Comm. Korean Math. Soc. **11** (1996), No. 4, pp. 1087–1094

THE N-TH PRETOPOLOGICAL MODIFICATION OF CONVERGENCE SPACES

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ABSTRACT. In this paper, we introduce the notion of the n-th pretopological modification. Also, we find some properties which hold between convergence quotient maps and n-th pretopological modifications.

1. Introduction

A convergence structure defined by Kent [4] is a correspondence between the filters on a given set X and the subsets of X which specifies which filters converge to points of X. This concept is defined to include types of convergence which are more general than that defined by specifying a topology on X. Thus, a convergence structure may be regarded as a generalization of a topology.

With a given convergence structure q on a set X, Kent [4] introduced an associated convergence structure which is called a pretopological modification.

Also, Kent [6] introduced a convergence quotient map, which is a quotient map for a convergence space.

In this paper, with a convergence structure q, we introduce notions of the filter $V_q^n(x)$ and the n-th pretopological modification of q which is denoted by $\pi_n(q)$, where $n \in N \cup \{\infty\}$.

In Theorem 7, we show that for a map $f\colon (X,q) \to (Y,p)$, $V_p(f(x)) = f(V_q(x))$ iff $V_p^n(f(x)) = f(V_q^n(x))$ for each $n \in N \cup \{\infty\}$.

Received April 27, 1996. Revised September 13, 1996.

1991 AMS Subject Classification: 54A05, 54A20.

Key words and phrases: convergence space(structure), convergence quotient map, n-th pretopological modification.

Supported by the Basic Science Research Institute Program, Ministry of Education, 1996, Project No. BSRI-96-1406.

In Theorem 10, we show that if p is pretopological and $f:(X,q) \to (Y,p)$ is a convergence quotient map, then $f:(X,\pi_n(q)) \to (Y,\pi_n(p))$ is also a convergence quotient map for each $n \in N \cup \{\infty\}$.

2. Preliminaries

A *convergence structure* q on a set X is defined to be a function from the set $F(X)$ of all filters on X into the set $P(X)$ of all subsets of X, satisfying the following conditions:

(1) $x \in q(\dot{x})$ for all $x \in X$;

(2) $\Phi \subset \Psi$ implies $q(\Phi) \subset q(\Psi)$;

(3) $x \in q(\Phi)$ implies $x \in q(\Phi \cap \dot{x})$,

where $\dot{x}$ denotes the principal ultrafilter containing $\{x\}$; Φ and Ψ are in $F(X)$. Then the pair (X,q) is called a *convergence space*. If $x \in q(\Phi)$, then we say that Φ *q-converges* to x. The filter $V_q(x)$ obtained by intersecting all filters which q-converge to x is called the *q-neighborhood filter* at x. If $V_q(x)$ q-converges to x for each $x \in X$, then q is said to be *pretopological* and the pair (X,q) is called a *pretopological convergence space.*

Let $C(X)$ be the set of all convergence structures on X, partially ordered as follows:

$$q_1 \leq q_2 \text{ iff } q_2(\Phi) \subset q_1(\Phi) \text{ for all } \Phi \in F(X).$$

If $q_1 \leq q_2$, then we say that q_1 is *coarser* than q_2, and q_2 is *finer* than q_1. By [5], we know that if q_1 is pretopological, then

$$q_1 \leq q_2 \text{ iff } V_{q_1}(x) \subset V_{q_2}(x) \text{ for all } x \in X.$$

For any $q \in C(X)$, we define a related convergence structure $\pi(q)$, as follows:

$$x \in \pi(q)(\Phi) \text{ iff } V_q(x) \subset \Phi.$$

In this case, $\pi(q)$ is called the *pretopological modification* of q, and the pairs $(X,\pi(q))$ is called the *pretopological modification* of (X,q).

PROPOSITION 1([4]). *$\pi(q)$ is the finest pretopological convergence structure coarser than q.*

Let f be a map from X into Y and Φ a filter on X. Then $f(\Phi)$ means the filter generated by $\{f(F) \mid F \in \Phi\}$.([1])

PROPOSITION 2. *Let $f: X \to Y$ be a map and $\{\Phi_i \mid i \in I\}$ a family of filters on $F(X)$. Then $f(\cap_{i\in I}\Phi_i) = \cap_{i\in I} f(\Phi_i)$.*

PROOF. Let $B \in f(\cap_{i\in I}\Phi_i)$. Then there exists $A \in \cap_{i\in I}\Phi_i$ such that $f(A) \subset B$. Thus $A \in \Phi_i$ and so $f(A) \in f(\Phi_i)$ for all $i \in I$. Finally, $f(A) \in \cap_{i\in I} f(\Phi_i)$ and so $B \in \cap_{i\in I} f(\Phi_i)$.

