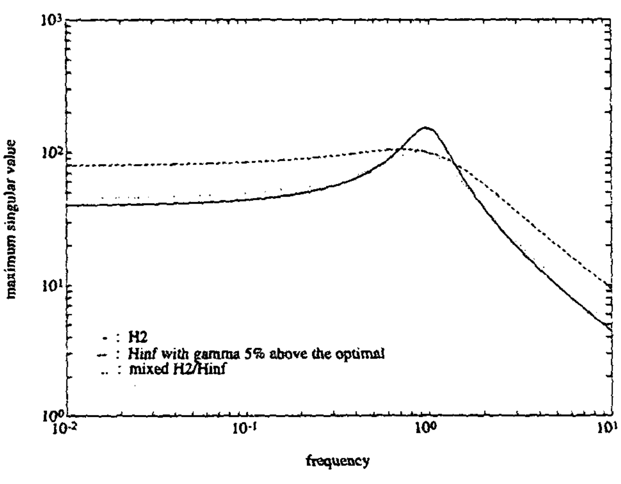

Fig. 1. Comparison of H_2, H_∞, and mixed H_2/H_∞ performances.

The solid line corresponds to the H_2 design, the dashed line corresponds to an H_∞ design that is 5% suboptimal, and the dotted line is a mixed H_2 and H_∞ design. These observations suggest that it would be nice to have a theory that directly handles both H_2 and H_∞ performance objectives

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at the same time. This motivates us to consider a more general problem which achieves this goal naturally and also gives a unified approach to solve both H_2 and H_∞ control problems. Of course, the real motivation for the mixed problem is that H_2 usually makes more sense for performance, but H_∞ is better for robustness to plant perturbations. Thus naturally we want a theory that handles both. The obvious advantage for a mixed norm is that it gives a natural trade-off between H_2 performance and H_∞ performance.

In this paper, a mixed H_2/H_∞ control problem is considered. This is the problem of finding an internally stabilizing controller that minimizes a mixed H_2/H_∞ performance measure subject to an inequality constraint on the H_∞-norm of another closed-loop transfer function. This problem can be interpreted and motivated as a problem of optimal nominal performance subject to a robust stability constraint.

Rotea and Khargonekar [2] have obtained some sufficient conditions for the solvability of the mixed H_2/H_∞ control problem in the state-feedback case. Bernstein and Haddad [3] and Ridgely and *et al.* [14] gave necessary conditions for optimality of controllers of a predefined order. Doyle *et al.* [4] and Zhou *et al.* [1] have considered a problem which is equivalent to the dual of the problem of Bernstein and Haddad. They have given necessary and sufficient conditions for the existence of an optimal controller but these are given in terms of coupled nonlinear matrix equations. At this time, there are no effective procedures for solving these equations other than certain homotopy methods developed by Richter [5] and Mariton and Bertrand [6]. Boyd *et al.* [7] have developed the convex programming approach to the mixed H_2/H_∞ control problem. They have reduced such controller synthesis problems to convex optimization problems over the infinite-dimensional space of stable transfer functions. Khargonekar and Rotea [8] further reduced the search space into a bounded set of real matrices. Although the convex programming approach offers a feasible numerialternative to the mixed H_2/H_∞ control problem, there is no completely analytic solution to this problem.

In this paper, we focus on the mixed H_2/H_∞ problem as formulated by Bernstein and Haddad [3]. The analytic (or semi-analytic) solution to this problem is derived on the framework of convex optimization given by Khargonekar and Rotea [8] using the relationship between the constrained extrema and Lagrange multipliers [9]. The paper is organized as follows. Section II is devoted to problem formulation on the framework of Khargonekar and Rotea [8]. The state-feedback problem and the conversion of the given problem into a convex optimization problem by Khargonekar and Rotea [8] are included in this section. In Section III, the reduced problem is solved analytically (or semi-analytically) for the scalar plant case and the multivariable plant case, respectively. We conclude this section with the simple example used by Khargonekar and Rotea [8] to compare our analytic method with the existing numerione. This illustrates some interesting features of our approach. Finally, some concluding remarks follow in Section IV.

III. Statement of the Problem

1. *Characterization of the Mixed H_2/H_∞ Performance Measure*

Consider the finite-dimensional LTI system G shown in Fig. 2.

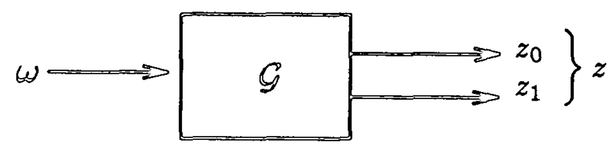


Fig. 2. Diagram for the definition of the mixed H_2/H_∞ performance measure.

Assume that G is internally stable [10] and that it is described by the following state-space model:

$$\begin{cases} \dot{x} = Fx + G\omega \\ z_0 = H_0 x + J_0 \omega \\ z_1 = H_1 x + J_1 \omega \end{cases} \tag{1}$$

where all the matrices are real and of compatible dimensions, and F is a stability matrix. Let the transfer matrix from ω to z is:

$$T_{z\omega} = \begin{bmatrix} T_{z_0\omega} \\ T_{z_1\omega} \end{bmatrix}$$

$\| T_{z_0\omega} \|_2 < \infty$ if and only if $J_0 = 0$ and, in this case, if L_c denotes the controllability Grammian of the pair (F,G), i.e., L_c satisfies [10]

