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A Geometrical Expansion Technique for Tolerance Approach to Sensitivity Analysis in Linear Programming

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Abstract

The *tolerance* approach to the sensitivity analysis in linear programming considers simultaneous and independent variations in the coefficients of the objective function or of the right-hand side terms and gives a region in which the coefficients and terms can be changed and still keeps the current optimal basis **B** for the original problem as an optimal basis for the perturbed problem. In this paper we describe a procedure that expands the region S obtained by the tolerance approach into a larger region R, so that more variations in the objective function coefficients or the right-hand side terms are permissible.

I. Introduction

Suppose that the following linear programming has been solved and an optimal basis is produced by the simplex method,

$$\begin{aligned} &\text{minimize } \sum_{j=1}^{n} c_j x_j \\ &s.t. \ \sum_{j=1}^{n} a_{ij} x_j = b_i, \ i=1,\ldots,m \\ &\qquad x_1,\ldots,x_n \geq 0. \end{aligned} \qquad (1)$$

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Suppose that one alters the coefficients **c** of the objective function or the right-hand side terms **b** in (1). A question to be raised is how much can one change them so that the current optimal basis remains optimal. Usually, the "ordinary" sensitivity analysis deals with a perturbation of one coefficient or one right-hand side term at a time [1, 2].

The tolerance approach [3, 4] considers simultaneous and independent variations in the coefficients of the objective function or in the right-hand side terms and produces a specific region S contained in a so-called *critical* region P, within which the coefficients or right-hand side terms can be changed, leaving the optimal basis intact.

The region S generated by the tolerance approach could be small if the values of the given cost coefficients or the right-hand side terms are near zero or if they are closer to the boundary of the critical region. The purpose of this paper is to describe a technique for expanding the region S obtained by the tolerance approach with additive variation into a larger region R such that $S \subseteq R \subseteq P$. If at least one expansion is possible, the expanded region allows more variations than the tolerance approach proposed by Wendell [4].

This paper is organized as follows. In section 2, we first describe an expansion technique on a set of inequalities. We then reconsider the tolerance approach in section 3. Section 4 demonstrates how a region S obtained by the tolerance approach can be expanded into a larger region R using the procedure(algorithm TAE) described in detail in section 2.3. Conclusions and topics of further study are discussed in Section 5.

Ⅱ. An Expansion Technique

Here we pose the following problem: Given a polytope P defined by the following set of inequalities,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ \vdots \qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned} \tag{2}$$

with at least one interior point y° in P, we would like to find a specific region R with $y^\circ \in R$ such that $R \subseteq P$. An approach would consist of finding an S contained in P and expanding S as much as possible; of course, y° must be in S.

2.1 Computing a Region S

By denoting the above polytope (2) as $\mathbf{Ax} \leq \mathbf{b}$ and calling this an 'x'-critical region, we want to find a vector α such that $\mathbf{A}(\mathbf{y}^\circ + \alpha) \leq \mathbf{b}$. This can be written as $\mathbf{A}\alpha \leq \mathbf{b} - \mathbf{Ay}^\circ = \tilde{\mathbf{b}}$, i.e.,

$$\sum_{j=1}^{n} a_{ij}\alpha_j \leq \tilde{b}_i \; for \; i=1,\ldots,m. \tag{3}$$

Similarly, we call (3) an 'α'-critical region. Note that $\tilde{b}_i$ is the ith component of $\tilde{\mathbf{b}}$ and each $\tilde{b}_i > 0$ since $\mathbf{y}^\circ$ is an interior point. To find a suitable α, we compute $\alpha_* \geq 0$ as the largest value such that so long as $-\alpha_* \leq \alpha_j \leq \alpha_*$, the inequalities in (3) should be satisfied. For any $\tilde{\alpha} \geq 0$, the largest that the left-hand side in (3) can ever get subject to $-\alpha_* \leq \alpha_j \leq \alpha_*$, $j=1,\ldots,n$ is $(\sum_{j=1}^{n} |a_{ij}|)\tilde{\alpha}$ for each i. Hence,

$$\alpha_* = \max\{\tilde{\alpha} \geq 0 : \sum_{j=1}^{n} |a_{ij}|\tilde{\alpha} \leq \tilde{b}_i\} \text{ for } i=1,\ldots,m.$$