Conversely, let $B \in \cap_{i\in I} f(\Phi_i)$. Then, for each $i \in I$, there exists $F \in \Phi_i$ such that $f(F) \subset B$. Since $F \subset f^{-1}(B)$, we obtain $f^{-1}(B) \in \Phi_i$ for each $i \in I$ and so $f^{-1}(B) \in \cap_{i\in I}\Phi_i$. While, since $B \supset f(f^{-1}(B)) \in f(\cap_{i\in I}\Phi_i)$, we obtain $B \in f(\cap_{i\in I}\Phi_i)$. This completes the proof.

Let f be a map from a convergence space (X, q) to a convergence space (Y, p). Then f is said to be *continuous* at a point $x \in X$, if the filter $f(\Phi)$ on Y p-converges to $f(x)$ for every filter Φ on X q-converging to x. If f is continuous at every point $x \in X$, then f is said to be continuous.

Let q and q' be in $C(X)$, and p and p' in $C(Y)$. Then, we know that if $q \leq q'$, $p \geq p'$ and $f: (X, q) \to (Y, p)$ is continuous, then $f: (X, q') \to (Y, p')$ is continuous.

PROPOSITION 3 ([6]). (1) *If $f: (X, q) \to (Y, p)$ is continuous at $x \in X$, then $V_p(f(x)) \subset f(V_q(x))$.*

(2) *If p is pretopological and $V_p(f(x)) \subset f(V_q(x))$, then $f: (X, q) \to (Y, p)$ is continuous at $x \in X$.*

Let (X, q) be a convergence space. Then the set function $I_q: P(X) \to P(X)$ is defined by as follows:

$$I_q(A) = \{x \in A \mid A \in V_q(x)\}$$

for each $A \subset X$. Then, I_q has the following properties:

$$(1)\ I_q(\emptyset) = \emptyset,\ I_q(A) \subset A$$
$$(2)\ I_q(X) = X$$
$$(3)\ I_q(A \cap B) = I_q(A) \cap I_q(B)$$
$$(4)\ A \subset B \text{ implies } I_q(A) \subset I_q(B)$$

for each $A, B \subset X$. But, in general, $I_q(I_q(A)) \neq I_q(A)$.

Also, we define a set function $I_q^n: P(X) \to P(X)$ for each $n \in N \cup \{\infty\}$, where N is the set of positive integers, as follows:

$$I_q^1(A) = I_q(A),$$
$$I_q^{n+1}(A) = I_q(I_q^n(A)) \text{ if } n \in N,$$
$$I_q^\infty(A) = \cap\{I_q^n(A) \mid n \in N\}.$$

It is clear that $I_q^n(A \cap B) = I_q^n(A) \cap I_q^n(B)$ for each $n \in N \cup \{\infty\}$ and $A, B \subset X$.

Indeed, I_q^n has all of the properties of a topological interior operator except idempotency.

Let $V_q^n(x) = \{A \subset X \mid x \in I_q^n(A)\}$. Then $V_q^n(x)$ is a filter on X for each $n \in N \cup \{\infty\}$, and we know that for each $n \in N$,

$$I_q^n(A) \supset I_q^{n+1}(A) \supset I_q^\infty(A) \text{ for each } A \subset X,$$

and

$$V_q^n(x) \supset V_q^{n+1}(x) \supset V_q^\infty(x) \text{ for each } x \in X.$$

Define a structure $\pi_n(q)$ for each $n \in N \cup \{\infty\}$ as follows:

$$x \in \pi_n(q)(\Phi) \quad \text{iff} \quad V_q^n(x) \subset \Phi$$

for each $\Phi \in F(X)$. It is not difficult to show that for each $n \in N \cup \{\infty\}$,

$$V_{\pi_n(q)}(x) = V_q^n(x) \text{ for each } x \in X,$$
$$I_{\pi_n(q)}(A) = I_q^n(A) \text{ for all } A \subset X$$

and for each $n \in N$,

$$q \geq \pi_n(q) \geq \pi_{n+1}(q) \geq \pi_\infty(q).$$

While, since $V_q(x) \subset \dot{x}$, we obtain $x \in \pi_n(q)(\dot{x})$ for each $x \in X$. Also $\Phi \subset \Psi \in F(X)$ implies $\pi_n(q)(\Phi) \subset \pi_n(q)(\Psi)$.