$$FL_c + L_c F' + GG' = 0$$

then

$$\| T_{z_0\omega} \|_2^2 = tr(H_0 L_c H_0').$$

Let the scalar γ be given and assume that $\| T_{z_1\omega} \|_\infty < \gamma$. Define $M := \gamma^2 I - J_1 J_1'$, then $M > 0$. It is well known [10] that there exists a real symmetric matrix Y such that

$$R(Y) := FY + YF' + (YH_1' + GJ_1')M^{-1}(H_1 Y + J_1 G') + GG' = 0 \tag{2}$$

and $F+(YH_1'+GJ_1')M^{-1}H_1$ is asymptotically stable. Moreover, Y satisfies

$$0 \le L_c \le Y \le \hat{Y}$$

where $\hat{Y}$ denotes any real symmetric solution to the quadratic matrix inequality $R(\hat{Y}) \le 0$ [11]. Thus, if the H_2-norm of $T_{z_0\omega}$ is finite, then

$$\| T_{z_0\omega} \|_2^2 = tr(H_0 L_c H_0') \le tr(H_0 Y H_0').$$

From this the following definition of the *mixed* H_2/H_∞ *performance measure* for the LTI G is derived [3].

$$J(T_{z\omega}) := \begin{cases} \infty & \text{if } J_0 \neq 0 \\ tr(H_0 Y H_0') & \textit{otherwise}. \end{cases}$$

Note that the mixed H_2/H_∞ performance measure $J(T_{z\omega})$ is also a function of the parameter γ. However we will not make this dependence explicit since γ will be remained fixed throughout this paper.

The following lemma provides an alternative characterization for the mixed H_2/H_∞ performance $J(T_{z\omega})$ that will be useful for establishing some of the results in this paper.

Lemma 1([8]) Consider the stable system G defined in (1) and let $T_{z\omega}$ denote the transfer matrix from ω to $z=(z_0, z_1)$. Let $\gamma > 0$ be given. Assume that $\| T_{z_1\omega} \|_\infty < \gamma$ and that $T_{z_0\omega}$ is strictly proper. Let R(·) be given in (2). Then

$$J(T_{z\omega}) = \inf \{ tr(H_0 Y H_0') : Y = Y' > 0 \text{ such that } R(Y) < 0 \}.$$

2. Synthesis Framework

The synthesis framework addressed in this paper follows the problem formulated by Khargonekar and Rotea [8] which was originally introduced by Bernstein and Haddad [3]. Consider the finite-dimensional LTI feedback system depicted in Fig.3, where the plant G and the controller C are given by some state-space models.

The signal ω denotes an exogenous input, while z_0 and z_1 denote controlled signals. The signals u and y denote the control input and the measured output, respectively. The transfer matrices of the plant and the controller are denoted by G and C, respectively. We denote the closed-loop transfer matrix by

$$T_{z\omega} = \begin{bmatrix} T_{z_0\omega} \\ T_{z_1\omega} \end{bmatrix}$$

where $T_{z_0\omega}$ and $T_{z_1\omega}$ denote the closed-loop transfer matrices from ω to z_0 and ω to z_1, respectively.

A controller C is called *admissible* if C internally stabilizes the plant G. *Internal stability* means that the states of G and C go to zero from all initial values when $\omega=0$. Since we will restrict our attention exclusively to proper, real-rational controllers which are stabilizable and detectable, these properties will be assumed throughout. The set of all admissible controllers C for the plant G is denoted by A(G). Note that $A(G) \neq \emptyset$ if and only if G is stabilizable from u and detectable from y.

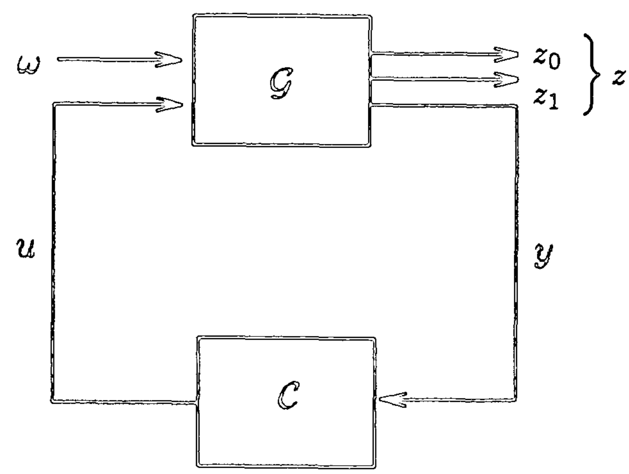


Fig. 3. The synthesis framework.

Consider the feedback interconnection shown in Fig.3, let γ be given, and define the following set of controllers:

$$A_\infty(G) := \{ C \in A(G) : \| T_{z_1\omega} \|_\infty < \gamma \} \tag{3}$$

where the subscript "∞" in the notation $A_\infty(G)$ stands for the constraint on the ∞-norm. Following Bernstein and Haddad [3], the optimal or suboptimal mixed H_2/H_∞ controller synthesis problem considered in this paper is defined as follows.

The Mixed H_2/H_∞ *Control Problem*: "Compute the mixed H_2/H_∞ optimal performance measure

$$\nu(G) := \inf \{ J(T_{z\omega}) : C \in A_\infty(G) \} \tag{4}$$

and find an optimal controller $C \in A(G)$ or, given any $\alpha > \nu(G)$, find a suboptimal controller $C \in A_\infty(G)$ such that $J(T_{z\omega}) < \alpha$".

Note that the performance measure in (4) is the same as one used in [3] and [8].