If we let

$$\gamma_i = \frac{\tilde{b}_i}{\sum_{j=1}^{n} |a_{ij}|}$$

for $i=1,\ldots,m$, then

$$\alpha_* = \min\{\gamma_1, \gamma_2, \ldots, \gamma_m\}. \tag{4}$$

This computation is done based on the Tchebycheff norm, and see [4] for its application on the development of the tolerence approach. If we let each component of α be α_*, i.e., $\alpha = (\alpha_*, \alpha_*, \ldots, \alpha_*)^T$, then $\mathbf{A}\alpha \leq \tilde{\mathbf{b}}$ and

$$S = \prod_{j=1}^{n} [y^\circ_j - \alpha_*, \; y^\circ_j + \alpha_*] \tag{5}$$

will be contained in P with $\mathbf{y}^\circ \in S$. Here, [a,b] denotes a closed interval and the product is the Cartesian product. Notice above that α is a vector, but α_* and $\tilde{\alpha}$ are scalars, and S represents a square in R^2, a cube in R^3, and so on. Also, y° becomes the center point of S.

To illustrate this point, consider the following example whose graph is given in Figure 1.

$$\text{Example 1:} \quad \begin{aligned} -2x_1 + x_2 &\leq 2 \\ x_1 + x_2 &\leq 3 \\ 4x_1 - 5x_2 &\leq 20 \\ -x_2 &\leq 2 \end{aligned}$$

By letting $\mathbf{y}^{\circ}=(2,\ 0)$, $\mathbf{A}\alpha\leq\mathbf{b}-\mathbf{A}\mathbf{y}^{\circ}=\tilde{\mathbf{b}}$ becomes

$$\begin{aligned} -2\alpha_1+\ \alpha_2&\leq 6\\ \alpha_1+\ \alpha_2&\leq 1\\ 4\alpha_1-5\alpha_2&\leq 12\\ -\ \alpha_2&\leq 2 \end{aligned} \tag{6}$$

and $\alpha^{\circ}_{*}=\alpha_{*}=\min\{\gamma_1=6/3,\ \gamma_2=1/2,\ \gamma_3=12/9,\ \gamma_4=2/1\}=1/2$. The range of S and its region are given below.

