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ON THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR A CLASS OF 1ST-ORDER NONLINEAR AUTOREGRESSIVE PROCESSES

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ABSTRACT. A class of nonlinear Markov processes on the real line is considered,and a functional central limit theorem is proved for the functions of bounded variation on the real line by identifying a broad subset of the range of the generator.

1. Introduction

Consider a 1-dimensional *Markov* process $\{Y_n : n \geq 0\}$ on $\mathbb{R}^1$ defined by

$$Y_{n+1} := f(Y_n) + \varepsilon_{n+1}(n \geq 0). \tag{1}$$

where f is $\mathbb{R}^1$-valued Borel measurable function on $\mathbb{R}^1$, and Y_0 is an arbitrarily specified random variable with values in $\mathbb{R}^1$, independent of the random forcing terms, $\{\varepsilon_n : n \geq 1\}$.

Let $\mathcal{B}^1$ denote the Borel sigma field on $\mathbb{R}^1$ and λ_1 Lebesgue measure on $(\mathbb{R}^1, \mathcal{B}^1)$. Then $(\mathbb{R}^1, \mathcal{B}^1, \lambda_1)$ is the state space of (1).

Let $p^{(n)}(x, dy)$ denotes the n-step transition probability of $Y_n (n \geq 1)$ and $p(x, dy) \equiv p^{(1)}(x, dy)$.

A probability measure π on $(\mathbb{R}^1, \mathcal{B}^1)$ is said to be *invariant* for $\{Y_n : n \geq 0\}$, or for $p(x, dy)$, if

$$\int_{\mathbb{R}^1} p(x, A)\pi(dx) = \pi(A), \quad \forall A \in \mathcal{B}^1. \tag{2}$$

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The Markov process is λ_1-*irreducible*, if for every $A \in \mathcal{B}^1, \lambda_1(A) > 0$ one has

(3) $$\sum_{n \geq 1} p^{(n)}(x, A) > 0.$$

It is simple to check that the process (1) is λ_1-irreducible if $\varepsilon_n (n \geq 1)$ has a density function which is positive a.e.(λ_1).

A set $B \in \mathcal{B}^1$ is said to be *small*(with respect to λ_1) if $\lambda_1(B) > 0$,and for every $A \in \mathcal{B}^1$ with $\lambda_1(A) > 0$ there exists $j \geq 1$ such that

(4) $$\inf_{x \in B} \sum_{n=1}^{j} p^{(n)}(x, A) > 0.$$

Note that every nonempty compact subset in $\mathbb{R}^1$ is small(see,e.g,Bhattacharya and Lee[3], Lemma 1).

A ϕ-irreducible aperiodic Markov process with transition probability $p(x, dy)$ is said to be (*Harris*) *ergodic* if there exists a probability measure π such that

(5) $$||p^{(n)}(x, dy) - \pi(dy)|| \longrightarrow 0 \quad \text{as} \quad n \to \infty, \forall x \in \mathbb{R}^1.$$

Here $|| \cdot ||$ denotes the variation norm on the Banach space of finite signed measure on $(\mathbb{R}^1, \mathcal{B}^1)$.

If the convergence in (5) is exponentially fast then the process is said to be *geometrically* (*Harris*) *ergodic*.

Recently there have been considerable works on kth-order nonlinear autoregressive models, most of which provide some verifiable criteria for geometric ergodicity (see, e.g., Chan and Tong [7], $Tj\phi$stheim [12], Bhattacharya and Lee [2],[3], Lee [9]).

If (5) holds then π is necessarily the unique invariant probability for $p(x, dy)$, and the process having π as the initial distribution is stationary.

We now assume that the process $\{Y_n : n \geq 0\}$ is (Harris) ergodic and that Y_0 has the unique invariant π as its distribution. Consider a real-valued function ψ on $\mathbb{R}^1$ such that $E\psi^2(Y_0) < \infty$. Write $\tilde{\psi} = \psi - \bar{\psi}$, where $\bar{\psi} = \int \psi d\pi$.

In $L^2(\mathbb{R}^1, \pi)$, consider the identity operator I and the transition operator T,

$$(Tg)(x) := \int g(y)p(x, dy).$$

Then $(T^m\tilde{\psi})(x) = (T^m\psi)(x) - \bar{\psi}$ for all $m \geq 0$. Also, $E\tilde{\psi}(X_0) = 0$.

The ergodicity of the process $\{Y_n : n \geq 0\}$ implies that the kernel of the operator $I - T$ is one dimensional,

$$Ker(I - T) = \{\lambda, 1\}_{\lambda \in R}$$

(see, e.g., Gordin and Lifsic [8], Bhattacharya [1]).

We shall need the following result of Gordin and Lifsic [8].

PROPOSITION 1. *Assume $p(x, dy)$ admits an invariant probability π and, under the initial distribution π, $\{Y_n\}$ is ergodic. Assume also that $\tilde{\psi} = \psi - \bar{\psi}$ is in the range of $I - T$. Then*

$$(6) \qquad n^{-1/2}\left[\sum_{j=0}^{[nt]}(\psi(Y_j) - \bar{\psi}) + (nt - [nt])(\psi(Y_{[nt]+1}) - \bar{\psi})\right] \; (t \geq 0)$$

converges weakly to a Brownian motion with mean zero and variance parameter $\|h\|_2^2 - \|Th\|_2^2$, where $(I - T)h = \tilde{\psi}$ and $[nt]$ is the integer part of nt.

Our main result is the following theorem.

