Free Vibrations and Buckling Loads of Stepped Columns

Step 기둥의 자유진동 및 좌굴하중

Lee, Byoung Koo* · Oh, Sang Jin** · Mo, Jeong Man**
이 병 구 · 오 상 진 · 모 정 만

적 요

이 논문은 step기둥의 자유진동 및 좌굴하중에 관한 연구이다. 축하중을 받는 변단면 기둥의 자유진동을 지배하는 편미분방정식을 이용하여 축하중을 받는 step기둥의 자유진동을 지배하는 상미분방정식을 유도하였다. 또한 이 자유진동을 지배하는 미분방정식을 이용하여 step기둥의 좌굴하중을 지배하는 상미분방정식을 유도하였다. 유도된 미분방정식들을 Heun방법과 Regula-Falsi방법을 이용하여 고유진동수 및 좌굴하중을 산출할 수 있는 수치해석방법을 개발하였다. 실제 수치해석 예에서는 2개의 step구간을 갖는 회전-회전, 회전-고정, 고정-고정 기둥에 대한 무차원 고유진동수와 무차원 변수들과의 관계 및 무차원 좌굴하중과 무차원 변수들과의 관계를 그림에 나타내었다.

I. Introduction

Since columns are basic structural forms in the various engineering fields, their statics and dynamics have been studied extensively. Recently, the columns with variable cross-section including the stepped columns have been used increasingly due to the economic, aesthetic and structural reasons, and etc. Thus, many researchers have been concerned with the static and dynamic behavior of the stepped columns. References[1,4-10] and their citations include the governing equations and the significant literature on this subject.

The main purpose of this paper is to investigate the free vibrations and buckling loads of stepped columns. The ordinary differential equation is derived for the free vibrations of linearly elastic stepped columns on the basis of partial differential equation of motion. The effect of axial load is included. Also, the ordinary differential equation governing the

* Professor, Wonkwang University
** Graduate Student, Wonkwang University

키워드: buckling load, free vibration, harmonic motion, Heun's method, mode shape, natural frequency, Regula-Falsi method, stepped column.

buckled shape of stepped columns is derived using the differential equation of free vibration. The governing equations are solved numerically by the Heun's method and determinant search method combined with Regula-Falsi method. The hinged-hinged, hinged-clamped and clamped-clamped end constraints with two stepped segmental sections are considered. In numerical examples, the lowest three natural frequencies and only the first buckling loads are calculated and presented as functions of non-dimensional system parameters.

II. Mathematical Model

Shown in Figure 1 depicts a stepped column with span length l which is supported by hinged or clamped ends. The column is subjected by the axial load P in which the compressive force is positive. The cross-section of column is a rectangle with constant breadth and stepped depth. As shown in this figure, the column is sectioned by several segments in which the axial length and depth of cross-section of ith segment are l_i and d_i, respectively, and k is the number of total segments. Here, two non-dimensional variables of m_i and n_i are defined as follows, respectively.

$$m_i = l_i / l \quad \cdots\cdots (1)$$

$$n_i = d_i / d_1 \quad \cdots\cdots (2)$$

where m_i is the segmental length parameter and n_i is the section ratio. It is noted that

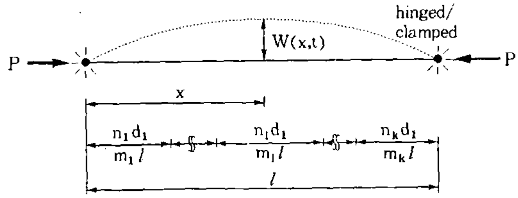

Fig. 1. Variables of stepped column with axial load and typical mode shape

n_1 is one in equation (2) because of $n_1 = d_1/d_1 = 1$. And it is clear that the ith segment lies from $x = l_1 + l_2 + l_3 + \cdots + l_{i-1}$ to $x = l_1 + l_2 + l_3 + \cdots + l_i$ in which x is the axial co-ordinate.