$$S=[1.5,\ 2.5]\times[-0.5,\ 0.5]$$

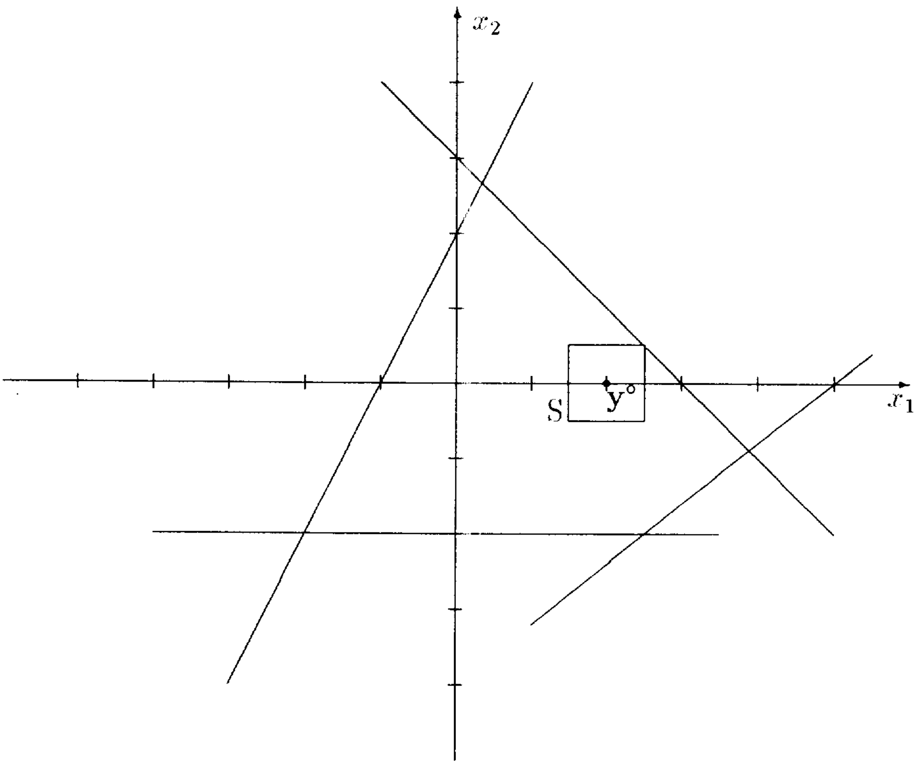

Figure 1.

2.2 Expansion of Region S

Next, we discuss an expansion R of S such that $S\subseteq R\subseteq P$. The basic idea is to find a new interior point $\mathbf{y}^{i}$ by moving away from the boundary of the polytope and to apply the procedure given in section 2.1. The expansion can be done as many times as possible by repeating the above steps until a termination criterion is satisfied.

Upon obtaining α_*°, we suppose that an ℓth inequality in (6) becomes tight(if more than one, choose arbitrarily) in the following sense:

$$|a_{\ell 1}|\alpha_*^\circ+|a_{\ell 2}|\alpha_*^\circ+\cdots+|a_{\ell n}|\alpha_*^\circ=\tilde{b}_\ell. \tag{7}$$

We will find a point $\mathbf{z}=(z_1, z_2,\ldots,z_n)$ belonging to the corresponding boundary of the ℓth inequality in Example 1 and which becomes a corner point(vertex) of S as follows.

Proposition 1 *If α_*° and S are obtained by (4) and (5), respectively, and ℓth inequality in (3) becomes tight in the sense of (7), then a point $\mathbf{z}=(z_1, z_2,\ldots, z_n)$ belonging to the boundary of the ℓth inequality of (2) and which becomes a corner point(vertex) of S is given by*

$$z_j = \begin{cases} y_j^\circ+\alpha_*^\circ & \text{if } a_{\ell j}>0 \\ y_j^\circ-\alpha_*^\circ & \text{if } a_{\ell j}<0 \\ y_j^\circ \text{ or } y_j^\circ+\alpha_*^\circ \text{ or } y_j^\circ-\alpha_*^\circ & \text{if } a_{\ell j}=0,\ j=1,\ldots,n. \end{cases}$$

Proof: The proof is easily established by noting that each left term in (7) contributes a nonnegative amount whose sum is equal to $\tilde{b}_\ell>0$. This implies, in particular, that if $a_{\ell j}<0$ for some j, then a negative amount $-\alpha_*^\circ$ must be multiplied to this coefficient, i.e.,

$$|a_{\ell j}|\alpha_*^\circ = \begin{cases} a_{\ell j}\alpha_*^\circ & \text{if } a_{\ell j}>0 \\ a_{\ell j}(-\alpha_*^\circ) & \text{if } a_{\ell j}<0. \end{cases}$$

Substituting $\tilde{b}_\ell=b_\ell-(a_{\ell 1}y_1^\circ+\cdots+a_{\ell n}y_n^\circ)$ in (7) and rearranging, one can see that the components of a point $\mathbf{z}$ that satisfies the boundary of the ℓth inequality in (2) must have the above form. ∎

In Example 1, for $\mathbf{y}^\circ=(2, 0)$, we have $\ell=2$ and $\alpha_*^\circ=\frac{1}{2}$. Since $a_{21}>0$ and $a_{22}>0$, we add $\frac{1}{2}$ to each y_1° and y_2° to obtain

$$\mathbf{z}=(2.5,\ 0.5).$$

This point belongs to the boundary of the second inequality of Example 1 (see Figure 2). Then we consider a direction eminating from the point $\mathbf{z}$ and passing through the point $\mathbf{y}^\circ$ and determine a point $\mathbf{y}^1$ in P as follows:

$$\mathbf{y}^1=\mathbf{y}^\circ+(\mathbf{y}^\circ-\mathbf{z})=\mathbf{z}+2(\mathbf{y}^\circ-\mathbf{z}).$$

Note that the distance between the points $\mathbf{z}$ and $\mathbf{y}^\circ$, $\mathbf{y}^\circ$ and $\mathbf{y}^1$ are the same. In example 1, $\mathbf{y}^1=(1.5,\ -0.5)$ (see Figure 2).

