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An Interactive Weight Vector Space Reduction Procedure for Bicriterion Linear Programming

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ABSTRACT

This paper develops a simple interactive procedure which can be efficiently used to solve a bicriteria linear programming problem. The procedure exploits the relatively simple structure of the bicriterion linear programming problem. Its application to a transportation problem is also presented. The results demonstrate that the method developed in this paper could be easily applicable to any bicriteria linear program in general.

1. Introduction

There exists a wide variety of decision-making situations in which trading off one objective against another is involved, such as time versus cost. Bicriterion mathematical programming has been successfully applied in such situations. Examples of such works include nutrition planning [2] and portfolio analysis [8]. In this article an interactive weight vector space reduction procedure is given for bicriterion linear programming problem (BLP) given by

Minimize $\{c^1x, c^2x\}$ subject to $x \in X$,

where $X=\{x \in R^n | Ax=b, x \geq 0\}$ is a nonempty compact set, A is $m \times n$ matrix and $b \in R^m$.

Since the two objectives are generally conflicting, it is very rare that the two objectives are simultaneously minimized. Therefore, a compromise solution must be obtained. Usually the most preferred compromise solution in the multiple criteria decision making problem is required to be an efficient (nondominated, Pareto-optimal) solution.

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Definition 1

A point $x^o \in X$ is an efficient solution of problem BLP if and only if there exists no $x \in X$ such that $c^k x \leq c^k x^o$ for all $k=1,2$ and $c^k x \neq c^k x^0$ for some $k = 1,2$.

Let $Z \in R^2$ be the set of all feasible criterion vectors where Z is the set of images of all $x \in X$.

Definition 2

A point $z^0 \in Z$ is called a nondominated solution of problem BLP if and only if there exists no $z \in Z$ such that $z \leq z^0$ and $z \neq z^0$.

Notice that bicriterion linear programming problem is a special case of the multiple objective linear programming problem.

There exist various methods for identifying all of the efficient solutions for a multiple objective linear programming problem (see [5],[6],[9] and references therein). Unfortunately, for large-scale problems, generating all possible efficient solutions is computationally prohibitive. These methods may also arise the problem that the decision maker (DM) should choose the most preferred one among overwhelmingly large number of candidate solutions. Interactive procedures try to overcome those problems. In this approach, the DM dynamically interacts with a computerized algorithm, and thereby explores the feasible decisions until he finds one that he prefers the most. Through practical experience, researchers and decision makers have learned that the interactive approaches do not have many of serious flaws which noninteractive approaches possess. It also has certain advantages that the noninteractive approches do not have. Chief among these is that the DM is actively and dynamically involved in the decision making process. He can thereby learn about his preferences and come to a decision in which he can have confidence. Furthermore, interactive procedures can be flexibly designed to yield a variety of attractive characteristics.

In the literature, there has been extensive research on interactive multiple objective linear programming in the last two decades (see [4],[7], and references therein). Obviously, the bicriterion linear programming problem can be solved using any of those interactive methods; but they do not exploit the relatively simple structure of bicriterion linear programming problems. A few algorithms of non-interactive approach have been developed by using the relatively simple structure of bicriterion linear problems. For example, Benson [1] suggested the method for identifying all of the efficient solutions [1] and Benson and Lee [3] suggested the method for solving the problem of optimizing over the efficient set of BLP problem by using the fact that the dimension of the outcome set of the bicriteria problem is at most two. However, for specialized interactive algorithms, there are still needs that exploit the relatively simple structure of bicriterion linear programming problems.

In this paper we present an interactive weight vector space reduction procedure which exploits the relatively simple structure of the bicriterion linear programming problem. The procedure has been designed to require a minimum amount of input from the DM and to simplify the decision making process by utilizing the relatively simple structure of the bicriterion linear program.