THEOREM 1. *Assume that the process $\{Y_n : n \geq 0\}$ in (1) is (Harris) ergodic and that the unique invariant probability π has a compact support containing the origin. Assume also that Y_0 has π as its distribution.*

Then for every ψ that may be expressed as the difference between two monotone nondecreasing functions in $L^2(\mathbb{R}^1, \pi)$, $\psi - \int \psi d\pi$ belongs to the range of $I - T$.

For the proof let us begin with two simple lemmas.

LEMMA 1. *Let μ be a probability measure on $(\mathbb{R}^1, \mathcal{B}^1)$ such that $\int x^2\mu(dx) < \infty$. Then*

$$\int x^2\mu(dx) - \left(\int x\mu(dx)\right)^2 = \frac{1}{2}\int\int (x-y)^2\mu(dx)\mu(dy).$$

PROOF. Expand the right-hand side and integrate. □

LEMMA 2. *Let $\psi \in L^2(\mathbb{R}^1, \pi)$. If $\sum_{n=0}^{\infty} \|T^n(\psi - \bar{\psi})\|_2 < \infty$, then $\psi - \bar{\psi}$ belongs to the range of $I - T$; indeed, $(I-T)h = \psi - \bar{\psi}$, where*

$$h = -\sum_{n=0}^{\infty} T^n(\psi - \bar{\psi}). \tag{7}$$

PROOF. Apply $(I-T)$ to the right side of (7). □

PROOF OF THEOREM 1. Let $\psi \in L^2(\mathbb{R}^1, \pi)$. be monotone nondecreasing. By Lemma 1,

$$\begin{aligned} &\|T(\psi - \bar{\psi})\|_2^2 \\ &= \int \left(\int (\psi(y) - \bar{\psi})p(x,dy)\right)^2 \pi(dx) \\ &= \int \left[\int (\psi(y) - \bar{\psi})^2 p(x,dy) - \frac{1}{2}\int\int (\psi(y) - \psi(z))^2 p(x,dy)p(x,dz)\right] \pi(dx) \\ &= \|\psi - \bar{\psi}\|_2^2 - \frac{1}{2}\int \left[\int\int (\psi(y) - \psi(z))^2 p(x,dy)p(x,dz)\right] \pi(dx). \end{aligned} \tag{8}$$

Let C_π denote the compact support of the distribution π. Then for $x \in C_\pi$,

$$\begin{aligned} &\int\int (\psi(y) - \psi(z))^2 p(x,dy)p(x,dz) \\ &\geq \int_{\{z\geq 0\}}\int_{\{y \leq 0\}} (\psi(y) - \psi(0))^2 p(x,dy)p(x,dz) \\ &+ \int_{\{z\leq 0\}}\int_{\{y \geq 0\}} (\psi(y) - \psi(0))^2 p(x,dy)p(x,dz) \\ &\geq min\{C_1, C_2\} \int (\psi(y) - \psi(0))^2 p(x,dy), \end{aligned} \tag{9}$$

where $C_1 = inf_{x\in C_\pi} p(x, [0,\infty) \cap C_\pi)$, $C_2 = inf_{x\in C_\pi} p(x, (-\infty, 0] \cap C_\pi)$.
Since C_π is small,C_1 and C_2 are positive(see,e.g., Bhattacharya and Lee (1995),Lemma 1). Hence

$$
\begin{aligned}
(10) \qquad & \int \left[\int \int (\psi(y) - \psi(z))^2 p(x, dy) p(x, dz) \right] \pi(dx) \\
& \geq \min\{C_1, C_2\} \int_{C_\pi} \left[\int (\psi(y) - \psi(0))^2 p(x, dy) \right] \pi(dx) \\
& = min\{C_1, C_2\} \int (\psi(y) - \psi(0))^2 \pi(dy) \\
& \geq \min\{C_1, C_2\} \|\psi - \bar{\psi}\|_2^2 \geq (1-\delta) \|\psi - \bar{\psi}\|_2^2,
\end{aligned}
$$

where $\delta = max\{1 - C_1, 1 - C_2\}$. Note that δ is less than 1.Using (10) in (8) one gets

$$
(11) \qquad \|T(\psi - \bar{\psi})\|_2 \leq c \|\psi - \bar{\psi}\|_2,
$$

where

$$
(12) \qquad c = (1 - \frac{1}{2}(1-\delta))^{\frac{1}{2}} < 1.
$$

Next note that if ψ is monotone nondecreasing,so is $T\psi$,and since T is a contraction on $L^2(\mathbb{R}^1, \pi)$,one has

$$
(13) \qquad \|T^n(\psi - \bar{\psi})\|^2 \leq c^n \|\psi - \bar{\psi}\|^2 \quad \forall\, n.
$$

It now follows from Lemma 2 that $\psi - \bar{\psi}$belongs to the range of I-T.□

COROLLARY 1. *Under the hypothesis of the Theorem above,(6) holds for every function of bounded variation ψ in $L^2(\mathbb{R}^1, \pi)$.*

REMARK 1. Under the mild extra condition on the process $\{Y_n\}$ that for each n, $\sup_x |P^{(n)}(x, dy) - \pi(dy)| \leq c\delta^n \pi(dy)$ for some constants $c > 0$, $0 < \delta < 1$, with a little additional work, one can directly prove that $\sum_{n=0}^{\infty} \|T^n(\psi - \bar{\psi})\|_2 < \infty$ for every ψ in $L^2(\mathbb{R}^1, \pi)$, that is, $\psi - \bar{\psi}$ belongs to the range of $I - T$.

Details concerning the assertions in the above Remark will appear elsewhere.

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