Now, we repeat the procedure given in section 2.1 by starting at $\mathbf{y}^1$, i.e., we replace $\mathbf{y}^0$ by $\mathbf{y}^1$.

For $\mathbf{y}^1=(1.5,\ -0.5)$, $\mathbf{A}\alpha \leq \mathbf{b}-\mathbf{A}\mathbf{y}^1=\tilde{\mathbf{b}}$ becomes

$$\begin{aligned} -2\alpha_1+\ \alpha_2 &\leq 5.5 \\ \alpha_1+\ \alpha_2 &\leq 2.0 \\ 4\alpha_1-5\alpha_2 &\leq 11.5 \\ -\ \alpha_2 &\leq 1.5. \end{aligned}$$

Then we calculate α_*^1 as in (4). At this point only one of the two possible cases can occur :

$$\text{(I)}\ \alpha_*^1 < 2\alpha_*^0$$
$$\text{(II)}\ \alpha_*^1 = 2\alpha_*^0.$$

If (I) occurs, we terminate the expansion process (see Section 2.3 for its meaning and other expansion strategies) and should be satisfied with the region $R=S$. However, if (II) results, the region S can be expanded into a larger region R that contains S. For the above problem, $\alpha_*^1=\min\{5.5/3,\ 2/2,\ 11.5/9,\ 1.5/1\}=1$. Note that $\alpha_*^1=2\alpha_*^0$ and hence it satisfies the second case (II). Of course, the same $\mathbf{z}$ is obtained. Then the range of the expanded region R becomes

$$R=[0.5,\ 2.5]\times[-1.5,\ 0.5].$$

It is obtained by taking $\pm\alpha_*^1$ amount in each axis from its center point $\mathbf{y}^1$. In general, at the ith step, the range of R is given as

$$R=\prod_{j=1}^{n}[y_j^i-\alpha_*^i,\ y_j^i+\alpha_*^i].$$

For the case above, $i=1$ and see Figure 2 below for the expansion from S to R. By computing $\mathbf{y}^2=\mathbf{z}+2(\mathbf{y}^1-\mathbf{z})=(0.5,\ -1.5)$, one can further test for a possible expansion, but it can be easily shown that $\alpha_*^2<2\alpha_*^1$ at $\mathbf{y}^2$. Hence, one can terminate the expansion process at this point.

2.3 An Expansion Algorithm

The procedure given in the previous subsection, estimating α_* in the 'α'-critical region and obtaining a region S in the 'x'-critical region in repetitive fasion, is summarized as a tolerance approach expansion algorithm.