The plan of this article is as follows. In Section 2 we present the necessary theoretical prerequisites for developing our methodology for solving the bicriterion linear programming problem. In Section 3 the methodology is presented. In Section 4 an application to the bicriterion transportation problem is presented. Concluding remarks are given in Section 5.

2. Theoretical Background

The procedure presented in this article is a weight vector space reduction method for solving the bicriterion linear programming problem. The method operates by iteratively asking the answers of the DM regarding the desirability of an increase in a certain objective function at the expense of a decrease in another objective function. From these responses, portions of weight vector space are eliminated from further consideration.

Let λ_i (i=1,2) be the positive scalar weight associated with ith objective. Without loss of generality, we will assume that each weighting vector $\lambda \in R^2$ is normalized so that its elements sum to one (in accordance with the L_1-norm). Let $\wedge$ denote the set of all such weighting vectors where $\wedge = \{ (\lambda^1, \lambda^2) \mid \lambda^1, \lambda^2 > 0, \lambda^1+\lambda^1=1 \}$.

With C the 2×n criterion matrix whose rows are the c^i (i=1,2), the weighted-sums linear programming is written min $\{\lambda^T Cx \mid x \in X\}$.

Based on the following result [7], it is widely known that the weighted-sums linear programming only produces an efficient solution of X. Let X_E denote the set of all efficient solutions of problem BLP.

Theorem 1

Let $x^0 \in X$. Then $x^0 \in X_E$ if and only if x^0 is an optimal solution for the weighted-sums LP min $\{\lambda^T Cx \mid x \in X\}$ for some $\lambda \in \wedge$.

By using weighted-sums method, we are able to convert a bicriterion linear programming into a single criterion linear programming that can be easily solved using standard available LP software. To make the method work, all that is needed is a "good" weighting vector in terms of the DM's utility. Our goal is to find such a "good" weighting vector. To do this, the weight vector space reduction approach is used in the procedure. The following result, which utilizes the relatively simple structure of the bicriterion linear program, will be used in reducing the weight vector space depending upon the responses of the DM.

Theorem 2

Let $0 < \lambda_1^1 < \lambda_1^2 < 1$. With the weight vector $\lambda^1=(\lambda_1^1, 1-\lambda_1^1)$ and $\lambda^2=(\lambda_1^2, 1-\lambda_1^2)$, individually solve the weighted-sums LP: min $\{ \lambda^T Cx \mid x \in X\}$. Let x^{1*} and x^{2*} be the optimal solution for the weighted-sums LP with λ^1 and λ^2, respectively. And let x^{1*} is not an optimal solution of the weighted-sums LP with λ^2, or x^{2*} is not an optimal solution of the weighted-sums LP with λ^1. Then $c^1x^{2*} < c^1x^{1*}$ and $c^2x^{2*} > c^2x^{1*}$.

Proof. From given assumption to x^{1*} and x^{2*}, we know the following two facts:

(1) $\lambda_1^1c^1x^{1*}+(1-\lambda_1^1)c^2x^{1*} < \lambda_1^1c^1x^{2*}+(1-\lambda_1^1)c^2x^{2*}$

(2) $\lambda_1^2c^1x^{2*}+(1-\lambda_1^2)c^2x^{2*} < \lambda_1^2c^1x^{1*}+(1-\lambda_1^2)c^2x^{1*}$.

From (1) and (2), $(\lambda_1^1-\lambda_1^2)(c^1x^{1*}-c^2x^{1*}) < (\lambda_1^1-\lambda_1^2)(c^2x^{2*}-c^2x^{2*})$

Thus, we know that $(\lambda_1^1-\lambda_1^2)\{(c^1x^{1*}-c^1x^{2*})+(c^2x^{2*}-c^2x^{1*})\} < 0$.

Now, suppose, to the contrary, $c^1x^{2*} \geq c^1x^{1*}$. We know that x^{1*} and x^{2*} are efficient solutions of X from theorem 1. From the definition of efficiency, $c^2x^{2*} \leq c^2x^{1*}$.