Figure 1 shows also the typical mode shape, dotted line, of column. At co-ordinate x and time t, the dynamic displacement is depicted as W(x, t). The column is assumed to be in harmonic motion, or the dynamic displacement W(x, t) is proportional to $\sin(\omega t)$ and w(x), where ω is the natural frequency, and w(x) is the amplitude which is function of x only. The harmonic motion is then

$$W(x, t) = w(x)\sin(\omega t) \quad \cdots\cdots (3)$$

The partial differential equation governing free vibration of tapered beam/column element with axial load P, given in reference[3)], is

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 W(x, t)}{\partial x^2}\right] + \rho A\frac{\partial^2 W(x, t)}{\partial t^2} + P\frac{\partial^2 W(x, t)}{\partial x^2} \quad \cdots\cdots (4)$$

where E is Young's modulus, ρ is mass density, A is cross-sectional area and I is area moment of inertia of cross-section. Since A and I are functions of x only, the equation (4) is developed as follows.

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+2E\frac{dI}{dx}\frac{\partial^3 W(x,t)}{\partial x^3}+\left(E\frac{d^2I}{dx^2}+P\right)$$

$$\frac{\partial^2 W(x,t)}{\partial x^2}+\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0 \quad \cdots\cdots\cdots\cdots (5)$$

Substituting each of $\partial^4W/\partial x^4$, $\partial^3W/\partial x^3$, $\partial^2W/\partial x^2$ and $\partial^2W/\partial t^2$ obtained from equation (3) into equation (5) gives the ordinary differential equation governing free vibration of tapered column. The result is

$$EI\frac{d^4 w(x)}{dx^4}+2E\frac{dI}{dx}\frac{d^3 w(x)}{dx^3}+\left(E\frac{d^2I}{dx^2}+P\right)$$

$$\frac{d^2 w(x)}{dx^2}-\rho A\omega^2 w(x)=0 \quad \cdots\cdots\cdots\cdots\cdots (6)$$

The respective A and I are A_i and I_i in the ith segment and dI/dx and d^2I/dx^2 are zero because I is not varied in the ith segment. In order to apply the equation (6) for tapered column to stepped one, this equation is re-written as follows.

$$EI_i\frac{d^4 w(x)}{dx^4}+P\frac{d^2 w(x)}{dx^2}-\rho A_i\omega^2 w(x)=0$$

for ith segment ······················· (7)

At hinged and clamped ends, the boundary conditions are, respectively,

$w(x)=0$ at hinged end ($x=0$ or l) ··· (8.1)

$\frac{d^2w(x)}{dx^2}=0$ at hinged end ($x=0$ or l) ··· (8.2)

$w(x)=0$ at clamped end ($x=0$ or l) ······ (9.1)

$\frac{dw(x)}{dx}=0$ at clamped end ($x=0$ or l) ··· (9.2)

where equations (8.2) and (9.2) assure that the bending moment at hinged end and rotation of cross-section at clamped end are zero, respectively.

To facilitate the numerical studies, the following non-dimensional system variables are introduced :

$$\xi=x/l \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (10)$$

$$\eta=w(x)/l \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (11)$$

$$p=P\,l^2/\pi^2 EI_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots (12)$$

$$c_j=\omega_j l^2\sqrt{\rho A_1/EI_1},\ j=1,2,3,\cdots \quad \cdots\cdots (13)$$

in which x and w(x) are normalized by span length l as ξ and η, p is load parameter, and c_j is frequency parameter where j is the mode number.

Dividing both sides of equation (7) by EI_i of the flexural rigidity of ith segment gives the following equation.

$$\frac{d^4 w(x)}{dx^4}+\frac{P}{EI_1\alpha_i}\frac{d^2 w(x)}{dx^2}-\frac{\rho A_1\beta_i}{EI_1\alpha_i}\omega^2 w(x)=0$$

for ith segment ······················· (14)

where,

$$\alpha_i=I_i/I_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (15)$$

$$\beta_i=A_i/A_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (16)$$

Since the cross-section is a rectangular shape whose breadth is constant, both values of α_i and β_i can now be written by the term of n_i defined in equation (2) as follows.

$$\alpha_i = n_i^3 \quad \cdots\cdots (17)$$

$$\beta_i = n_i \quad \cdots\cdots (18)$$

Substituting equations (17) and (18) into equation (14) and using equations (10)-(13) give the non-dimensional ordinary differential equation which governs the free vibration of stepped column element with axial load. The result is

$$\frac{d^4\eta}{d\xi^4} = -\frac{\pi^2 p}{n_i^3}\frac{d^2\eta}{d\xi^2} + \frac{c_j^2}{n_i^2}\eta \quad \text{for ith segment} \quad \cdots\cdots (19)$$