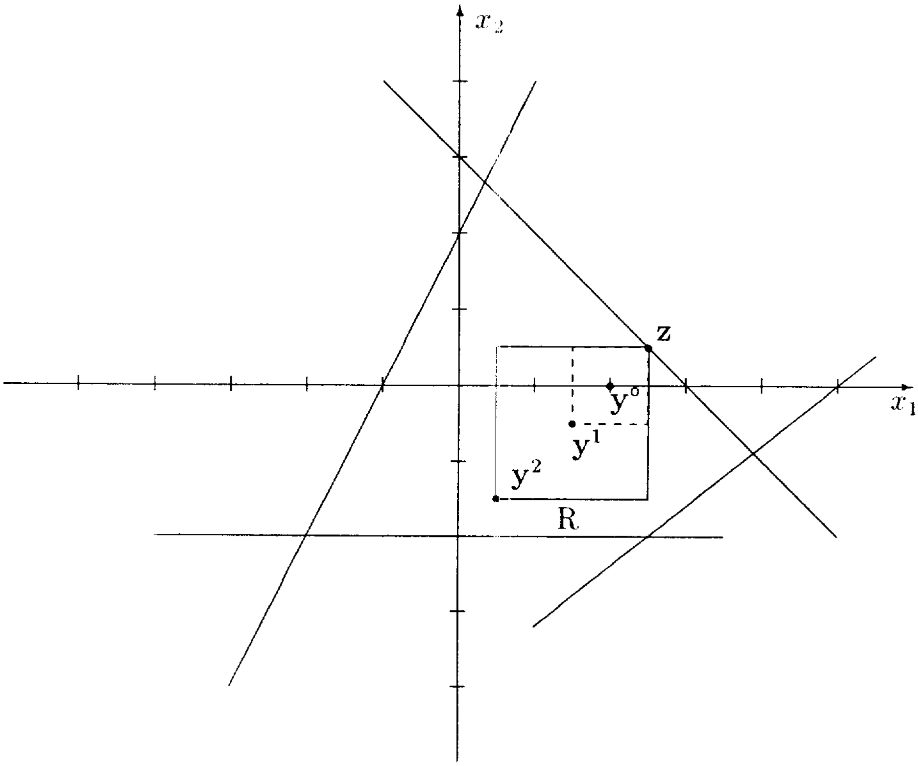

Figure 2.

Algorithm TAE

begin

Given a polytope $P(\mathbf{Ax} \leq \mathbf{b})$, an interior point $\mathbf{y}^\circ$, $NMAX$, and a step length κ

STEP **0** : $i \leftarrow 0$

Form $\mathbf{A}\alpha \leq \mathbf{b} + \mathbf{Ay}^\circ = f(\mathbf{y}^\circ)$

Compute α°_*, S, and $\mathbf{z}$, as in (4), (5) and Proposition 1

STEP 1: $i \leftarrow i+1$

$R \leftarrow S$

$\mathbf{y}^i = \mathbf{z} + \kappa(\mathbf{y}^{i-1} - \mathbf{z})$

Form $\mathbf{A}\alpha \leq \mathbf{b} + \mathbf{Ay}^i = f(\mathbf{y}^i)$

Compute α^i_* as in (4)

STEP 2 : **If** termination criterion is satisfied or $i = NMAX$, exit with R

If $\alpha^i_* = \kappa\alpha^{i-1}_*$ **then**

Compute S at $\mathbf{y}^i$ using (8)

Go to STEP 1

end

end

The above algorithm terminates when the following criterion is satisfied.

Termination Criterion :

$$\alpha_*^i < \kappa\alpha_*^{i-1}$$

For the illustration in Section 2.2, we used $\kappa=2$. By setting an upper limit for the number of iterations, *NMAX*, the algorithm terminates in a finite number of steps. This is useful if the critical region is unbounded and requires an infinite number of expansion iterations. An expansion is also possible, however, if $0<\alpha_*^i<\kappa\alpha_*^{i-1}$, but one can visualize that an explicit range of R can not be obtained easily.

In n-dimensional space, there are 2^n-1 possible directions for an expansion of S. For example, when $n=2$, three directions can be considered, see Figure 2. In Step 1 of the algorithm TAE, we always use the "diagonal direction" eminating from the point $\mathbf{z}$ and passing through the center point $\mathbf{y}^i$. This direction is easy to be described and the explicit range of the region R can be efficiently computed.

The step length κ in the expression $\mathbf{y}^i=\mathbf{z}+\kappa(\mathbf{y}^{i-1}-\mathbf{z})$ determines the size of R at each iteration. One can use many different values of κ, for example, $\kappa=2$, $\kappa=(i+1)^2$, $\kappa=2^i$, and so on. For the numerical experiment in this paper, we only used $\kappa=2$, a linear step length expansion. For n-dimensional space, however, the size of the expanded region R becomes 2^n times the size of the previous region S if $\kappa=2$.