This says that $(\lambda_1^1-\lambda_1^2)\{(c^1x^{1*}-c^1x^{2*})+(c^2x^{2*}-c^2x^{1*})\} \geq 0$. It is contradiction. Thus, $c^1x^{2*} < c^1x^{1*}$. From the definition of efficiency, $c^2x^{2*} > c^2x^{1*}$.

Theorem 2 is necessary for the validation of the Step 3 in the method presented in next section.

3. The Method

The method operates by iteratively asking the answers of the DM regarding the desirability of an increase in a certain objective function at the expense of a decrease in another objective function. From these responses, portions of weight vector space $\Lambda=\{(\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 > 0, \lambda_1+\lambda_2=1 \}$ are eliminated from further consideration. The purpose of this procedure is to locate the efficient extreme point of greatest utility in a fixed number of iterations. By denoting l_i and u_i as the minimum and maximum value of λ_i $(i=1,2)$, respectively, the steps of the method are given as follows.

Initialization

(a) Rescale (normalize) the objective functions.

(b) Let the DM specify the objective function which is to be given the priority in reducing the range of the associated weight in a controlled fashion. Without loss of generality, assume that this function is z^1.

(c) Let the DM specify the final iteration interval $[l_1, u_1]$ width w.

(d) Let $[l_1^{(0)}, u_1^{(0)}]=[0, 1]$.

(e) By individually optimizing each objective function, we obtain two efficient points, x^i, i=1,2, to the original problem, and its associated images, two nondominated criterion vectors z^i, i=1,2. Have the DM review these two nondominated criterion vectors. If the DM wishes to stop with one of them, the procedure terminates. Otherwise, let the DM know the ideal criterion vector $z^* = (z_1^1, z_2^2)$.

(f) Let k=1.

At iteration k ($k \geq 1$),

Step 1. Let $\lambda_1^{(k)} = (l_1^{(k-1)}+u_1^{(k-1)})/2$ and $\lambda_2^{(k)} = (1-\lambda_1^{(k)})$.

Step 2. With $\lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)})$, solve the weighted-sums LP: min $\{ \lambda^{(k)T}Cx \mid x \in X\}$.
Let $x^{(k)}$ be the optimal solution and $z^{(k)}$ be its associated image.

Step 3. Present the DM with the nondominated criterion vector $z^{(k)}$. If the DM states that

(1) he is satisfied with all components of $z^{(k)}$ by comparing with z^*, then stop with $(z^{(k)}, x^{(k)})$ as the final solution.

(2) he wants to improve $z_1^{(k)}$, then set $l_1^{(k)} = (l_1^{(k-1)}+u_1^{(k-1)})/2$ and $u_1^{(k)} = u_1^{(k-1)}$.

(3) he is willing to sacrifice $z_1^{(k)}$ to improve $z_2^{(k)}$, then set
$l_1^{(k)} = l_1^{(k-1)}$ and $u_1^{(k)} = (l_1^{(k-1)}+u_1^{(k-1)})/2$.

Step 4. If $(u_1^{(k)}-l_1^{(k)}) \leq w$, then stop with $(z^{(k)}, x^{(k)})$ as the final solution. Otherwise, go to Step 1 with k=k+1.

Because there may be magnitudes of difference among the criterion values generated by the different objectives, the objectives are rescaled (normalized). Several ways are available for rescaling the objective functions [7].

The method concentrates on the interval of the associated weight with one of the objective functions, say z_1. The length of this interval is given by (u_1-l_1). At each iteration the length of this interval, u_1-l_1, is reduced according to the bisection search method. In addition, u_2-l_2, the length of the interval of the associated weight with z_2, is also reduced. The procedure terminates when $(u_1-l_1) \leq w$. The value of w is prespecified by the DM at the initialization of the method. With the specification of the final iteration interval width w, it is guaranteed that the procedure terminates in a finite number of iterations. In fact the theoretical convergence of the procedure may not be a major concern in many practical applications of interactive methodology since the DM may be satisfied with a "good enough" or "satisfactory" solution.