Let $x \in \pi_n(q)(\Phi)$. Then $V_q^n(x) \subset \Phi$. Since $V_q^n(x) \subset \dot{x}$, we obtain $V_q^n(x) \subset \Phi \cap \dot{x}$ and so $x \in \pi_n(q)(\Phi \cap \dot{x})$. Also, $x \in \pi_n(q)(V_q^n(x)) = \pi_n(q)(V_{\pi_n(q)}(x))$. Thus, $\pi_n(q)$ is a pretopological convergence structure on X, which is called the *n-th pretopological modification* of q. Also, $(X, \pi_n(q))$ is called the *n-th pretopological modification* of (X, q).

PROPOSITION 4. *For $q \in C(X)$, $\cap\{V_q^n(x) \mid n \in N\} = V_q^\infty(x)$.*

PROOF. Let $A \in V_q^\infty(x)$. Then $x \in I_q^\infty(A)$ and so $x \in I_q^n(A)$ for all $n \in N$. Thus, $A \in V_q^n(x)$ for all $n \in N$.

Conversely, let $A \in V_q^n(x)$ for all $n \in N$. Then $A \in V_q^\infty(x)$. This completes the proof.

PROPOSITION 5. *Let $f: (X, q) \to (Y, p)$ be a map and $n \in N \cup \{\infty\}$. Then the following are equivalent:*

(a) $V_p^n(f(x)) = f(V_q^n(x))$ *for each* $x \in X$.

(b) $f^{-1}(I_p^n(B)) = I_q^n(f^{-1}(B))$ *for each* $B \subset Y$.

PROOF. First, assume that (a) is true, and let $x \in f^{-1}(I_p^n(B))$. Then $f(x) \in I_p^n(B)$ and so $B \in V_p^n(f(x)) = f(V_q^n(x))$. Thus, $f^{-1}(B) \in V_q^n(x)$ and so $x \in I_q^n(f^{-1}(B))$. Finally, $f^{-1}(I_p^n(B)) \subset I_q^n(f^{-1}(B))$. The reverse inequality is proved by the counter-order.

Next, assume that (b) is true, and let $B \in V_p^n(f(x))$. Then $f(x) \in I_p^n(B)$ and so $x \in f^{-1}(I_p^n(B)) = I_q^n(f^{-1}(B))$. Thus $f^{-1}(B) \in V_q^n(x)$ and so $B \in f(V_q^n(x))$. Finally, $V_p^n(f(x)) \subset f(V_q^n(x))$. The reverse inequality is proved by the counter-order. This completes the proof.

Let (X, q) be a convergence space, Y a nonempty set, and a map $f: (X, q) \to Y$ a surjection. The *convergence quotient structure* p on Y is defined by specifying that for any $y \in Y$ and $\Psi \in F(Y)$,

$$y \in p(\Psi) \text{ iff there exist } x \in f^{-1}(y) \text{ and } \Phi \in F(X) \text{ such that } \Psi \supset f(\Phi) \text{ and } x \in q(\Phi).$$

In this case, $f:(X,q) \to (Y,p)$ is called a *convergence quotient map* and the pair (Y,p) is called a *convergence quotient space*.

Kent [6] proved that for a surjection $f:(X,q) \to (Y,p)$, f is a convergence quotient map if and only if p is the finest convergence structure on Y relative to which f is continuous.

PROPOSITION 6 ([6]). *If $f:(X,q) \to (Y,p)$ is a convergence quotient map, then, for each $y \in Y$, $V_p(y) = \cap\{f(V_q(x)) \mid x \in f^{-1}(y)\}$.*

3. Main Results

THEOREM 7. *Let $f:(X,q) \to (Y,p)$ be a map. Then the following are equivalent:*

(a) $V_p(f(x)) = f(V_q(x))$.

(b) $V_p^n(f(x)) = f(V_q^n(x))$ *for each* $n \in N \cup \{\infty\}$

PROOF. It is clear that (b) implies (a). We will use the induction to prove that (a) implies (b). Assume that $V_p^k(f(x)) = f(V_q^k(x))$, and let $B \in V_p^{k+1}(f(x))$. Then $f(x) \in I_p^{k+1}(B) = I_p(I_p^k(B))$ and so $I_p^k(B) \in V_p(f(x)) = f(V_q(x))$. By the assumption and Proposition 5, $f^{-1}(I_p^k(B)) = I_q^k(f^{-1}(B)) \in V_q(x)$. Thus $x \in I_q(I_q^k(f^{-1}(B)) = I_q^{k+1}(f^{-1}(B))$ and so $f^{-1}(B) \in V_q^{k+1}(x)$. Finally, $B \in f(V_q^{k+1}(x))$. This means $V_p^{k+1}(f(x)) \subset f(V_q^{k+1}(x))$. The reverse inequality is proved by the counter-order.