Also, the non-dimensional boundary conditions of equations (8.1)-(9.2) are obtained by equations (10) and (11). The results are

$$\eta = 0 \text{ at hinged end}(\xi = 0 \text{ or } 1) \quad \cdots\cdots (20.1)$$

$$\frac{d^2\eta}{d\xi^2} = 0 \text{ at hinged end}(\xi = 0 \text{ or } 1) \quad \cdots (20.2)$$

$$\eta = 0 \text{ at clamped end}(\xi = 0 \text{ or } 1) \quad \cdots (21.1)$$

$$\frac{d\eta}{d\xi} = 0 \text{ at clamped end}(\xi = 0 \text{ or } 1) \cdots (21.2)$$

When a compressive load P coincides with the first buckling load B, the respective frequency ω_j of $j=1$ becomes zero. Thus, substituting $c_j=0$ and $p=b$ into equation (19) gives the differential equation (22) which governs the buckled mode shape of stepped column.

$$\frac{d^4\eta}{d\xi^4} = -\frac{\pi^2 b}{n_i^3}\frac{d^2\eta}{d\xi^2} \quad \text{for ith segment} \quad \cdots (22)$$

in which b is the first buckling load parameter defined as

$$b = B\, l^2/\pi^2 EI_1 \quad \cdots\cdots (23)$$

III. Numerical Methods

Based on the above analysis, two general FORTRAN 77 computer programs were written to calculate the frequency parameter c_j and its corresponding mode shape $\eta = \eta_j(\xi)$, and the first buckling load parameter b, respectively. The Heun's method was used to integrate the differential equations and then, the determinant search method combined with Regula-Falsi method was used to determine both eigenvalues of c_j and b. For the sake of completeness, the numerical methods are summarized as follows. First is the free vibration problem.

(1) Specify the column geometry(end constraint, m_i and n_i for $i=1, 2, 3, \cdots, k$, and p) and set of two homogeneous boundary conditions accordance with the end constraint, which are either equations (20.1) and (20.2) or (21.1) and (21.2).

(2) Consider fourth order system, equation (19), as two initial value problem whose initial values are the two homogeneous boundary conditions $\xi=0$, as chosen in step (1). Then, assume a trial frequency parameter c_j in which first trial value is zero.

(3) Using Heun's method[2)], integrate equation (19) from $\xi=0$ to 1. Perform two

separate integrations, one for each of the two boundary conditions.

(4) From the Heun's solution, evaluate at $\xi=1$ the determinant D of coefficient matrix for the boundary conditions of either equations (20.1) and (20.2) or (21.1) and (21.2). If $D=0$, then the trial value of c_j is an eigenvalue. If not, then increment c_j and repeat the above calculations.

(5) In each iteration, note the sign of D. If D changes sign between two consecutive trials, then the eigenvalue lies between these last trial values of c_j.

(6) Use the Regula-Falsi method[2] to compute the advanced trial c_j based on its two previous values.

(7) Terminate the calculations and print the value of c_j and the corresponding mode shape η_i when the convergency criteria are met.

Figure 2 shows the flow diagram for above algorithm. The second is the buckling load

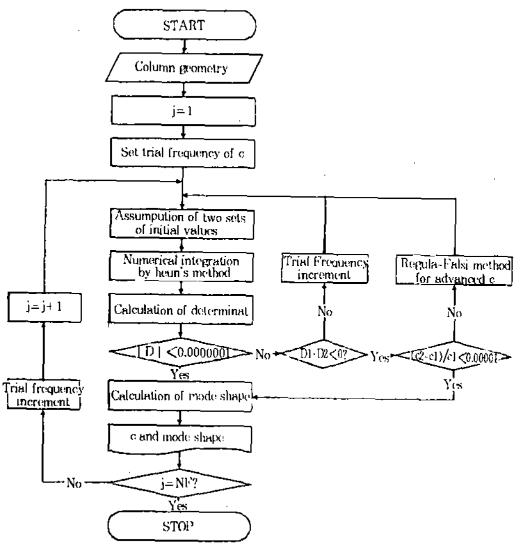

Fig. 2. Flow diagram

problem. Same procedure mentioned above is used for a given column geometry(end constraint, n_i and m_i for $i=1, 2, 3, \cdots, k$). And it is clear that the eigenvalue in equation (22) is the buckling load parameter b.