Ⅲ. The Tolerance Approach

In this section we reconsider the tolerance approach proposed by Wendell[4]. From the linear program given in (1), we consider the following perturbed problem.

$$\begin{aligned} &\text{minimize } \sum_{j=1}^{n}(c_j+\alpha_j c'_j)x_j \\ &\text{s.t. } \sum_{j=1}^{n} a_{ij}x_j=b_i,\ i=1,\ldots,m \\ &\qquad x_1,\ldots,x_n \geq 0. \end{aligned} \tag{9}$$

We denote $\mathbf{B}$ an optimal basis of (1) and IR the index set of the nonbasic variables. Let $\mathbf{y}_j=\mathbf{B}^{-1}\mathbf{a}_j$, as usual. Then, $\mathbf{B}$ remains an optimal basis in (9) if

$$\sum_{i=1}^{m}(c_{B_i}+\alpha_{B_i}c'_{B_i})y_{ij}-(c_j+\alpha_j c'_j)\leq 0 \tag{10}$$

for each $j\in IR$, where c_{B_i}, c'_{B_i}, α_{B_i} denote the corresponding basic variable coefficients in the vectors $\mathbf{c}$, $\mathbf{c}'$, α, respectively. Note that if $\mathbf{c}'=\mathbf{1}$(i.e., each $c'_j=1$), then α becomes an additive parameter and would represent additive variation in the coefficients. And if $\mathbf{c}'=\mathbf{c}$, then α would represent multiplicative variation.

A maximum allowable tolerance on the ***multiplicative*** variation in the objective function coefficients is denoted by α_*^{mult} ($\alpha_*^{mult}>0$ is the largest value such that whenever $-\alpha_*^{mult}\leq\alpha_j\leq\alpha_*^{mult}$ for each $j=1,\ldots,n$, the basis $\mathbf{B}$ is still an optimal basis for the perturbed problem (9)) and is given by

$$\alpha_*^{mult}=\text{minimum}\ \frac{c_j-z_j}{\sum_{i=1}^{m}|c'_B y_{ij}|+|c'_j|}\quad j\in IR^+,$$

where $IR^+=\{j\in IR:\ \sum_{i=1}^{m}|c'_B y_{ij}|+|c'_j|>0\}$, see [1, 4] for more detail. A maximum allowable tolerance on the ***additive*** variation would then be

$$\alpha_*^{add}=\text{minimum}\ \frac{c_j-z_j}{\sum_{i=1}^{m}|y_{ij}|+|1|}\quad j\in IR^+,$$

where $IR^+=\{j\in IR:\ \sum_{i=1}^{m}|y_{ij}|+|1|>0\}$.

For an illustration, consider the following problem given in [1],

$$\begin{aligned}\text{minimize}\ & -2x_1+x_2-x_3\\ \text{s.t.}\ & x_1+x_2+x_3 \leq 6\\ & -x_1+2x_2 \leq 4\\ & x_1,\ x_2,\ x_3 \geq 0.\end{aligned} \tag{11}$$

For this problem, the optimal simplex tableau is given by

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-3	-1	-2	0	-12
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

The minimum is attained at $\mathbf{x}^*=(6, 0, 0, 0, 10)$. The index set for the basic variable $IB=\{1,$

5} and for the nonbasic variable $IR=\{2, 3, 4\}$. For the multiplicative variation, we get $\alpha_*^{mult} = \text{minimum}\{3/3, 1/3, 2/2\}=1/3$. This implies that the objective coefficients can vary within a tolerance of $33\frac{1}{3}$ percent and the given basis will remain optimal. If we let $\mathbf{c}^\circ=(-2, 1, -1, 0, 0)$ be the original objective function coefficients, then a region T containing $\mathbf{c}^\circ$, in which the basis $\mathbf{B}$ is still an optimal basis, is given by

$$T=[-8/3, -4/3]\times[2/3, 4/3]\times[-4/3, -2/3]\times[0, 0]\times[0, 0].$$

Note that when $c_4=c_5=0$, no variation is permissible.