3. State-Feedback Problem

We consider the case where the plant to be controlled is given by a state-space model in which the state vector is available for feedback. Consider the mixed H_2/H_∞ synthesis problem defined in Section II for the following plant:

$$G_{sf} := \begin{cases} \dot{x} = Ax + B_1\omega + B_2 u \\ z_0 = C_0 x + D_0 u \\ z_1 = C_1 x + D_1 u \\ y = x \end{cases} \tag{5}$$

All the matrices in (5) are real and of compatible dimensions. Let G_{sf} denote the transfer matrix of (5). Note that we exclude the feedthrough terms to make the presentation simple as before.

Given a plant G_{sf} and an internally stabilizing controller C, the mixed H_2/H_∞ cost of the closed-loop system is a function of the transfer matrix T_{zw} only for a fixed γ. Since T_{zw} depends only on the transfer matrices G_{sf} and C, let

$$J(G_{sf}, C) := j(T_{zw}(G_{sf}, C))$$

denote the mixed H_2/H_∞ cost corresponding to the feedback system.

Now, given the plant G_{sf} defined in (5), we are interested in the computation of constant state-feedback matrices for the minimization of $J(G_{sf}, K)$ since the infimum of the mixed H_2/H_∞ performance measure over all dynamic full information feedback controllers equals the infimum over all static state-feedback controllers [8].

D. Conversion to the Convex Optimization Problem

In this section, conversion of the static state-feedback problem to the convex optimization problem formulated by Khargonekar and Rotea [8] will be shown.

We assume that $\gamma=1$ without loss of generality. For the state-feedback plant defined in (5), let n=dim(x) and q=dim(u). Let Σ denote the set of $n\times n$ real symmetric matrices and define

$$\Omega = \{(W, Y) \in R^{q \cdot n} \times \Sigma : Y > 0\}. \qquad (6)$$

Note that Ω is an open strictly convex subset of $R^{q\cdot n}\times\Sigma$. Given $(W, Y) \in \Omega$, define

$$f(W, Y) := tr((C_0 Y + D_0 W) Y^{-1} (C_0 Y + D_0 W)') \qquad (7)$$

and for $(W, Y) \in R^{q\cdot n}\times\Sigma$, let

$$Q(W, Y) := AY + YA' + B_2 W + W' B_2' + B_1 B_1' + (C_1 Y + D_1 W)' (C_1 Y + D_1 W). \qquad (8)$$

Define also the set of real matrices

$$\Phi(G_{sf}) := \{(W, Y) \in \Omega : Q(W, Y) < 0\} \qquad (9)$$

and consider the optimization problem

$$\sigma(G_{sf}) := \inf f(W, Y) : (W, Y) \in \Phi(G_{sf}). \qquad (10)$$

Then the mixed H_2/H_∞ synthesis problem is reduced to finding *W*, *Y* which minimizes *f(W,Y)* in (7) restricted to the condition of (9) and the state-feedback gain $K := WY^{-1}$ is obtained [8]. The resulting controller becomes suboptimal. The optimal controller can be obtained by changing the inequality in (9) into equality, since (8) is the transformed Riccati equation in the formulation of performance measure [8] and $J(G_{sf}, K = WY^{-1}) = f(W, Y)$ in this case.

Although the *convex feasibility program* method gives a numerisolution to this problem, there is no completely analytic solution until now. Moreover there are no guarantees that the optimum is achieved so a reasonable strategy in this case for the convex programming approach is to repeat the iteration and to stop when there is no appreciable improvement in the mixed H_2/H_∞ cost $J(G_{sf}, K)$.

The objective of this paper is to provide some sufficient conditions for the existence of optimal controller and analytic (or semi-analytic) solutions via Lagrange multiplier method [9].

III. Analytic Approach to State-Feedback Problems

In this section, we will develop an analytic (or semi-analytic) approach for solving the static state-feedback problem. The main results of this paper are given in the next two theorems.

A. Scalar Plant Case

Consider the mixed H_2/H_∞ synthesis problem defined in Section II for the following scalar plant:

$$G_{sf} := \begin{cases} \dot{x} = ax + b_1\omega + b_2 u \\ z_0 = c_0 x + d_0 u \\ z_1 = c_1 x + d_1 u \\ y = x \end{cases} \qquad (11)$$

All the constants in (11) are real. Note again that we exclude the feedthrough terms to make the presentation simple.

For the scalar plant, it means that the exogenous input ω, the control input u, and the measured output y are scalar variables. Thus the controlled signals z_0 and z_1 may be vector quantity. In this case, z_0 and z_1 in (11) can be rewritten as

$$\begin{cases} z_0 = C_0 x + D_0 u \\ z_1 = C_1 x + D_1 u \end{cases} \qquad (12)$$

For the scalar plant in (11) including (12), the static (sub)optimal controller gain *k* can be found by the following Theorem 1.

Theorem 1: Consider the system G_{sf} defined in (11).For this system, except the case of $d_0^2 + d_1^2 = 0$ and $b_2 c_0 \neq 0$, (sub)optimal state-feedback gain *k* can be found explicitly by solving the following cubic equation of *k*.

$$\alpha k^3 + \beta k^2 + \gamma k + \delta = 0 \qquad (13)$$

where $\alpha = d_0^2 d_1^2 \eta$, $\beta = (b_2 + 3(c_1 d_1)\eta) d_0^2$, $\gamma = 2(c_0 d_0)(c_1 d_1)\eta + 2(a + c_1^2 \eta) d_0^2 - c_0^2 d_1^2 \eta$, and $\delta = 2(a + c_1^2\eta)(c_0 d_0) - (b_2 + (c_1 d_1)\eta) c_0^2$.