In initialization step (e), if the possibility exists that some objectives have a alternative optima, a way to assure that only efficient solutions are generated is to lexicographically minimize the objective [7]. Since the ideal criterion vector z^* is a good reference point for assessing the quality of a candidate criterion vector, the information of the ideal criterion vector z^* would help the DM in responding the required tradeoff question at Step 3.

At the beginning of each iteration k the interesting interval of $(u_1\text{-}l_1)$ is given by $(u_1^{(k-1)}\text{-}l_1^{(k-1)})$. In Step 1, a bisecting search method is employed in search for the improved weight vector in terms of the DM's utility.

In Step 2, we know from theorem 1 that the generated solution is guaranteed to be an efficient extreme solution.

In Step 3, if the DM wants to make tradeoff, portions of weight vector space $\Lambda=\{(\lambda_1, \lambda_2)\,|\,\lambda_1, \lambda_2 > 0, \lambda_1+\lambda_2=1\}$ are eliminated depending on the DM's responses from further consideration. Depending on the DM's responses, the method reduces $(u_1\text{-}l_1)$ by eliminating a subset of weight vector space Λ. This step can be validated by theorem 2. For example, if the DM wants to improve $z_1^{(k)}$, then the minimum value of $\lambda_1^{(k)}$, $l_1^{(k)}$, is determined by the average of $l_1^{(k-1)}$and $u_1^{(k-1)}$. This indicates that the subinterval of Λ, $\{(\lambda_1, \lambda_2)\,|\,l_1^{(k-1)} \leq \lambda^1 < l_1^{(k)}, 1\text{-}l_1^{(k)} < \lambda_2 \leq 1\text{-}l_1^{(k-1)}\}$, is eliminated from further consideration. And if he is willing to sacrifice $z_1^{(k)}$ to improve $z_2^{(k)}$, then the maximum value of $\lambda_1^{(k)}$, $u_1^{(k)}$, is determined by the average of $l_1^{(k-1)}$ and $u_1^{(k-1)}$. This indicates that the subinterval of Λ, $\{(\lambda_1, \lambda_2)\,|\,u_1^{(k)} < \lambda_1 \leq u_1^{(k-1)}, 1\text{-}u_1^{(k-1)} \leq \lambda_2 < 1\text{-}u_1^{(k)}\}$, is eliminated from further consideration.

In Step 4, the method checks if the width of the interval $[l_1, u_1]$ has been reduced to a prespecified level. The process continues, if the DM does not find a satisfactory solution, until Λ has been reduced to a small enough region for a final solution to be identified. Since the interval of $(u_1^{(k-1)}\text{-}l_1^{(k-1)})$ is reduced by half of that in each kth iteration, the maximum number of iteration k can not exceed beyond certain level $(k < (-\ln w\ /\ \ln 2)\ )$ and this ensures the finiteness.

Because of the way weight vector space is contracted, the method has an error-correcting capability. That is, the method can adjust to changes in the DM's aspirations during the course of the solution procedure as long as they are not severe.

4. An Application to the Bicriterion Transportation Problem

The classical single objective transportation problem refers to a special class of linear programming problems in which the constraints exhibit a particular mathematical structure. As multiplicity of criteria is relevant in a variety of linear programming problems, two objectives are relevant in certain transportation problems. For example, the two objectives may be minimization of the total cost and minimization of time of transportation. In some situations, the second objective could represent unfulfilled demand, underused capacity, reliability of delivery, safety of delivery, product deterioration or many others. If the DM would like to minimize these two objectives simultaneously, a point will likely be reached where a further reduction of the value in a certain objective function may only be obtained at the expense of increasing the value in another objective function. Thus, in general the objectives will also be conflicting.

