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PROPERTIES OF WEAKLY STAR REDUCIBLE SPACES

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ABSTRACT. We show that every ultrapure space is weakly star reducible, and that every countably compact weakly star reducible space is compact. We also pose open problems

1. Introduction

In the present paper, we are interested in studying weak covering properties in the presence of a countable compact condition.

A space X is said to be *isocompact* if every closed countably compact subset of X is compact. The most obvious example of isocompact spaces is a Lindelöf space. Among the classes of spaces having the isocompactness property are neighborhood $\mathcal{F}$-spaces, ([6]), spaces satisfying property θL ([5]), weak $[\omega_1, \infty)^r$-refinable spaces ([12]), $\delta\theta$-penetrable spaces ([3]), and pure spaces ([1]).

In [10], Masami Sakai introduced a new large class of isocompact spaces, called "κ-neat spaces". This class contains all of the above mentioned classes.

In [14], Wicke and Worrell defined a covering property, called star reducible, possessed by all $\delta\theta$-refinable countably subparacompact spaces (Remark 1.4. in [14]) and also introduced weak star reducibility which is obviously weaker than star reducibility.

The purpose of this paper is to show that every ultrapure space is weakly star reducible (Theorem 3.1), and that every countably compact weakly star reducible space is compact (Theorem 3.4).

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This paper is organized as follows: Section 1 is an introduction. Section 2 consists of preliminaries which involve definitions and basic implications of weak covering properties. Section 3 is devoted to major results and their related problems.

Throughout this paper, we use the following notation: For any set $A \subset X$ and a collection $\mathcal{U}$ of subsets of X, $st(A,\mathcal{U})$ (the *star* of $\mathcal{U}$ about A) denotes the set $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

If $x \in A$, $st(\{x\},\mathcal{U})$ is simply denoted by $st(x,\mathcal{U})$.

$ord(x,\mathcal{U}) = |\{U \in \mathcal{U} : x \in U\}|$,

$[\mathcal{U}]^{<\omega} = \{\mathcal{K} \subset \mathcal{U} : \mathcal{K}$ is finite $\}$, and $[\mathcal{U}]^{\omega} = \{\mathcal{K} \subset \mathcal{U} : \mathcal{K}$ is countable $\}$.

2. Preliminaries

We establish some convenient terminology used throughout the rest of this paper. As far as topological concepts are concerned, we follow [7].

DEFINITION 2.1. ([2], [12]) A space X is said to be *weakly θ-refinable* (resp. *weakly $\delta\theta$-refinable*) if for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$ of $\mathcal{U}$ such that if $x \in X$ there is some $n \in \omega$ with $0 < ord(x,\mathcal{G}_n) < \omega$ (resp. $0 < ord(x,\mathcal{G}_n) \leq \omega$).

Moreover, if each $\mathcal{G}_n$ covers X, then X is said to be *θ-refinable* (resp. *$\delta\theta$-refinable* or *submeta-Lindelöf*).

DEFINITION 2.2. ([1]) A countable family $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ of collections of subsets of a space X is called an *interlacing* on X if $\bigcup\bigcup\mathcal{V} = X$ and for each $n \in \omega$, each $V \in \mathcal{V}_n$ is open in $\bigcup\mathcal{V}_n$.

An interlacing $\mathcal{V}$ is called *suspended* (resp. *δ-suspended*) from a family $\mathcal{H}$ of subsets of a space X if for every $n \in \omega$ and $x \in \bigcup\mathcal{V}_n$, there is a finite family $\mathcal{K} \in [\mathcal{H}]^{<\omega}$ (resp. a countable family $\mathcal{K} \in [\mathcal{H}]^{\omega}$) such that $st(x,\mathcal{V}_n) \cap (\bigcap\mathcal{K}) = \emptyset$.

A space X is called *pure* (resp. *ultrapure*,) if for each free closed ultrafilter (resp. free closed collection,) $\mathcal{F}$ on X there is an interlacing which is δ-suspended from $\mathcal{F}$.

REMARK 2.3. (1) Note that ultrapure implies pure.

(2) In the case of ultrafilters with the countable intersection property (c.i.p.), the terms suspended and δ-suspended coincide ([8]).

The following theorem is due to Arhangel'skiĭ.