All the solutions of k and η are restricted to

$$(c_1 + d_1 k)^2 \eta^2 + 2(a + b_2 k)\eta + b_1^2 \leq 0 \text{ and } \eta > 0 \qquad (14)$$

and the optimum occurs when equality holds. This optimal solution always exists if $c_1 d_1 \neq 0$.

For the case of multiple solutions, we choose one which minimizes

$$\eta(c_0 + d_0 k)^2 \qquad (15)$$

For the case of (12), all the above things are still hold by defining $c_0 := \sqrt{C_0'C_0}$, $d_0 := \sqrt{D_0'D_0}$, $(c_0 d_0) := (C_0' D_0)$, $c_1 := \sqrt{C_1'C_1}$, $d_1 := \sqrt{D_1'D_1}$, and $(c_1 d_1) := (C_1' D_1)$ and by changing the corresponding values into vector values in each if-conditions, compatibly.

Proof: See the Appendix.

Remark 1: At first glance, it seems that the proposed solution is not analytic at all since the gain k and its constraint equation contain another variable η. However it can be solved analytically (or semi-analytically) through two steps: first, represent k in terms of η and substitute this k into the constraint equation (for optimal case) and solve it for η --- which is an algebraic equation of η of order less than 6 if $d_0 d_1 = 0$ (otherwise this step may or may not require a certain zero finding algorithm), then substitute this η into the original formula of k.

B. Multivariable Plant Case

Consider the mixed H_2/H_∞ synthesis problem for the multivariable plant in (5) which is rewritten in the following.

$$G_{sf} := \begin{cases} \dot{x} = Ax + B_1\omega + B_2 u \\ z_0 = C_0 x + D_0 u \\ z_1 = C_1 x + D_1 u \\ y = x \end{cases} \tag{16}$$

Suppose that the variables in (16) are vectors and the constants are real matrices of compatible dimensions. The feedthrough terms are omitted for simple presentation as before.

For the multivariable plant in (16), the static (sub)optimal feedback controller K can be found by the following Theorem 2.

Theorem 2: Consider the system G_{sf} defined in (16). If $\frac{\partial Q}{\partial Y}$ is nonsingular then the (sub)optimal state-feedback controller K is given by $K = WY^{-1}$, where $\frac{\partial Q}{\partial Y}$ is nn × nn matrix as

$$\begin{vmatrix} \frac{\partial Q_{11}}{\partial Y_{11}} & \cdots & \frac{\partial Q_{nn}}{\partial Y_{11}} \\ \vdots & & \vdots \\ \frac{\partial Q_{11}}{\partial Y_{nn}} & \cdots & \frac{\partial Q_{nn}}{\partial Y_{nn}} \end{vmatrix}$$

and $\frac{\partial \hat{Q}_{ij}}{\partial Y_{rs}}$ is the cofactor of $\frac{\partial Q_{ij}}{\partial Y_{rs}}$ in $\frac{\partial Q}{\partial Y}$. Moreover the elements of W can be found from

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial Q_{ij}}{\partial W_{uv}}\frac{\partial \hat{Q}_{ij}}{\partial Y_{rs}}\right)\frac{\partial f}{\partial Y_{rs}} = \left|\frac{\partial Q}{\partial Y}\right|\frac{\partial f}{\partial W_{uv}} \tag{17}$$

where $u = 1, \cdots, q$ and $v = 1, \cdots, n$ with $q = \dim(u)$, $n = \dim(x)$, and

$$\frac{\partial f}{\partial Y_{rs}} = [\ C_0'C_0 - (D_0 WY^{-1})'(D_0 WY^{-1})]_{rs} \tag{18}$$

$$\frac{\partial f}{\partial W_{uv}} = [\ 2D_0'(C_0 + D_0 WY^{-1})]_{uv} \tag{19}$$

Further Y is restricted to

$$AY + YA' + B_2 W + W'B_2' + B_1 B_1' + (C_1 Y + D_1 W)'(C_1 Y + D_1 W) \le 0 \tag{20}$$

with Y = Y' > 0, and the optimum occurs when equality holds in (20).

Proof: See the Appendix.

In this case, we can infer about the sufficient condition for the existence as follows.

conjecture 1: If both $C_1'C_1$ and $D_1'D_1$ are either positive definite or negative definite then the optimal controller K exists.

C. Illustrative Example

In this section, we consider a simple example in order to highlight some interesting features of the suggested method. We will study the mixed H_2/H_∞ synthesis problem for the same plant which was used by Khargonekar and Rotea [8], to compare each approach.

Consider the following scalar plant:

$$G := \begin{cases} \dot{x} = -x + \omega + u \\ z_0 = u \\ z_1 = [\ x'\ u'\ ]' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = x \end{cases}$$

The parameter γ is chosen to be 1. The mixed H_2/H_∞ optimal performance in terms of the constant controller gain K for the plant G above is given by

$$\nu(G) = \inf\{J(G, K) = K^2 Y : (K, Y) \in \Phi_1\} \tag{21}$$

where

$$\Phi_1 = \{(K, Y) \in R \times R : Y > 0 \text{ and } Y^2(1 + K^2) + 2(K-1)Y + 1 < 0\}.$$

The constraint set Φ_1 is plotted in Fig.4.