The bicriterion transportation problem may be stated mathematically as:

$$\text{Minimize} \quad z_1=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^1 x_{ij}$$

$$z_2=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^2 x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij}=a_i, \quad i=1,2,\ldots m.$$

$$\sum_{i=1}^{m} x_{ij}=b_j, \quad j=1,2,\ldots n.$$

$$x_{ij}\geq 0 \text{ for all } (i,j)$$

where c_{ij}^1 = cost of transporting a unit from source i to destination j,

c_{ij}^2 = time of transporting a unit from source i to destination j,

a_i = availability at i,

b_j = requirement at j,

x_{ij} = amount transported from i to j.

To illustrate the procedure presented in the previous section, consider the bicriteria linear transportation problem which has the following characteristics.

Supplies: a_1 = 10, a_2 = 16, a_3 = 18,

Demands: b_1 = 16, b_2 = 8, b_3 = 12, b_4 = 8.

$$c^1 = \begin{matrix} 2 & 1 & 3 & 4 \\ 6 & 8 & 1 & 4 \\ 4 & 6 & 8 & 9 \end{matrix} \qquad c^2 = \begin{matrix} 4 & 3 & 6 & 8 \\ 2 & 5 & 10 & 2 \\ 6 & 1 & 4 & 3 \end{matrix}$$

In order to simulate the actions of a decision maker, a simple utility function of the form $U(z_1, z_2) = -0.1z_1z_2 - 10z_1$ is assumed.

Initialization

(a) Assume the given objective functions are already rescaled.

(b) Let z_1 be the objective function which is to be given the priority in reducing the range of the associated weight.

(c) Assume the DM specify $w = 0.05$.

(d) Let $[l_1^{(0)}, u_1^{(0)}]=[0,1]$.

(e) We found the following:

$x^1 = (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34})$

$= (0, 8, 0, 2, 0, 0, 12, 4, 16, 0, 0, 2)$

$z^1 = (126, 270)$,

$x^2 = (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34})$

$= (8, 2, 0, 0, 8, 0, 0, 8, 0, 6, 12, 0)$

$z^2 = (230, 124)$.

Suppose the DM is not satisfied with these solutions, continue by letting the DM know the ideal criterion value $z^* = (z_1^1, z_2^2) = (126, 124)$.

(f) Let $k=1$.

Iteration 1

Step 1. $(\lambda_1^{(1)}, \lambda_2^{(1)}) = (0.5, 0.5)$.

Step 2. $x^{(1)} = (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34})$

$= (10, 0, 0, 0, 6, 0, 2, 8, 0, 8, 10, 0)$

$z^{(1)} = (218, 136)$.

Step 3. From the DM's utility function it is reasonable to assume that the DM is not satisfied with the criterion vector $z^{(1)}$ and he wants to improve the first criterion vector components $z_1^{(1)}$. Set $l_1^{(1)} = 0.5$ and $u_1^{(1)} = 1$.

Step 4. Since $(u_1^1 - l_1^1) > w$, go to Step 1 with $k=2$.

Iteration 2

Step 1. $(\lambda_1^{(2)}, \lambda_2^{(2)}) = (0.75, 0.25)$.

Step 2. $x^{(2)} = (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34})$

$= (0, 6, 4, 0, 0, 0, 8, 8, 16, 2, 0, 0)$.

$z^{(2)} = (134, 236)$.

Step 3. The DM may be satisfied with all components of $z^{(2)}$ and the procedure terminates.

5. Concluding Remarks

Many researchers have suggested interactive procedures which can be used to simplify the decision making process for the DM. However, there is still a need for specialized interactive algorithms that exploit the relatively simple structure of bicriterion linear programming problems.

This paper has presented an interactive procedure which is a particularly straightforward and easy to use in solving any bicriterion linear programming problems. This procedure has been designed to require a minimum amount of input from the DM. Its application to the bicriterion transportation problem demonstrates that a satisfactory solution can be identified in relatively few iterations.

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