THEOREM 2.4. *Every weakly $\delta\theta$-refinable space X is ultrapure.*

PROOF. See ([1]).

The following definition is a covering property which is weaker than weakly $\delta\theta$-refinable.

DEFINITION 2.5. ([12]) A space X is said to be *weakly* $[\omega_1, \infty)^r$-*refinable* if for any open cover $\mathcal{U}$ of X, of uncountable regular cardinality, there exists an open refinement which can be expressed as $\bigcup\{\mathcal{G}_\alpha : \alpha \in \Gamma\}$, where $|\Gamma| < |\mathcal{U}|$ and if $x \in X$ there is some $\alpha \in \Gamma$ such that $0 < ord(x, \mathcal{G}_\alpha) < |\mathcal{U}|$.

THEOREM 2.6. *([12]) If X is countably compact, weakly $[\omega_1, \infty)^r$-refinable, then X is compact.*

Theorem 2.6 says that weakly $[\omega_1, \infty)^r$-refinable spaces are isocompact.

There are other weak covering properties which imply isocompactness. For example, Davis in [5] studied 'property θL' and showed that this property generalizes weakly $\delta\theta$-refinability and implies isocompactness. For other conditions which force a countably compact space to be compact, see [11].

THEOREM 2.7. ([1]) *Every countably compact, pure space is compact.*

PROOF. See [11].

Theorem 2.7 shows that every pure space is isocompact.

DEFINITION 2.8. ([14]) A cover $\mathcal{U}$ of a space X is called *regularly rigid* if no subcollection of $\mathcal{U}$ of cardinality less than $|\mathcal{U}|$ covers X and $|\mathcal{U}|$ is regular or $1 < |\mathcal{U}| < \omega$.

DEFINITION 2.9. ([14]) A space X is called *star reducible* if for every regularly rigid open cover $\mathcal{H}$ of X, there exists a sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of open covers of X such that for all $p \in X$ there exist $n \in \omega$ and $\mathcal{H}' \subset \mathcal{H}$ such that $|\mathcal{H}'| < |\mathcal{H}|$ and $\mathcal{H}'$ covers $st(p, \mathcal{G}_n)$.

The following definition is essentially based on Definition 4.8 in [14].

DEFINITION 2.10. A space X is called *weakly star reducible* if for every uncountable regularly rigid open cover $\mathcal{U}$ of X there is a collection $\mathcal{V}$ of collections of subsets of X such that:

(i) $\bigcup\bigcup\mathcal{V} = X$,

(ii) $|\mathcal{V}| < |\mathcal{U}|$,

(iii) for all $\mathcal{G} \in \mathcal{V}$ and for all $G \in \mathcal{G}$, G is open in $\bigcup\mathcal{G}$, and

(iv) for all $p \in X$, there exist $\mathcal{G} \in \mathcal{V}$ and $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| < |\mathcal{U}|$ and $st(p, \mathcal{G}) \subset \bigcup\mathcal{U}'$.

REMARK 2.11. (1) Weakly star reducibility is obviously weaker than star reducibility.

(2) Every developable space is star reducible (Remark 1.5 in [14]) and thus weakly star reducible.

Define for each free closed ultrafilter $\mathcal{H}$ on X with c.i.p., $\lambda(\mathcal{H}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{H}, \bigcap\mathcal{F} = \emptyset\}$. Note that $\lambda(\mathcal{H})$ is an uncountable regular cardinal.

DEFINITION 2.12. ([10]) Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p. and κ be a cardinal number. A system $\langle\{X_\gamma\}, \{\mathcal{V}_\gamma\}, \{f_\gamma\}\rangle_{\gamma\in\Gamma}$ is called a *κ-neat system* for $\mathcal{H}$ if the following are satisfied:

(1) $|\Gamma| < \lambda(\mathcal{H})$.

(2) $\{X_\gamma\}_{\gamma\in\Gamma}$ is a cover of X and $\mathcal{V}_\gamma$ is an open collection of X such that $X_\gamma \subset \bigcup\mathcal{V}_\gamma$ for each $\gamma \in \Gamma$.

(3) Each f_γ is a function from X_γ to $\mathcal{V}_\gamma$ such that if $A \subset X_\gamma$, $|A| \leq \kappa$ and $f_\gamma|A$ is injective, then the closure of A in $\bigcup\mathcal{V}_\gamma$ is contained in $\bigcup_{x\in A} f_\gamma(x)$.