From this figure, it is clear that this set is not convex. Furthermore Φ_1 is unbounded. Note also that the cost function $J(G, K) = K^2 Y$ is not convex on $\{(K, Y) \in R \times R : Y > 0\}$. Hence, this formulation of the synthesis problem gives rise to a nonconvex programming problem with unbounded domain.

i) Method 1. (The existing convex programming approach) The objective function $f(W, Y) = \frac{W^2}{Y}$ in (10) is convex on $\Omega = \{(W, Y) \in R \times R : Y > 0\}$. Moreover, the constraint set Φ defined in (9) is guaranteed to be convex and bounded. For this example, the constraint set Φ defined in (9) is a circle given by

$$\Phi = \{(W, Y) \in R \times R : Y > 0 \text{ and } (W+1)^2 + (Y-1)^2 < 1\}.$$

Solutions to the suboptimal synthesis problem of finding a controller $C \in A_\gamma(G)$ such that J(G,C) < α, can be found by intersecting the constraint set Φ with the level sets

$$\Lambda_\alpha = \left\{(W, Y) \in R \times R : Y > 0; \text{ and } \frac{W^2}{Y} < \alpha\right\}.$$

This intersection is shown in Fig.5 (a)-(d) for α = 1, 0.1, 0.01, 0.

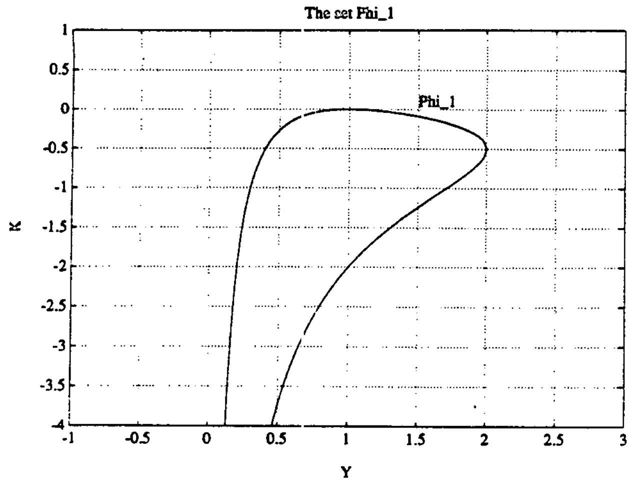
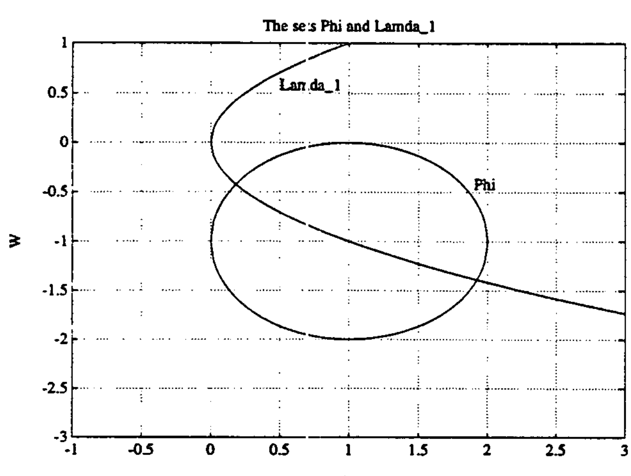


Fig. 4. The constraint set Φ_1 in Y-K plane.

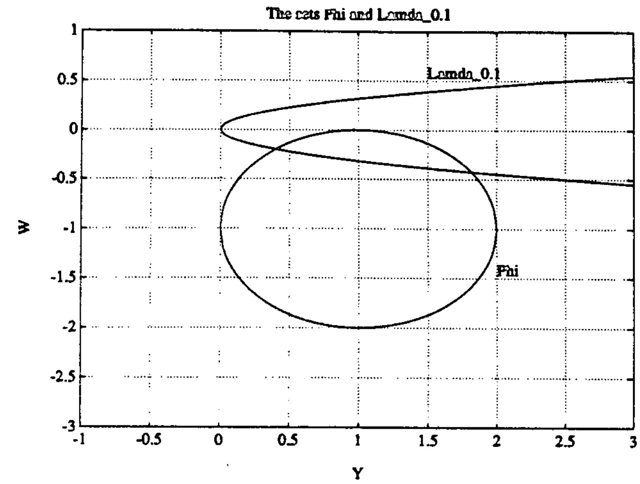
From these figures, by decreasing α, it can also be seen that the optimal H_2/H_∞ performance measure for this problem is $v(G)=0$, which corresponds to (W_{opt}, Y_{opt}) = (0,1). The unique controller that attains this optimal performance is the constant gain K_{opt} = 0. Note that, even though this gain stabilizes the closed-loop, it is not in $A_\infty(G)$, for $\| T_{z_1\omega}(G, K_{opt}) \|_\infty = 1$. Instead, it is attained by a controller C $\in$ A(G) such that $\| T_{z_1\omega}(G, C) \|_\infty = \gamma$. In general if the two objectives $\| T_{z_1\omega}(G, C) \|_\infty$ and *J(G,C)* are "truly competing" with each other, optimal controllers will be at the boundary of the ∞-norm constraint [8].

ii) **Method 2. (The suggested analytic solution)** Applying Theorem 1 to this problem, the (sub)optimal controller K is obtained as follows:

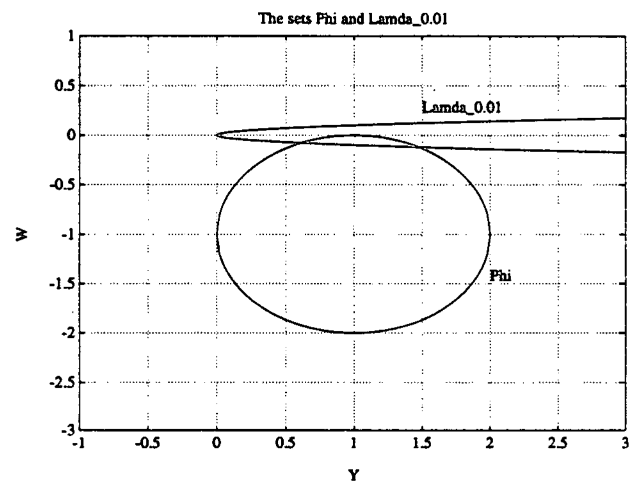
$$K := \begin{cases} \dfrac{-1-\sqrt{1+8Y-8Y^2}}{2Y} & \text{for } 0 < Y \le 1 \\ \dfrac{-1+\sqrt{1+8Y-8Y^2}}{2Y} & \text{for } 1 \le Y \le \dfrac{2+\sqrt{6}}{4} \end{cases} \qquad (22)$$



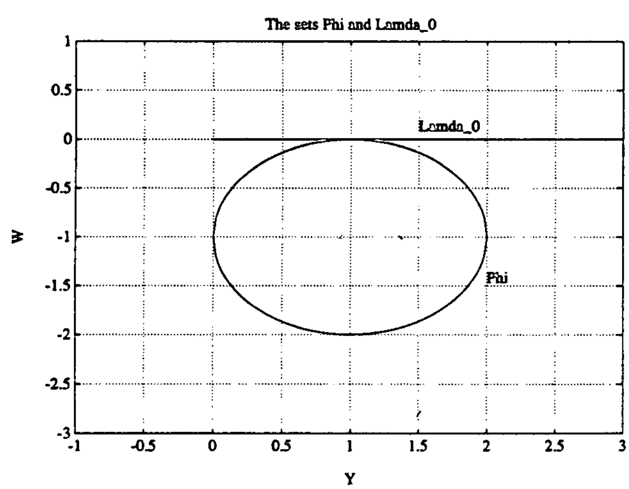
(a)



(b)

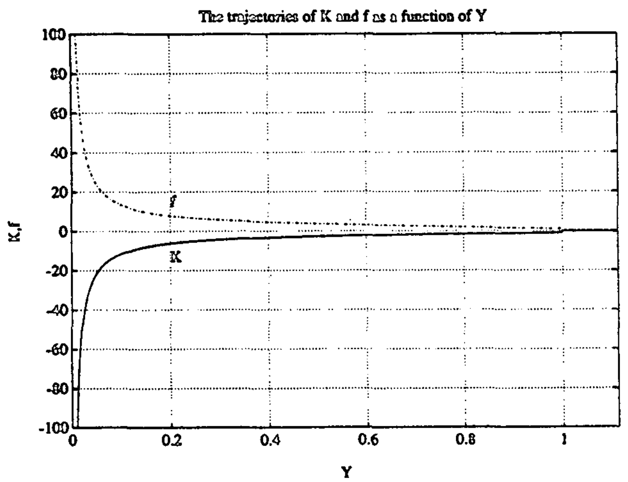


(c)

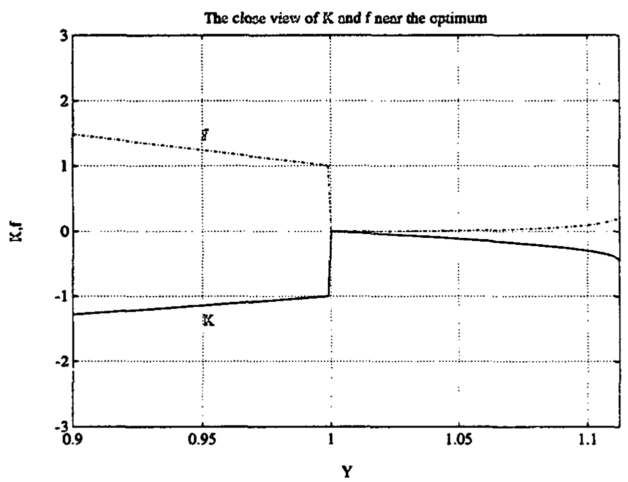


(d)

Fig. 5. The set Φ and Λ_α : (a) α=1 (b) α=0.1 (c) α=0.01 (d) α=0



(a)



(b)

Fig. 6. The (sub)optimal controller K and the cost function f as a function of Y : (a) Overall view in the full range of Y (b) Close view near the optimum.

where K and Y are used instead of k and η. The optimum is obtained for Y = 1 since the equality holds at this point in the constraint formula of (14). This optimum of K_{opt} is 0 which makes the cost function $\nu(G)$ in (21) smaller between K's in {0,-1} from (22). This can be confirmed as follows. Since $\nu(G) \propto K^2$, the behavior of |K| is important in determining the minimum of $\nu(G)$. K is monotonically increasing in the negative axis, or |K| is monotonically decreasing as Y increases in the interval [0,1]. Moreover K is monotonically decreasing in the negative axis, or |K| is monotonically increasing as Y increases in the interval $[\,1,\frac{2+\sqrt{6}}{4}\,]$. Hence the minimum of the mixed H_2/H_∞ performance measure $\nu(G)$ occurs when Y = 1 and the optimal controller K_{opt} = 0 is obtained. These are illustrated in Fig.6. Of course we know that the optimal controller is not included in $A_\infty(G)$ but in $A(G)$ since $\| T_{z_1\omega}(G,K_{opt}) \|_\infty = 1$. Anyhow we can obtain the optimal controller K_{opt} and the other suboptimal controller K at the same time, without any iterative graphiprocedure as was done in Method 1.

