Theoretical analysis of the properties of the pseudomedian filter

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Abstract

The pseudomedian filter was designed to be a computationally effecient alternative to the median filter. However, a thorough analysis ofthe pseudomedian filter reveals some important differences between its response and that of the median filter. This paper develops some formal properties of the pseudomedian filter. The root signal analysis demonstrates the close relatonship between the median and pseudomedian filters while pointing out important and useful differences.

Ⅰ. Introduction

Linear filtering is a well-established method for extracting signals from noisy environments. There are many tools available for analyzing linear filters, and the properties of these filters are well-understood. Nevertheless in some situations a linear filter, even an optimal linear filter, does not perform adequately. In these cases, the special properties of certain nonlinear filters are required. Unfortunately, nonlinear filters are much more difficult to analyze than linear filters, and as a result the properties of many nonlinear filters are poorly understood.

Median filtering is one of the most common nonlinear techniques used in signal processing. Among the first to demonstrate the benefits of taking medians of sample data were Borda and Frost [12], and the median filter itself is generally ascribed to Turkey [11]. The properties of the median filter have been studied in quite some depth since these early uses [6, 7, 8, 9, 10]. The edge-preserving and impulse-removing properties of this filter are usually the most desirable features, and although the median filter is not conceptually complex, its computation can become quite cumbersome. The problem of efficient computation of the filter led Pratt, Cooper, and Kabir [1] to propose the "pseudomedian" filter, a filter with properties similar to the median filter, which can be more efficiently calculated. The pseudomedian filter often produces results that are very similar indeed to those produced by the median fulter, and many of the theoretical properties of the psedomedian filter are the same or nearly the same as the corresponding properties of the median filter. However, the response of the pseudomedian filter to high-frequency osillations and the impulses is quite different from the response of the median filter. In many circumstances, the response of the pseudomedian filter is preferable.

This paper develops some formal properties of the pseodomedian filter. Gallagher and Wise [2] and Tyan [3] first demonstrated the root signal set(that is, the set of signals unchanged by filtering) for the median filter. The root signal analysis of the pseudomedian filter demonstrates the close relationship between the median and pseudomedian filters while pointing out important and useful differences.

Ⅱ. Pseodomedian Filter Definition

The median filter is described as operating on a discrete signal. A window of width $2N+1$ sample points slides across the signal. The output of the filter is the median of the $2N+1$ values in the window, and this output is the filtered value at the sample in the center of the window.

The pseudomedian filter is also described for a discrete signal and a window width $2N+1$. However, the output of the pseudomedian operator, PMED, is the average of the maximum of the minima and the minimum of the maxima of the $N+1$ sliding subsequences of length $N+1$ in the window. This definition is illustrated by the equations below for $N=1$ and $N=2$.

$$\mathrm{PMED}\{a, b, c\} = \frac{1}{2}\max\{(\min\{a, b\}, \min\{b, c\})\} + \frac{1}{2}\min\{(\max\{a, b\}, \max\{b, c\})\}$$

where the values in the window(width $=2N+1=3$) are $\{a, b, c\}$

$$\mathrm{PMED}\{a, b, c, d, e\} = \frac{1}{2}\max\{\min\{a, b, c\}, \min\{b, c, d\}, \min\{c, d, e\}\} + \frac{1}{2}\min\{(\max\{a, b, c\}, \max\{b, c, d\}, \max\{c, d, e\})\}$$

where the values in the window(width $=2N+1=5$) are $\{a, b, c, d, e\}$

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Once angain, the filtered output for the window is the filtered value at the sample in the center of the window. The relationship of the definition of the pseudomedian to the median is illustrated by noting that the median can be defined as the maximum of the minima (or, equivalently, the minimum of the maxima) of all subsequences of length N + 1 in the window. There are $\binom{2N+1}{N+1}$ such subsequences, and so the pseudomedian uses only a small subset of all possible subsequences in the window. It is worthwhile to notice that each of the subsequences used in the pseudomedian has the value at the center of the window as an element; this indicates that the pseudomedian will exhibit a more "center-weighted" response than the median.

Several observations about the pseudian filter can be made at this point. First, since the pseudomedian is defined as an average of two signal values, its output is not limited to values in the unfiltered signal. For example, an unfiltered signal consisting only of integer values does not necessarily result in a pseudomedian-filtered output of only integer values, as would be the case for the median-filtered output. This averaging method also indicates that the pseudomedian filter have a "more linear" response than the median filter. Second, the pseoudomedian filter has many of the same general properties of the median filter: it exhibits a "low-pass" type of response in many cases, and preserves edges while reducing impulse noise. Finally, the pseudomedian filter can be implemented by an algorithm that is theoretically of lower order than most algorithms for the median filter. The fast median filtering algorithm developed by Huang, et al. [5] is a very fast implementation of the median filter, however, and a similar implementation of the pseudomedian filter would not be as efficient. Other algorithms for the median filter are either similar in speed or noticeably slower than most implementations of the pseudomedian filter.