In numerical examples, the hinged-hinged, hinged-clamped and clamped-clamped end constraints with two stepped segments are considered, and the lowest three frequency parameter c_j ($j=1, 2, 3$) and only the first buckling load parameter b are calculated. All calculations were carried on a notebook computer with graphics support.

Ⅳ. Numerical Examples and Discussion

The first series of numerical studies are shown in Table-1. These studies serves to validate the analysis presented herein. Table-1 shows that the numerical results of this study quite agree with the reference values. Figure 3 shows the c_j versus p curves for $k=2$, $n_2=0.8$ and $m_1=m_2=0.5$. It is shown in this figure that the frequency parameters $c_j(j=1, 2, 3)$ decrease as the load parameter p is increased, other parameters remaining constant. This holds true for the hinged-hinged, hinged-clamped and clamped-clamped end constraints. Further, it is observed for these three columns that the p values marked by □ on the horizontal axis are the buckling load parameter b for given geometries of columns. Thus, the values of c_2 and c_3 after these values of b are meaningless since the columns already have been buckled at the p values of b, and not shown in this figure. Also, it is seen that as the end

Table-1. Comparison of results between references and this study

• free vibration problem

geometry of column	end constraints*	j	frequency parameter, c_j SAP90	this study
$k=2$ $n_1=0.7$ $m_1=m_2=0.5$ $p=0.$	h-h	1	8.17	8.02
		2	33.89	33.42
		3	74.93	72.90
	h-c	1	13.67	13.27
		2	42.01	41.44
		3	87.95	86.57
	c-c	1	18.62	18.16
		2	52.92	52.13
		3	101.23	99.35

• buckling load problem

geometry of column	end constraints*	buckling load parameter, b Ref.[11]	this study
uniform	h-h	1.00	1.00
	h-c	2.05	2.05
	c-c	4.00	4.00

*h : hinged c : clamped

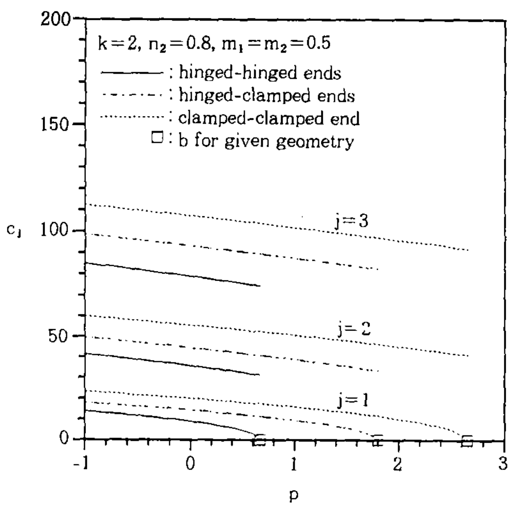

Fig. 3. c_j versus p curves for $k=2$, $n_2=0.8$ and $m_1=m_2=0.5$

constraint increases on all three column geometries, from hinged-hinged to hinged-clamped to clamped-clamped, each value of c_j

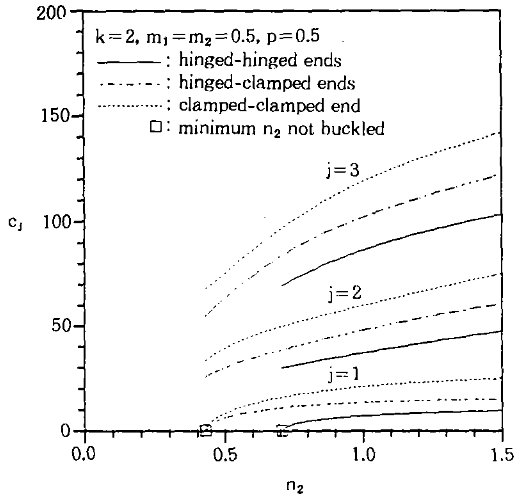

Fig. 4. c_j versus n_2 curves for $k=2$, $m_1=m_2=0.5$ and $p=0.5$

increases.

It is shown in Figure 4, for which $k=2$, $m_1=m_2=0.5$ and $p=0.5$, that the frequency parameeters($j=1, 2, 3$) increase as the section ratio n_2 is increased. In this figure, the □ marks on the horizontal axis are the minimum n_2 values not buckled for given geometries of columns. It is noted that the minimum values of n_2 for hinged-clamped and clamped-clamped end constraints are nearly equal to each other. Also, the c_2 and c_3 values before the minimum n_2 values are not shown since the columns are buckled in these range of n_2 values.