IV. Conclusions

In this paper, we considered a mixed H_2/H_∞ control problem on the framework of convex optimization given by Khargonekar and Rotea [8]. The analytic (or semi-analytic) solutions of (sub)optimal state-feedback controller are derived for the scalar plant case and the multivariable plant case, respectively. The proposed analytic solution is compared with the existing numerione through an illustrative example. More details about multivariable case and output-feedback case can be found in [13].

Since we presented an analytic (or semi-analytic) solution, a certain zero finding algorithm might be required for some cases (as mentioned in the Remark 1). Thus a completely analytic solution should be further researched. Moreover a substantial research on the existence conditions of optimal solution for the multivariable plant case is still needed.

Appendix

A. Proof of Theorem 1

Recall that the given mixed H_2/H_∞ control problem reduces to the following convex optimization problem: Find ε and η such that

$$\{f(\varepsilon,\eta) = (c_0\eta + d_0\varepsilon)^2\ \eta, > 0\} \tag{23}$$

is minimized subject to

$$g(\varepsilon,\eta) = 2a\eta + 2b_2\varepsilon + b_1^2 + (c_1\eta + d_1\varepsilon)^2 \le 0 \tag{24}$$

where f and g comes from (7), (8). Then the (sub)optimal controller is $k = \frac{\varepsilon}{\eta}$ and the optimum occurs when equality holds in (24). Let

$$g(\varepsilon,\eta) = n \le 0$$

where n is any nonpositive constant. Then

$$g(\varepsilon,\eta) - n = 0$$

Applying the Lagrange multiplier method [9]:

$$\frac{\partial f}{\partial \varepsilon} = \lambda \frac{\partial g}{\partial \varepsilon} \tag{25}$$

$$\frac{\partial f}{\partial \eta} = \lambda \frac{\partial g}{\partial \eta} \tag{26}$$

Moreover,

$$\begin{cases} \frac{\partial f}{\partial \varepsilon} = 2c_1d_0+2d_0^2\varepsilon\eta \\ \frac{\partial f}{\partial \eta} = c_0^2-frac\, d_0^2\varepsilon^2\eta^2 \\ \frac{\partial g}{\partial \varepsilon} = 2(b_2+c_1d_1\eta+d_1^2\varepsilon) \\ \frac{\partial g}{\partial \eta} = 2(a+c_1^2\eta+c_1d_1\varepsilon) \end{cases}$$

Substituting these into (25), (26) with $k=\frac{\varepsilon}{\eta}$ gives

$$c_0d_0+d_0^2k = \lambda(b_2+(c_1d_1+d_1^2k)\eta)$$
$$c_0^2-d_0^2k^2 = 2\lambda(a+(c_1^2+c_1d_1k)\eta)$$

By eliminating λ and rearranging the terms with respect to k, we obtain

$$\alpha k^3+\beta k^2+\gamma k+\delta=0 \qquad (27)$$

where
$\alpha=d_0^2d_1^2\eta$, $\beta=(b_2+3(c_1d_1)\eta)d_0^2$,
$\gamma=2(c_0d_0)(c_1d_1)\eta+2(a+c_1^2\eta)d_0^2-c_0^2d_1^2\eta$,
and $\delta=2(a+c_1^2\eta)(c_0d_0)-(b_2+(c_1d_1)\eta)c_0^2$. This equation can be solved to obtain the explicit form of k as follows.

If $d_0=0$ and $d_1\neq 0$ then (27) becomes

$$-c_0^2d_1^2\eta k-(b_2+(c_1d_1)\eta)c_0^2=0$$
$$k=-\left(\frac{b_2}{d_1^2\eta}+\frac{c_1}{d_1}\right).$$

If $d_0\neq 0$ and $d_1=0$ and $b_2=0$ then (27) becomes

$$k=-\frac{(a+c_1^2\eta)c_0d_0}{(a+c_1^2\eta)d_0^2}=-\frac{c_0}{d_0}.$$

If $d_0\neq 0$ and $d_1=0$ and $b_2\neq 0$ then (27) becomes

$$b_2d_0^2k^2+2(a+c_1^2\eta)d_0^2k+2(a+c_1^2\eta)c_0d_0-b_2c_0^2=0.$$

This has two real roots,

$$k=\frac{-(a+c_1^2\eta)\pm\sqrt{(a+c_1^2\eta)^2-b_2\left[2(a+c_1^2\eta)\frac{c_0}{d_0}-\frac{c_0^2b_2}{d_0^2}\right]}}{b_2}.$$

If $d_0\,d_1\neq 0$ then the resulting equation becomes cubic. Let $p=\frac{1}{3}\left(\frac{\gamma}{\alpha}-\frac{\beta^2}{3\alpha^2}\right)$, $q=-\frac{1}{2}\left(\frac{\delta}{\alpha}-\frac{\beta\gamma}{3\alpha^2}+\frac{2\beta^3}{27\alpha^2}\right)$, and $k=v-\frac{\beta}{3\alpha}$, then (27) becomes

$$v^3+3pv-2q=0. \qquad (28)$$

(28) has one real solution v_1 and two complex solutions $v_{2,3}$:

$$v_1=\sqrt[3]{q-\sqrt{p^3+q^2}}+\sqrt[3]{q+\sqrt{p^{3+q^2}}} \qquad (29)$$

$$v_{2,3}=-\frac{1}{2}v_1\pm j\frac{\sqrt{3}}{2}\left(\sqrt[3]{q-\sqrt{p^3+q^2}}-\sqrt[3]{q+\sqrt{p^3+q^2}}\right) \qquad (30)$$

Let $p^3+q^2=ae^{j\theta}$ and substitute this into (30). Then, by separating (30) into real and imaginary parts, we can classify the following three cases.