Ⅲ. Root Signal Theory

The pseudomedian filter operates on a discrete window in a sliding-window fashion, where the output at a particular point is the pseudomedian of the values in 2N + 1 wide window centered at the point. That is,

$$\{y_k\} = PMED\{x_{k-N},\ldots,x_k,\ldots,x_{k-N}\}$$

The filter is assumed to be operating on a signal that extends to infinity in both directions, such that the filter window is always filled with signal values. However, the analysis presented below also holds true for extended signals. These signals are infinite in length and have N constant points equal to the first value in the signal appended to appended to the beginning of the signal, and N constant points equal to the last value appended to the end.(See also [2].)

The signal characteristics defined below create a precise vocabulary for the theorems presented in this paper. The terms apply to a signal to be operated on by a filter of window width 2N + 1.

1. A *constant neighborhood* is an area of N + 1 or more consecutive points that have the same value.

2. An *edge* is a monotonic sequence of points between constant neighborhoods such that the edge and constant neighborhoods combined are monotonic.

3. An *impulse* is an area of N or fewer points that is surrounded on either side by identically-valued constant neighborhoods.

4. An *oscillation* in any area that is not included in a constant neighborhood, edge, or impulse.

5. A *root signal* of a filter is a signal that is unchanged by that filter.

The first below associated two of the above definitions with a single property.

Lemma 1: A signal of arbitrary length consists only of constant neighborhoods and edges if and only if each subsequence of length N + 2 in the signal is monotonic.

Proof: Suppose a signal consists only of constant neighborhoods and edges. Since edges are by definition monotonic and are separated by constant neighborhoods, a monotonic subsequence of the signal must contain a constant neighborhood. But since a constant neighborhood has at least N + 1 consecutive equally valued points, any length N + 2 subsequence including a constant neighborhoods and edges must be monotonic.

Now suppose that every length N + 2 subsequence of a signal is monotonic. For a change in trend to occur(that is, from nondecreasing to nonincreasing or vice versa), there must be a subsequence of at least N + 1 equally valued points between the sections of increase and decrease. Thus the areas of equal value between trend changes are constant neighborhoods, and the monotonic regions between these areas are edges. Therefore all signals where every length N + 2 subsequence is monotonic consist only of constant neighborhoods and edges.

QED

Two simple observations concerning the pseudomedian filter may now be made. The value in the center of the filter window is important because this value is in every subsequence taken while computing the pseudomedian. Since this is so, no subsequence can have its minimum be greater than this value, and none can have its maximum be less than than value. Therefore the maximum of the minima is restricted to values greater than or equal to the value in the center of the window, and likewise the minimum of the maxima must be less than or equal to the value in the center of the window.

Observation 1: The minimum of the length $N+1$ subsequences taken for a length $2N+1$ pseudomedian is greater than or equal to the value of the point at the center of the window.

Observation 2: The maximum of the minima of the length $N+1$ subsequences taken for a length $2N+1$ pseudomedian is less than or equal to the value of the point at the center of the window.

The pseudomedian is thus the average of two values, one of which is greater than or equal to the value in the center of the filter window, and the other of which is less than or equal to the value in the center of the window.

The following two properties are important features of the pseudomedian filter and are shared with the median filter. They follow directly from the above observations.

Property 1: The pseudomedian of a monotonic sequence of length $2N+1$ is the value in the center of the sequence. That is, if $x_{-N} \leq \ldots \leq x_0 \leq \ldots \leq x_N$, then

$$PMED\{x_{-N},\ldots,x_0,\ldots,x_N\}=x_0$$

Proof: The maximum of the first subsequence in the window is x_0, which is the value in the center of the window. Since at least one subsequence has x_0 as its maximum, by Observation 1 the maximum of the minima is x_0.

Therefore, $PMED\{x_{-N},\ldots,x_k,\ldots,x_{k+N}\}=x_0$

QED

Property 2: The pseudomedian of a sequence of length $2N+1$ that contains a constant neighborhood is the value of the constant neighborhood.

Proof: Since a constant neighborhood is an area of at least $N+1$ consecutive equally valued points, then at least one of the length $N+1$ subsequences taken while computing the pseudomedian has its maximum and its minimum equal to the value of the constant neighborhood. By Observation 1, the minimum of the maxima is the value of the center point, and by Observation 2, the maximum of the minima is the value of the center point. The center point must be within the constant neighborhood, because the constant neighborhood is more than half the length of the window. Therefore, the pseudomedian of a length $2N+1$ sequence containing a constant neighborhood is the value of the constant neighborhood.