Figure 5 shows the c_j versus m_1 curves for $k=2$, $n_2=0.7$ and $p=0$, in which the frequency parameters $c_j(j=1, 2, 3)$ increase as the segmental length parameter m_1 is increased. Since the columns become uniform columns when the m_1 reaches at 1.0, the c_j values marked by □ are those of uniform columns which are validated by comparisons witn the references values[3)].

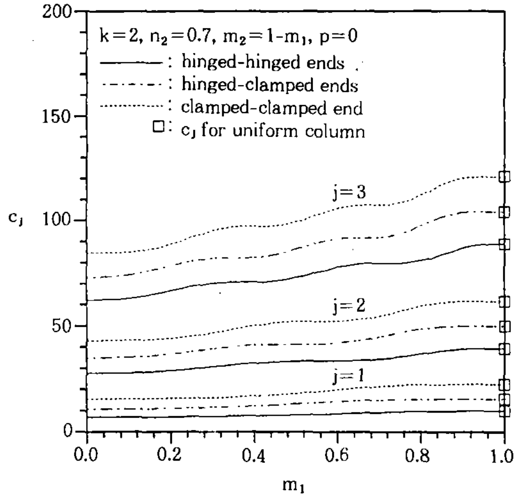

Fig. 5. c_j versus m_1 curves for $k=2$, $n_2=0.7$ and $p=0$

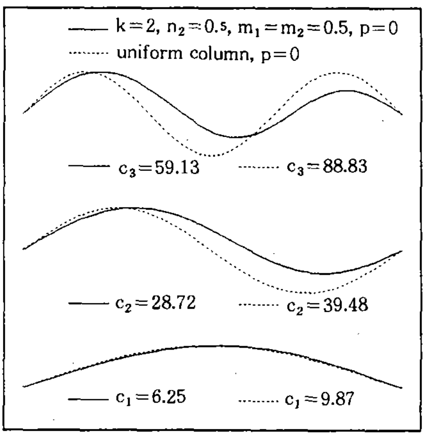

Fig. 6. Typical mode shapes of stepped and uniform columns

For comparison purpose, the typical mode shapes of stepped column for $k=2$, $n_2=0.5$, $m_1=m_2=0.5$ and $p=0$, and uniform columns for $p=0$ are shown in Figure 6. It is shown that there exists big discrepancies from the two mode shapes in the aspects of both the amplitudes and nodal points.

Figure 7 shows the b versus n_2 curves for $k=2$ and $m_1=m_2=0.5$. The buckling load

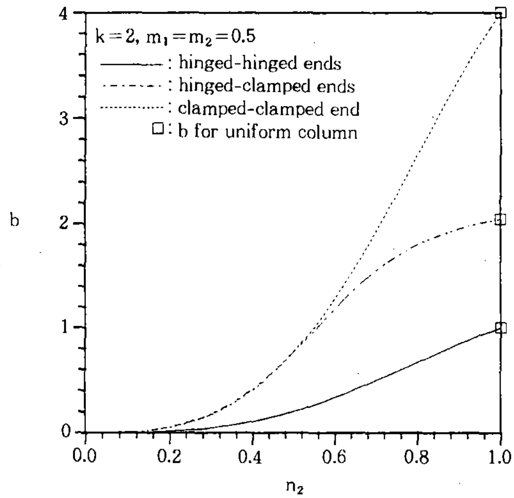

Fig. 7. b versus n_2 curves for $k=2$ and $m_1=m_2=0.5$

parameters b increase as the section ratio n_2 is increases, other parameters remaining constant. It is noted that the increasing rate for the clamped-clamped end constraint after approximately $n_2=0.5$ is very steep comparing with the other two curves. And the buckling load parameters b of uniform columns marked by □ are checked by the reference values[11]. And it is seen that as the end constraint increases on all three column geometries, from hinged-hinged to hinged-clamped to clamped-clamped, each value of b increases.