(4) For each $\gamma \in \Gamma$ and $x \in X_\gamma$ there exists $H \in \mathcal{H}$ such that $f_\gamma(x) \cap X_\gamma \cap H = \emptyset$.

A space X is called a *κ-neat space* if for each free closed ultrafilter $\mathcal{H}$ on X with c.i.p. there exists a κ-neat system for $\mathcal{H}$. An ω-neat space is merely called a *neat space*.

PROPOSITION 2.13. *(Proposition 2.3 in [10])*
The following spaces are neat. Moreover, the implications (a) $\to$ (b) and (d) $\to$ (e) hold.

(a) neighborhood $\mathcal{F}$-spaces.

(b) spaces satisfying property θL.

(c) weakly $[\omega_1, \infty)^r$-refinable spaces.
(d) $\delta\theta$-penetrable spaces.
(e) pure spaces.

THEOREM 2.14. *(Theorem 2.6 in [10]) Every neat space is isocompact.*

3. Main theorems and related problems

THEOREM 3.1. *Every ultrapure space X is weakly star reducible.*

PROOF. Let $\mathcal{U}$ be an uncountable regularly rigid open cover of X and let $\mathcal{F} = \{X - U : U \in \mathcal{U}\}$. Then $\mathcal{F}$ is a closed collection on X with $\bigcap \mathcal{F} = \emptyset$.

Since X is ultrapure, there is a countable collection $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ of collections of subsets of X such that

(a) $\bigcup\bigcup \mathcal{V} = X$,

(b) for all $n \in \omega$ and $V \in \mathcal{V}_n$, V is open in $\bigcup \mathcal{V}_n$, and

(c) for all $n \in \omega$ and $x \in \bigcup \mathcal{V}_n$, there exists $\mathcal{K} \in [\mathcal{F}]^\omega$ such that $st(x, \mathcal{V}_n) \cap (\bigcap \mathcal{K}) = \emptyset$.

To claim that this $\mathcal{V}$ works, we need to check conditions (i), (ii), (iii), and (iv) of Definition 2.10.

The condition (i) is just (a) above. Clearly, $|\mathcal{V}| < |\mathcal{U}|$ and thus (ii) is satisfied. (iii) follows directly from (b). To check (iv), let $x \in X$. Then $x \in \bigcup \mathcal{V}_n$ for some $\mathcal{V}_n \in \mathcal{V}$ (since $\bigcup\bigcup \mathcal{V} = X$ by (a)). So by (c) there exists a countable subcollection $\mathcal{K}$ of $\mathcal{F}$ such that $st(x, \mathcal{V}_n) \cap (\bigcap \mathcal{K}) = \emptyset$, i.e., $st(x, \mathcal{V}_n) \subset X - (\bigcap \mathcal{K})$. Hence there exists a countable subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that $st(x, \mathcal{V}_n) \subset \bigcup \mathcal{U}'$. Therefore X is weakly star reducible.

This proves Theorem 3.1.

COROLLARY 3.2. *Every weakly $\delta\theta$-refinable space X is weakly star reducible.*

PROOF. It directly follows from Theorem 2.4 and Theorem 3.1.

PROPOSITION 3.3. *Every closed subspace of a weakly star reducible space is weakly star reducible.*

PROOF. Suppose that X is weakly star reducible and $F \subset X$ is closed. Let $\mathcal{K}$ be an uncountable regularly rigid open cover of F. For each $K \in \mathcal{K}$ choose an open set O_K in X such that $O_K \cap F = K$. Then $\{O_K : K \in \mathcal{K}\} \cup \{X - F\}$ is an uncountable regularly rigid open cover of X. Since X is weakly star reducible, there is a collection $\mathcal{V}$ of collections of subsets of X satisfying the conditions of Definition 2.10. Then $\mathcal{V}|F = \{\mathcal{G}|F : \mathcal{G} \in \mathcal{V}\}$, where $\mathcal{G}|F = \{G \cap F : G \in \mathcal{G}\}$, is the desired collection.

THEOREM 3.4. *Every countably compact weakly star reducible space is compact.*

PROOF. Suppose X is a countably compact weakly star reducible space which is not compact. Then there is an open cover $\mathcal{U}$ of X of minimal cardinality κ such that $\mathcal{U}$ has no finite subcover.