(*i*) if $p^3+q^2<0$ and $p<0$ then (30) becomes two different real roots,

$$v_{2,3}=-2\sqrt{-p}\cos\left[\frac{\pi}{3}\pm\left(\frac{1}{3}\arctan\left(\frac{\sqrt{-(p^3+q^2)}}{q}\right)+n\pi\right)\right]$$

$$\text{where } n=\begin{cases}0 & \text{if } q+\sqrt{a}\cos\frac{\theta}{2}\geq 0\\ 1 & \text{otherwise.}\end{cases}$$

(*ii*) if $p^3+q^2=0$ then (30) becomes one real root (double root),

$$v_{2,3}=-\sqrt[3]{q}.$$

(*iii*) if $p^3+q^2<0$ and $p\geq 0$, or $p^3+q^2>0$ then(30) becomes two complex conjugate roots. Combining these three cases with one real root in (29) results in explicit form of k in this case.

The constraint (24) with the condition of η in (23) can be rewritten as follows using $k=\frac{\varepsilon}{\eta}$,

$$g(\varepsilon,\eta)=(c_1+d_1k)^2\eta^2+2(a+b_2k)\eta+b_1^2\leq 0 \text{ and } \eta>0$$

which is (14). Further, f in (23) can be rewritten as

$$f(\varepsilon,\eta)=\eta(c_0+d_0k)^2$$

which is (15).

If $c_1d_1\neq 0$ then the constraint set in (24) becomes bounded region such as circle or ellipse and then f has always a minimum on that region [12]. Thus the optimal solution always exists if $c_1d_1\neq 0$.

The preservation of all the above procedures for the case of (12) comes from the fact that the constraint g and the cost function f are still remained as scalar quantity. This completes the proof.

B. Proof of Theorem 2

Recall that the given mixed H_2/H_∞ control problem reduces to the convex optimization problems in (7), (8). Then the (sub)optimal controller is given by $K=WY^{-1}$ and the optimum occurs when equality holds in (8). Let $Q(W,Y)=N\leq 0$ where N is any negative semidefinite matrix. Applying $\frac{\partial}{\partial W}tr[AWB]=A'B'$, $\frac{\partial}{\partial W}tr[AW'B]=BA$, $\frac{\partial}{\partial W}tr[AWBW]=A'W'B'+B'W'A'$, and $\frac{\partial}{\partial W}tr[AWBW']=A'WB'+AWB$, we obtain

$$\frac{\partial f}{\partial W}=2D_0'(C_0+D_0WY^{-1}) \qquad (31)$$

and using $\frac{\partial(trWX^{-1})}{\partial X}=-X^{-1}WX^{-1}$,

$$\frac{\partial f}{\partial Y}=C_0'C_0-(D_0WY^{-1})'(D_0WY^{-1}). \qquad (32)$$

Define

$$\frac{\partial Q}{\partial W}:=\left[\frac{\partial Q}{\partial W_{ij}}\right]$$

$$:=\begin{bmatrix} \left[\frac{\partial Q_{kl}}{\partial W_{11}}\right] & \cdots & \left[\frac{\partial Q_{kl}}{\partial W_{1n}}\right] \\ \vdots & & \vdots \\ \left[\frac{\partial Q_{kl}}{\partial W_{q1}}\right] & \cdots & \left[\frac{\partial Q_{kl}}{\partial W_{qn}}\right] \end{bmatrix}$$

where $\left[\frac{\partial Q_{kl}}{\partial W_{ij}}\right] = \begin{bmatrix} \frac{\partial Q_{11}}{\partial W_{ij}} & \cdots & \frac{\partial Q_{1n}}{\partial W_{ij}} \\ \vdots & & \vdots \\ \frac{\partial Q_{n1}}{\partial W_{ij}} & \cdots & \frac{\partial Q_{nn}}{\partial W_{ij}} \end{bmatrix}$,

$$\Lambda := [\lambda_{ij}],$$

and define an operator '*' as

$$\Lambda * \frac{\partial Q}{\partial W} = [\lambda_{ij}] * \left[\frac{\partial Q}{\partial W_{rs}}\right]$$

$$:= \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\frac{\partial Q_{ij}}{\partial W_{rs}}\right]$$

$$:= \begin{bmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\frac{\partial Q_{ij}}{\partial W_{11}} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\frac{\partial Q_{ij}}{\partial W_{1n}} \\ \vdots & & \vdots \\ \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\frac{\partial Q_{ij}}{\partial W_{q1}} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\frac{\partial Q_{ij}}{\partial W_{qn}} \end{bmatrix}$$

Defining similarly for the case of $\frac{\partial Q}{\partial Y}$ and applying the Lagrange multiplier method [9] for this multiple constraints case :

$$\frac{\partial f}{\partial W} = \Lambda * \frac{\partial Q}{\partial W}$$

$$\frac{\partial f}{\partial Y} = \Lambda * \frac{\partial Q}{\partial Y}$$

Substituting and reconstructing the elements of matrices into column vectors, we can obtain (17)--(20) through the similar procedures as in the proof of Theorem 1.

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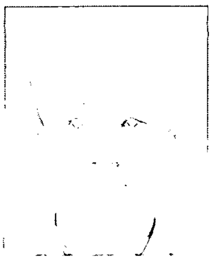

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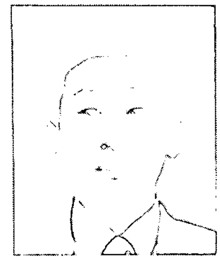

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