QED

The following lemma establishes a sufficient condition for a signal to be root th pseudomedin filter.

Lemma: A signal consisting only of constant neighborhoods and edges is a root of a pseudomedian filter of window width $2N+1$.

Proof: For a signal to be a root, the pseudomedian of each window must be the value in the center of the window. For a given signal, there are two possible cases for each $2N+1$ length window.

Case I: The points in the window are monotonic.

By Property 1, the pseudomedian is the value in the center of window.

Case II: The points in the window are nonmonotonic.

Since edges are monotonic and are separated by constant neighborhoods, a monotonic window must contain a constant neighborhood. By Property 2, the pseudomedian of such a window is the value of the constant neighborhood. The center point of the window must be within the constant neighborhood, and therefore the pseudomedian of the window is equal to the value in the center of the window.

QED

The next lemma is a more restricitve version of Lemma 6.A.1 proved in Tyan[3] for the median filter. The additional restrictions enforced by the pseudomedian filter are the basis for proving that the root signals from Lemma 2 are in fact necessary and sufficient conditions, whereas there exists and additional class of infinite-length root signals for the median filter.

Lemma3: Let $n<m$ and let $x_n<x_i<x_m$ for all $n<i<m$. If $PMED\{x_{n-p},\ldots,x_{n+p}\}=x_n$ and if $PMED\{x_{m-q},\ldots,x_{m+q}\}=x_m$, where $n-p \leq m-q$ and $n+p \leq m+q$, then $x_j \leq x_n$ for all $n-p \leq j<n$ and $x_j \leq x_m$ for all $m<j \leq m+q$.

Recast in less symbolic language for $p=q=N$, Lemma 3 states that if a signal is strictly increasing from point n to point m, and $PMED\{x_{n-p},\ldots,x_{n+p}\}=x_n$ and PMED

$\{x_{m-q},...,x_{m+q}\} = x_m$, then the signal is monotonically nondecreasing from point n-N to point m + N.

Proof of Lemma 3: See Figure 1 below for a diagram of the signal segment under consideration. Consider first the window from n-p to n + p. The proof will be given for the case where the maximum of the minima and minimum of the maxima of the length p + 1 subsequences must be x_n. (The value of x_n may be achieved by averaging two unequal values when computing the pseudomedian, but signals that achieve this result do not fit the hypothesis of the theorem. The proof for this case is omitted.) Since x_n is the center point of the window, if there is at least one length p + 1 subsequence with x_n as its maximum, by Observation 1 the minimum of the maxima will be x_n. For this to be true, the first subsequence in the window must have x_n as its maximu, since all other subsequences include $x_{n+1} > x_n$. Thus $x_j \le x_n$ for all $n-p \le j < n$.

Now consider the window from m-q to m + q. Again, the minimum of the maxima and the maximum of the mimima must both be x_m. Since x_m is the center point of the window, if there is at least one q + 1 subsequence with x_m as its minimum, by Observation 2 the maximum of the minima will be x_m. All subsequences in the window except the rightmost one include $x_{m-1} < x_m$, so the rightmost subsequence must have x_m as its minimum. Therefore, $x_j \le x_m$ for all $m < j \le m + q$.

QED

The last two lemmas needed to demonstrate the necessary conditions for root for signals also derive from Tyan's work on median filters [3]. Lemma 4 is a restatement of Tyan's Theorem 6.2(with virtually identical proof) and Lemma 5 is related to Tyan's Theorem 6.3.

Lemma 4: If there is at least one monotonic subsequence of length N + 1 in a root signal of a pseudomedian filter of length 2N + 1, then every length N + 2 subsequence of the root signal is monotonic.

Proof: Follows from Lemma 3. Proof for median filter given in Tyan [3] applies, with Lemma 3 substituted for the corresponding median filter lemma.

Lemma 5: Every root signal of a length 2N + 1 pseudomedian filter has at least one subsequence of length N + 1 that is monotonic.

Proof: Consider a root signal $\{x_k\}$ of a length 2N + 1 pseudomedian filter. Assume that this root does not have any monotonic subsequences of length N + 1. Consider a particular transition to a higher value at, say, k = 0, so $x_0 < x_1$. Then Lemma 3 requires that $x_i \le x_0$ for each i where-$N \le i < 0$. Thus $x_{-N} \le ... \le x_1 \le x_0$ is a monotonic subsequence of the length N + 1, which contradicts the assumption that the root signal $\{x_k\}$ does not have any monotonic subsequences for length N + 1. Therefore, every root signal has at least one subsequence of length N + 1 that is monotonic.

QED

The five lemmas proved above combine to show a necessary and suffincient condition for root signals (finite/extended or infinite) for a length 2N + 1 pseudomedian filter.