For an additive variation, we obtain $\alpha_*^{add}=\text{minimum}\{3/5, 1/3, 2/3\}=1/3$ and

$$S=[-7/3, -5/3]\times[2/3, 4/3]\times[-4/3, -2/3]\times[-1/3, 1/3]\times[-1/3, 1/3].$$

Ⅳ. Expansion Technique on Additive Variation

Letting $c'_{B_i}=c'_j=1$ for all i and j in (10) gives

$$\sum_{i=1}^{m} (c_{B_i}+\alpha_{B_i})y_{ij}-(c_j+\alpha_j)\le 0$$

for each $j\in IR$. We can think of $\sum_{i=1}^{m} c_{B_i} y_{ij}-c_j\le 0$ as an 'c'-critical region, and if we let $\mathbf{c}^\circ=\mathbf{c}$, then $\sum_{i=1}^{m}(c_{B_i}^\circ+\alpha_{B_i})y_{ij}-(c_j^\circ+\alpha_j)\le 0$ becomes an 'α'-critical region with $\mathbf{c}^\circ$ as an initial interior point of the 'c'-critical region. Notice that this becomes exactly the same situation described in Section 2, and the algorithm TAE can be applied on these critical regions. Rearranging the above expression gives

$$\sum_{i=1}^{m} y_{ij}\alpha_{B_i}-\alpha_j\le c_j^\circ-\sum_{i=1}^{m} c_B^\circ y_{ij}=f(c^\circ) \tag{12}$$

for each $j\in IR$.

4.1 Numerical Illustration

We consider a numerical experiment that is intended to provide an illustration of the performance of the algorithm TAE. We again use the example in (11) as a test problem. For this problem we set $\kappa=2$, and the following initial 'α'-critical region is obtained :

$$\begin{aligned}
\alpha_1 - \alpha_2 \qquad\qquad + 3\alpha_5 &\le 3\\
\alpha_1 \qquad - \alpha_3 \qquad + \alpha_5 &\le 1\\
\alpha_1 \qquad\qquad - \alpha_4 + \alpha_5 &\le 2,
\end{aligned}$$

where each column corresponds to the variable x_i. By the algorithm TAE in section 2.3, $\alpha_*^\circ = \min\{3/5,\ 1/3,\ 2/3\} = 1/3$ and $\mathbf{z} = (-5/3,\ 4/3,\ -4/3,\ 1/3,\ 1/3)$, taking $z_j = y_j^\circ + \alpha_*^\circ$ whenever $a_{0j} = 0$. Then, $\mathbf{c}^1 = \mathbf{z} + 2(\mathbf{c}^\circ - \mathbf{z}) = (-7/3,\ 2/3,\ -2/3,\ -1/3,\ -1/3)$. This gives

$$S = [-7/3,\ -5/3] \times [2/3,\ 4/3] \times [-4/3,\ -2/3] \times [-1/3,\ 1/3] \times [-1/3,\ 1/3].$$

To expand the region, we replace $\mathbf{c}^\circ$ by $\mathbf{c}^1$ in (12), which gives

$$\begin{aligned} \alpha_1 - \alpha_2 \qquad\qquad + 3\alpha_5 &\le 4 \\ \alpha_1 \qquad - \alpha_3 \qquad + \alpha_5 &\le 2 \\ \alpha_1 \qquad\qquad - \alpha_4 + \alpha_5 &\le \frac{7}{3}. \end{aligned}$$

Again, $\alpha_*^\circ = \min\{4/5,\ 2/3,\ 7/9\} = 2/3$ and note that (II) is satisfied. Now, $\mathbf{c}^2 = \mathbf{z} + 2(\mathbf{c}^1 - \mathbf{z}) = (-3,\ 0,\ 0,\ -1,\ -1)$ and trying to expand once more, the right-hand side vector gives (6, 4, 3) and $\alpha_*^2 = \min\{6/5,\ 4/3,\ 3/3\} = 1$. At this point, note that $\alpha_*^2 < 2\alpha_*^1$, and hence, we stop the procedure of the expansion algorithm. The region R is obtained from (8), replacing $\mathbf{y}^i$ by $\mathbf{c}^1$ and using $\alpha_*^1 = 2/3$, as

$$R = [-3,\ -5/3] \times [0,\ 4/3] \times [-4/3,\ 0] \times [-1,\ 1/3] \times [-1,\ 1/3].$$

Notice that $S \subseteq R$. When no variation on a certain coefficient or a set of coefficients is desired, then one can set the corresponding $\alpha_k's$ to zero and the expansion can be done on the remaining set of the coefficients.