In that case $n = \infty$, let $B \in V_p^\infty(f(x))$. Then $f(x) \in I_p^\infty(B)$ and so $f(x) \in I_p^n(B)$ for each $n \in N$. Thus $B \in V_p^n(f(x)) = f(V_q^n(x))$ for each $n \in N$. By Proposition 2, $B \in \cap\{f(V_q^n(x)) \mid n \in N\} = f(\cap\{V_q^n(x) \mid n \in N\}) = f(V_q^\infty(x))$. Finally, $V_p^\infty(f(x)) \subset f(V_q^\infty(x))$. The reverse inequality is proved by the counter-order. This completes the proof.

COROLLARY 8. *If $f:(X,q) \to (Y,p)$ is continuous, then for each $n \in N \cup \{\infty\}$, $f:(X,\pi_n(q)) \to (Y,\pi_n(p))$ is continuous.*

PROOF. It is clear that although "$=$" is replaced by "$\subset$" in the above Proposition 5 and Theorem 7, the statements are true. Consider that $\pi_n(q)$ is pretopological for each $n \in N \cup \{\infty\}$. Since $V_{\pi_n(p)}(f(x)) = V_p^n(f(x))$ and $V_{\pi_n(q)}(x) = V_q^n(x)$, by Proposition 3, the proof is complete.

THEOREM 9. *Let $f:(X,q) \to (Y,p)$ be continuous. Then the following hold:*

(1) *If q is pretopological and for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that $V_p(y) = f(V_q(x))$, then p is pretopological and $f:(X,q) \to (Y,p)$ is a convergence quotient map.*

(2) *If p is pretopological and $f:(X,q) \to (Y,p)$ is a convergence quotient map, then for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that $V_p(y) = f(V_q(x))$.*

PROOF. (1) Suppose that for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that $V_p(y) = f(V_q(x))$. Since q is pretopological, we obtain $x \in q(V_q(x))$. From the continuity of $f:(X,q) \to (Y,p)$, we obtain that $y = f(x) \in p(f(V_q(x))) = p(V_p(y))$ and so p is pretopological.

Let $f:(X,q) \to (Y,r)$ be a convergence quotient map. Then $p \leq r$. While, let $\Psi \in F(Y)$ and $y \in p(\Psi)$. Then $\Psi \supset V_p(y) = f(V_q(x))$ for some $x \in f^{-1}(y)$. Since $x \in q(V_q(x))$ and $f:(X,q) \to (Y,r)$ is a convergence quotient map, we obtain $y \in r(\Psi)$. Thus $p(\Psi) \subset r(\Psi)$ and so $p \geq r$. Finally, $p = r$. The proof is complete.

(2) Let $y \in Y$. Since p is pretopological, we obtain $y \in p(V_p(y))$. Since $f:(X,q) \to (Y,p)$ is a convergence quotient map, there exist $x \in f^{-1}(y)$ and $\Phi \in F(X)$ such that $V_p(y) \supset f(\Phi)$ and $x \in q(\Phi)$. Thus, $V_q(x) \subset \Phi$ and so $V_p(y) \supset f(V_q(x))$. Since $f:(X,q) \to (Y,p)$ is continuous, we obtain $V_p(y) \subset f(V_q(x))$. Finally, $V_p(y) = f(V_q(x))$. This completes the proof.

THEOREM 10. *If p is pretopological and $f:(X,q) \to (Y,p)$ is a convergence quotient map, then the following hold for each $n \in N \cup \{\infty\}$:*

(1) *For each $y \in Y$, there exists $x \in f^{-1}(y)$ such that $V_p^n(y) = f(V_q^n(x))$.*

(2) *$f:(X,\pi_n(q)) \to (Y,\pi_n(p))$ is a convergence quotient map.*

(3) *For each $y \in Y$, $V_p^n(y) = f(\cap\{V_q^n(x) \mid x \in f^{-1}(y)\})$.*

PROOF. (1) By Corollary 8, $f:(X,\pi_n(q)) \to (Y,\pi_n(p))$ is continuous. Since $f:(X,q) \to (Y,p)$ is a convergence quotient map and p is pretopological, by Theorem 9 (2), for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that $V_p(y) = f(V_q(x))$. Thus, by Theorem 7, $V_p^n(y) = f(V_q^n(x))$ for each $n \in N \cup \{\infty\}$.

(2) Since $V_{\pi_n(p)}(y) = f(V_{\pi_n(q)}(x))$ and $\pi_n(q)$ is pretopological, by Theorem 9 (1), $f:(X,\pi_n(q)) \to (Y,\pi_n(p))$ is a convergence quotient map.

(3) By the above (2) and Proposition 6, the proof is complete.

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