Note that $\kappa > \omega$ [because X is countably compact]. Also if $\gamma < \kappa$ and $\mathcal{U}' \subset \mathcal{U}$ with $|\mathcal{U}'| = \gamma$, then $\mathcal{U}'$ does not cover X.

CLAIM 1. κ is regular, i.e., $cf(\kappa) = \kappa$.

Suppose $cf(\kappa) = \lambda < \kappa$. Enumerate $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$. Then there exists an increasing function $f : \lambda \to \kappa$ such that $f(\alpha) \nearrow \kappa$ for every $\alpha < \lambda$. Let $W_\alpha = \bigcup\{U_\beta : \beta < f(\alpha)\}$. Then $\mathcal{W} = \{W_\alpha : \alpha < \lambda\}$ is an open cover of X and $\lambda < \kappa$ implies that $\mathcal{W}$ has a finite subcover $\{W_{\alpha_0}, W_{\alpha_1}, \ldots, W_{\alpha_n}\}$. Assume $\alpha_n > \alpha_i$ for $i < n$ so that $W_{\alpha_0} \subset W_{\alpha_1} \subset \ldots \subset W_{\alpha_n}$. This says that $X \subset W_{\alpha_n}$. So $\{U_\beta : \beta < f(\alpha_n)\}$ is a cover of X of cardinality $f(\alpha_n) < \kappa$ which is impossible. Hence $cf(\kappa) = \kappa$, proving Claim 1.

Hence $\mathcal{U}$ is an uncountable regularly rigid open cover of X. Using weak star reducibility, there is a collection $\mathcal{V}$ of collections of subsets of X such that

(1) $|\mathcal{V}| < \kappa$,

(2) for every $\mathcal{G} \in \mathcal{V}$ and for every $G \in \mathcal{G}$, G is open in $\bigcup \mathcal{G}$,

(3) for every $x \in X$, there exist $\mathcal{G} \in \mathcal{V}$ and $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| < \kappa$ and $st(x, \mathcal{G}) \subset \bigcup \mathcal{U}'$.

Now let $F_\alpha = X - U_\alpha$ for each U_α and $\alpha < \kappa$, and let $\mathcal{F}_0 = \{F_\alpha : \alpha < \kappa\}$. Then $\mathcal{F}_0$ is a centered family of closed sets such that $\bigcap \mathcal{F}_0 = \emptyset$. So there exists a free closed ultrafilter $\mathcal{F}$ on X such that $\mathcal{F}_0 \subset \mathcal{F}$. Let $\lambda(\mathcal{F}) = \min\{|\mathcal{F}'| : \mathcal{F}' \subset \mathcal{F}, \bigcap \mathcal{F}' = \emptyset\}$. Then $\lambda(\mathcal{F}) = \kappa$.

(*) Note that if $\gamma < \lambda(\mathcal{F})$, $\mathcal{F}' \subset \mathcal{F}$, and $|\mathcal{F}'| = \gamma$, then $\bigcap \mathcal{F}' \in \mathcal{F}$. [If $\gamma < \omega$, then clearly $\bigcap \mathcal{F}' \in \mathcal{F}$ as $\mathcal{F}$ a filter. So we may assume that $\gamma \geq \omega$. Choose $A \in \mathcal{F} - \mathcal{F}'$. Then $\mathcal{F}' \cup \{A\}$ has cardinality γ. By the definition of $\lambda(\mathcal{F})$, $\bigcap(\mathcal{F}' \cup \{A\}) \neq$. Hence $\bigcap \mathcal{F}' \in \mathcal{F}$.]

Enumerate $\mathcal{V} = \{\mathcal{G}_\beta : \beta < \gamma\}$, where $\gamma < \kappa$. For each $\beta < \gamma$, we let

$$C_\beta = \{x \in X : x \in \bigcup \mathcal{G}_\beta, st(x, \mathcal{G}_\beta) \subset \bigcup \mathcal{U}' \text{for some} \mathcal{U}' \subset \mathcal{U} \text{with} |\mathcal{U}'| < \kappa\}.$$

Claim 2. There is some $\beta < \gamma$ such that C_β intersects each element of $\mathcal{F}$.