Next, for the expansion of the right-hand side terms, consider the following perturbed problem from (1) :

$$\begin{aligned} &\text{minimize } \sum_{j=1}^{n} c_j x_j \\ &\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j = b_i + \beta_i b_i',\ i = 1, \ldots, m \qquad (13) \\ &\qquad x_1, \ldots, x_n \ge 0. \end{aligned}$$

Let $\mathbf{B}$ be an optimal basis of (1), and the components of $\mathbf{B}^{-1}$ as B_{ij}^{-1} for $i, j = 1, \ldots, m$. Then, $\mathbf{B}$ will be an optimal basis for the above perturbed problem (13) if

$$\sum_{j=1}^{m} B_{ij}^{-1}(b_j + \beta_j b_j') \ge 0,\ i = 1, \ldots, m. \qquad (14)$$

Letting $b_j' = 1$ for each j in (14) gives the additive variation, and we have

$$\sum_{j=1}^{m} -B_{ij}^{-1}\beta_j \le \bar{b}_i,\ i = 1, \ldots, m, \qquad (15)$$

where $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$. The maximum allowable tolerances on the additive variation is given by

$$\beta_*^{add} = \text{minimum} \; \frac{\bar{b}_j}{\sum_{i=1}^{m} |-B_{ij}^{-1}|}$$
$$i \in IB^+,$$

where $IB^+ = \{j \in IB : \sum_{i=1}^{m} |-B_{ij}^{-1}| > 0\}$. For the problem in (11), $\mathbf{b}^\circ = \mathbf{b} = (6, 4)$ and the optimal basis and its inverse are

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and the 'β'-critical region of the parameter is given as

$$\begin{aligned} -\beta_1 \qquad & \leq 6 \\ -\beta_1 \quad -\beta_2 & \leq 10. \end{aligned}$$

Following the algorithm TAE in section 2.3, $\beta_*^\circ = 5$ and $\mathbf{z} = (1, -1)$. Then, $\mathbf{b}^1 = (11, 9)$ and this gives $S = [1, 11] \times [-1, 9]$.

To expand the region, we replace $\mathbf{b}$ in (15) by $\mathbf{b}^1$, which gives

$$\begin{aligned} -\beta_1 \qquad & \leq 11 \\ -\beta_1 \quad -\beta_2 & \leq 20. \end{aligned}$$

Again, $\beta_*^1 = 10$ and (Ⅱ) is satisfied. Now, $\mathbf{b}^2 = (21, 19)$ and $R = [1, 21] \times [-1, 19]$. Trying to expand once more, the right-hand side vector becomes (21, 40) and $\beta_*^2 = 20$ which also satisfies (Ⅱ). For this particular problem, this process can be continued indefinitely(letting $NMAX = \infty$) and the final expanded region R can be obtained as

$$R = [1, +\infty) \times [-1, \infty).$$

When the right-hand side terms are within this range, the optimal basis is unchanged for the perturbed problem.

Ⅴ. Concluding Remarks

While the tolerence approach is trying to find a range in which the objective function coefficients or the right-hand side terms can be changed on a "one-time" basis, the expansion method, if possible, is trying to expand the region as many times as possible starting from the result obtained by the tolerence approach. This strategy is especially advantageous if the initial vectors $\mathbf{c}^{\circ}$ or $\mathbf{b}^{\circ}$ are closer to the boundary of the critical region(polytope). Although the results are not shown in this paper, we have also tested the algorithm TAE on a set of other problems and obtained similar results as expected.

When a value of the objective function coefficient or of the right-hand side term is zero, no range of the variation can be obtained by the multiplicative variation, but the additive variation assigns exactly the same amount of the range to each of the component, whether it is zero or not. Also, for the multiplicative variation, it is possible, in the expansion process, that the values of the components of the newly found interior point $\mathbf{c}^{i}$ or $\mathbf{b}^{i}$ become zero or almost zero. If this happens, virtually no variation is permissible on these components since the tolerance approach with the multiplicative variation gives percentage variations based on the values of the coefficients or the right-hand side terms. Hence, some other strategies must be explored. Further work also includes application of the algorithm TAE on codes that can solve practical problems.

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