It is shown in Figure 8 for $k=2$ and $n_2=0.7$ that the buckling loads b increase as the segmental length parameter m_1 is increased, other parameters remaining constant. The b values of □ marks are the b values of uniform columns which are also validated by the reference values[11]. In this figure, it is seen that the increasing rate of b versus m_1 curve is especially very high at the range approximately from $m_1=0.5$ to 0.8.

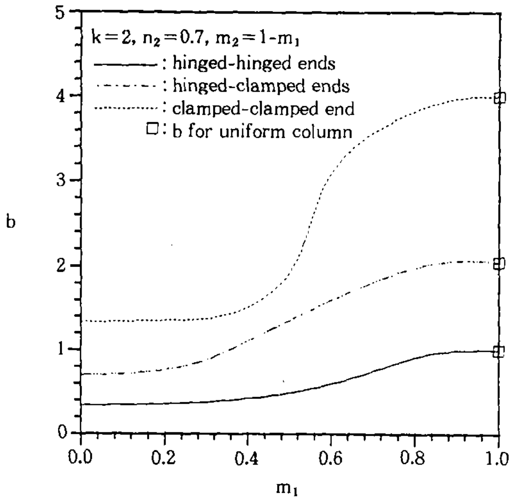

Fig. 8. b versus m_1 curves for k=2 and n_2=0.7

V. Conclusions

The numerical methods developed herein for computing frequencies and buckling loads of the stepped column were found to be especially robust and reliable over a wide and practical range of system parameters. As the results of theoretical analysis and numerical examples for the columns with two stepped segments, the conclusions are drawn as follows.

1. The non-dimensional differential equation which governs the free vibrations of stepped column with axial load is derived as the equation (19).

2. The non-dimensional differential equation governing the buckled shape of stepped column is also derived as the equation (22).

3. The frequency parameter decreases as the load parameter is increased.

4. The frequency parameter increases as the section ratio is increased.

5. The buckling load parameter increases as the section ratio is increased.

6. Both values of frequency parameter and buckling load parameter increase as the end constraint increases from hinged-hinged to hinged-clamped to clamped-clamped end constraints.

References

1. Balasubramanian, T. B. and G. Subramanian, 1985, "On the Performance of a Four-Degree of Freedom per Node Element for Stepped Beam Analysis and Higher Frequency Estimation", Journal of Sound and Vibration, Vol. 99, pp. 563-567.
2. Carnahan, H, H. A. Luther and J. O. Wilkes, 1969, Applied Numerical Methods, John Wiley and Sons.
3. Clough, R. W. and J. Penzien, 1993, Dynamics of Structures, Second Edition, McGraw-Hill.
4. Jang, S. K. and C. W. Bert, 1989, "Free Vibration of Stepped Beams : Exact and Numerical Solutions", Journal of Sound and Vibration, Vol. 130, pp. 342-346.
5. Klein, L., 1974, "Transverse Vibration of Non-Uniform Beams", Journal of Sound and Vibration, Vol. 137, pp. 491-505.
6. Lee, B. K., S. J., Oh and J. M. Mo, 1995, "Approximate Methods for Natural Frequencies and Buckling Loads of Beam-Columns with Intermediate Multiple Elastic Springs", Proceedings of the 6th International Conference of Computing in Civil and Building Engineering, Berlin, Germany, Vol. 2, pp. 1155-1160.
7. Levison, M., 1976, "Vibration of Stepped

Strings and Beams", Journal of Sound and Vibration, Vol. 49, pp. 287-291.

8. Rossi, R.E., P. A. A. Laura and R. H. Gutierrez, 1990, "A Note on Transverse Vibration of a Timoshenko Beam of Non-Uniform Thickness Clamped at One End and carrying a Concentrated Mass at the Other", Journal of Sound and Vibration, Vol. 143, pp. 491-502.
9. Ryu, B. J., S. H. Song, G. S. Lee and T. S. Myung, 1995, "Analytical and Experimental Analysis on the Transverse Vibration of a Stepped Timoshenko Beam", Proceedings of the International Conference on Structural Dynamics, Vibration, Noise and Control, Hong Kong, Vol. 1, pp. 173-177.
10. Tabel, N. J. and E. W. Suppiger, 1961, "Vibration of Stepped Beams", Journal of Aerospace Science, Vol. 128, pp. 295-298.
11. Timoshenko, S. P. and J. M. Gere, 1961, Theory of Elastic Stability, McGraw-Hill.

(접수일자 : 1996년 7월 9일)