Theorem 1: A necessary and sufficient condition for a signal to be invariant under pseudomedian filtering is that the signal consist only of constant neighborhoods and edges.

Proo: Any signal, and thus any potential root signal of a length 2N + 1 pseudomedian filter, can be ascribed to one of following cases:

Case I: At least one length N + 1 subsequence of the signal is monotonic; or

Case II: The signal has no monotonic subsequences of length N + 1.

By Lemma 4, all root signals belonging to Case I are everywhere monotonic of length N + 2; that is, every length N + 2 subsequence of a root signal satisfying Case I is monotonic. By Lemma 1, all such signals consist only of constant neighborhoods and edges.

By Lemma 5, there are no root signals that satisfy Case II.

All (root and non-root) signals must satisfy one of the above cases, but all root signals of the pseudomedian filter must satisfy Case I, and furthermore must only consist of constant neighborhoods and edges. Therefore, all root signals of the pseudomedain filter consist solely of constant neighborhoods and edges. By Lemma 2, all signals consisting of only constant neighborhoods and edges are root signals of the pseudomedian filter.

QED

Theorem 1 is a powerful result that holds both for finite-length signals that are extended as described earlier and for "infinite-length" signals. It is the same result as the necessary and sufficient condition for extended finite-length root signals of the median filter shown in

Gallagher and Wise [2]. This leads immediately to the results below:

Corollary 1: An extended finite-length root signal of a median filter of length 2N+1 is a root signal of a pseudomedian filter of length 2N+1.

Corollary 2: A root signal of a pseudomedian filter of length 2N+1 is a signal of a median filter of length 2N+1.

Recall that an extended finite-length signal has N constant points equal to the first value in the signal appended to the beginning of the signal, and N constant points equal to the last value appended to the end. The root signal set of the pseudomedian filter is thus identical to that of the median filter for finite-length signals. However, if the set of roots under consideration is extended to include signals that extend to infinity in either direction, another type of root signal of the median filter emerges. These "Type II" or "fast fluctuating" roots are necessarily bivalued; that is, they take on only two values, and do not contain any constant neighborhoods [3]. There is also a related set of signals that may be termed "oscillatory" roots of the median filter; these signals are not invariant to each pass of a median filter, but the original signal will recur after two or more passes of the filter. The pseudomedian filter does not have any fast-fluctuating or oscillatory roots.

In practice, all signals are finite in length, but Type II and oscillatory median filter roots are still important. Even a small section of such a root within a finite signal creates unattenuated oscillations in the output of the median filter; only at the endpoints of the section does the filter cause any change. The pseudomedian filter, however, usually reduces such bi-valued fast-fluctuating sections to a constant DC level in one pass by averaging the two values in the section throughout.

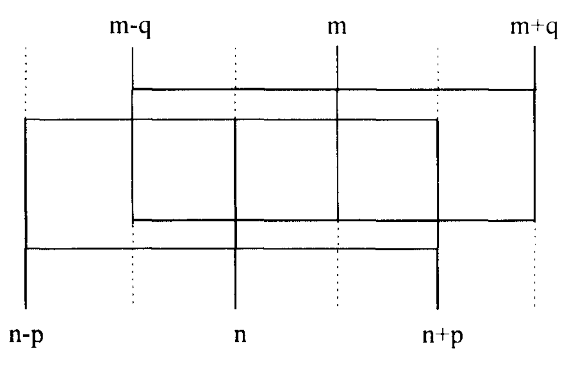

Fig 1. Diagram of signal segment considered for Lemma 3.

Ⅳ. Convergence to Root Signals

Another topic of interset in root signal analysis is convergence of a repeatedly filtered signal to a root signal. For the median filter, Gallagher and Wise [2] have shown that a signal of length L will become a root signal after at most(L-2)/2 successive passes of a median filter. No such upper limit on the number of passes required to yield a root signal can be derived for the pseudomedian filter. Indeed, some signals (for example, any impulse) will never in theory the reduced to a root signal by successive passes of a pseudomedian filter. However, each successive pass results in a signal that is "closer" to a root signal than the previous pass, and as the number of passes becomes very large, the resulting signal approaches a root signal. These observations indicate that respeated pseudomedain filtering will result in either:

i. a root signal after a finite number of passes; or

ii. a sequence of signals that converges to a root signal.

Note that the resulting root signal in either case often is not identical to the corresponding median filter root signal.

Ⅴ. Conclusion

The root signal analysis in this paper demonstrates some provocative similarities and differences between the pseudomedian and median filters. Although the root signal sets of the two filters are identical when restricted to finite-length signals, the root signals to which the filters converge for a given input signal are not the same. The properties of the filters revealed in this paper are theoretically interesting, but more important, are practically useful in predicting and understanding the effects of the filters on signals and images.

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