If not, then for each $\alpha < \gamma$ there is an $F_\alpha \in \mathcal{F}$ such that $C_\alpha \cap F_\alpha = \emptyset$. Since $\gamma < \kappa$, $\bigcap\{F_\alpha : \alpha < \gamma\} \in \mathcal{F}$ by the above (*). Since $C_\beta \cap \bigcap\{F_\alpha : \alpha < \gamma\} = \emptyset$ for all $\beta < \gamma$, it follows that $\bigcap\{F_\alpha : \alpha < \gamma\} = \emptyset$, a contradiction. *This proves Claim 2.*

Let C_η intersect each element of $\mathcal{F}$. For each $G \in \mathcal{G}_\eta$, let $W(G)$ be an open set in X such that $W(G) \cap (\bigcup \mathcal{G}_\eta) = G$ (using (2)). Put $\mathcal{W}_\eta = \{W(G) : G \in \mathcal{G}_\eta\}$.

Claim 3. There exists an $F \in \mathcal{F}$ such that $F \subset \bigcup \mathcal{W}_\eta$.

If not, then for every $F \in \mathcal{F}$ $F \cap (X - \bigcup \mathcal{W}_\eta) \neq \emptyset$. So $X - \bigcup \mathcal{W}_\eta \in \mathcal{F}$. Since $C_\eta \subset \bigcup \mathcal{G}_\eta$ and $\bigcup \mathcal{G}_\eta \subset \bigcup \mathcal{W}_\eta$, we have $C_\eta \cap (X - \bigcup \mathcal{W}_\eta) \subset C_\eta \cap (X - \bigcup \mathcal{G}_\eta) = \emptyset$. So $C_\eta \cap (X - \bigcup \mathcal{W}_\eta) = \emptyset$, a contradiction. *This proves Claim 3.*

We now work with the set F above and look at $F \cap C_\eta \neq \emptyset$. By [9, Theorem 18, p. 8], there exists a set $D \subset F \cap C_\eta$ such that

(a) no member of $\mathcal{G}_\eta$ contains two distinct points of D

(b) $F \cap C_\eta \subset \bigcup\{st(x, \mathcal{G}_\eta) : x \in D\} = st(D, \mathcal{G}_\eta)$.

Claim 4. D is a closed discrete subset of X

i) By (a), D is discrete since for each $G \in \mathcal{G}_\eta$, G is open in $\bigcup \mathcal{G}_\eta$ (by (2)).

ii) D is closed in X; if x were an accumulation point of D, then since $x \in F \subset \bigcup \mathcal{W}_\eta$, there is $G \in \mathcal{G}_\eta$ such that $x \in W(G)$. Since $W(G)$ is open in X, it must contain infinitely many points of D, but since $W(G) \cap D = G \cap D$, we have $|G \cap D| \geq \omega$ for some $G \in \mathcal{G}_\eta$, which contradicts the fact that each $G \in \mathcal{G}_\eta$ contains at most one element of D. So $D = cl(D)$, which shows that D is closed. *This proves Claim 4.*

Since X is countably compact, the closed discrete set D must be finite. Since $D \subset C_\eta$, κ is regular, and $F \cap C_\eta \subset \bigcup\{st(x, \mathcal{G}_\eta) : x \in D\}$ by (b), we have that some subcollection $\mathcal{U}'' \subset \mathcal{U}$ covers $F \cap C_\eta$ and $|\mathcal{U}''| < \kappa$ (using (3)). This implies that $\bigcap\{F_\delta : \delta < \alpha\} \cap F \cap C_\eta = \emptyset$ for some $\alpha < \kappa$. Since $\bigcap\{F_\delta : \delta < \alpha\} \cap F \in \mathcal{F}$, this yields a contradiction.

This completes the proof of Theorem 3.4.

COROLLARY 3.5. *Every weakly star reducible space is isocomapct.*

PROOF. It directly follows from Proposition 3.3 and Theorem 3.4.

QUESTION 3.6. *Is every weakly $[\omega_1, \infty)^r$-refinable space weakly star reducible.*

QUESTION 3.7. *When does weak star reducibility imply α-realcompactness (= closed complete)?*

It is known from [2] that if X is T_1 and weakly $\delta\theta$-refinable (resp. weakly θ-refinable), and if every closed discrete subset of X is countable (resp. of nonmeasurable cardinality), then X is α-realcompact.

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