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A SOLUTION OF EINSTEIN'S UNIFIED FIELD EQUATIONS

JONG WOO LEE AND KYUNG TAI CHUNG

ABSTRACT. In this paper, we obtain a solution of Einstein's unified field equations on a generalized n-dimensional Riemannian manifold X_n.

1. Introduction

Einstein [2] proposed a new unified field theory that would include both gravitation and electromagnetism. The intent of this theory may be characterized as a set of geometrical postulates for the space-time X_4. The geometrical consequences of these postulates have been developed very far by a number of mathematicians, such as Chung [1], Eisenhart [3], Geroch [4], Hlavatý [5], Mishra [8], Wrede [9], etc. In particular, Hlavatý [5] gave its mathematical foundation, and Wrede [9] studied Principles A and B of this theory on an n-dimensional generalized Riemannian manifold X_n, for the first time. Recently, Chung [1] and Lee [6,7] introduced the concepts of SE-connection $\Gamma^{\nu}_{\lambda\mu}$ and SE(k)-connection $\Gamma^{\nu}_{\lambda\mu}$ in a simple and useful form, and studied Einstein's unified field equations.

2. Preliminaries

This section is a brief collection of basic concepts and results which are needed in our subsequent considerations in the present paper.

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U;x^{\nu}\}$, where, here and

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in the sequel, Greek indices run over the range {1,2, ..,n } and follow the summation convention. The manifold X_n is endowed with a general real non-symmetric tensor $g_{\lambda\mu}$, called the *basic tensor*, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$(2.1) \qquad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where we assume that

$$(2.2) \qquad (a) \quad det(g_{\lambda\mu}) \neq 0 \quad , \quad (b) \quad det(h_{\lambda\mu}) \neq 0.$$

We may define a unique tensor $h^{\lambda\nu}=h^{\nu\lambda}$ by

$$(2.3) \qquad h_{\lambda\mu}h^{\lambda\nu} = \delta_{\mu}{}^{\nu}.$$

The tensors $h_{\lambda\mu}$ *and* $h^{\lambda\nu}$ *will serve for raising and/or lowering indices of tensors on* X_n *in the usual manner.* The manifold X_n is assumed to be connected by a general real connection $\Gamma^{\nu}_{\lambda\mu}$ which may also be decomposed into its symmetric part $\Lambda^{\nu}_{\lambda\mu}$ and skew-symmetric part $S_{\lambda\mu}{}^{\nu}$, called the *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$:

$$(2.4) \quad \Lambda^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{(\lambda\mu)} = \frac{1}{2}(\Gamma^{\nu}_{\lambda\mu} + \Gamma^{\nu}_{\mu\lambda}), \quad S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} = \frac{1}{2}(\Gamma^{\nu}_{\lambda\mu} - \Gamma^{\nu}_{\mu\lambda}).$$

Einstein's unified field theory on X_n is governed by the following set of equations:

$$(2.5) \qquad \partial_{\nu} g_{\lambda\mu} - g_{\alpha\mu}\Gamma^{\alpha}_{\lambda\nu} - g_{\lambda\alpha}\Gamma^{\alpha}_{\nu\mu} = 0 \quad (\partial_{\nu} = \frac{\partial}{\partial x^{\nu}}),$$

and

$$(2.6) \qquad (a)\ S_{\lambda} = S_{\lambda\alpha}{}^{\alpha} = 0, \quad (b)\ R_{[\lambda\mu]} = \partial_{[\lambda}P_{\mu]}, \quad (c)\ R_{(\lambda\mu)} = 0,$$

where P_{μ} is an arbitrary vector, called the *Einstein vector*, and $R_{\lambda\mu}$ is the contracted curvature tensor $R^{\alpha}_{\lambda\mu\alpha}$ of the curvature tensor $R^{\omega}_{\lambda\mu\nu}$:

$$(2.7) \qquad R^{\omega}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\omega}_{\lambda\nu} - \partial_{\nu}\Gamma^{\omega}_{\lambda\mu} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\omega}_{\alpha\mu} - \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\omega}_{\alpha\nu}.$$

It has been shown by Hlavatý [5] that if the equation (2.5) admits a solution $\Gamma^{\nu}_{\lambda\mu}$, then this solution must be of the form:

$$\Gamma^{\nu}_{\lambda\mu} = \{\lambda{}^{\nu}{}_{\mu}\} + U^{\nu}_{\lambda\mu} + S_{\lambda\mu}{}^{\nu}, \tag{2.8}$$

where $\{\lambda{}^{\nu}{}_{\mu}\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$, and

$$U^{\nu}_{\lambda\mu} = 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}. \tag{2.9}$$

3. A solution of (2.5)

The equations (2.8) and (2.9) reduce the investigation of $\Gamma^{\nu}_{\lambda\mu}$ to the study of its torsion tensor $S_{\lambda\mu}{}^{\nu}$. Hence in order to know the connection $\Gamma^{\nu}_{\lambda\mu}$ in (2.5), it is necessary and sufficient to know the tensor $S_{\lambda\mu}{}^{\nu}$.

THEOREM 3.1. *If the system (2.5) admits a solution $\Gamma^{\nu}_{\lambda\mu}$ on X_n such that its torsion tensor $S_{\lambda\mu}{}^{\nu}$ is of the form*

$$S_{\lambda\mu}{}^{\nu} = k_{\lambda\mu}Y^{\nu} \tag{3.1}$$

for some nonzero vector Y_{ν}, then it must be of the form

$$\Gamma^{\nu}_{\lambda\mu} = \{\lambda{}^{\nu}{}_{\mu}\} - 2k_{(\lambda}{}^{\nu}Z_{\mu)} + k_{\lambda\mu}Y^{\nu}, \tag{3.2}$$

where $\{\lambda{}^{\nu}{}_{\mu}\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$, and

$$Z_{\mu} = k_{\mu\alpha}Y^{\alpha}. \tag{3.3}$$

PROOF. Since the system (2.5) admits a solution on X_n, it is of the form (2.8). Since its torsion tensor $S_{\lambda\mu}{}^{\nu}$ is of the form (3.1), the tensor (2.9) is given by

$$U^{\nu}_{\lambda\mu} = h^{\nu\alpha}(k_{\alpha\lambda}Y^{\beta}k_{\mu\beta} + k_{\alpha\mu}Y^{\beta}k_{\lambda\beta}) = -2k_{(\lambda}{}^{\nu}Z_{\mu)} \tag{3.4}$$

making use of (3.3). Substituting (3.1) and (3.4) into (2.8), we obtain (3.2).

REMARK 3.2. In virtue of Theorem 3.1, in order to know the connection $\Gamma^{\nu}_{\lambda\mu}$ in the system (2.5), it is necessary and sufficient to know the vector Y_ν which defines the connection (3.2).

THEOREM 3.3. *The connection (3.2) on X_n is a solution of the system (2.5) if and only if the vector Y_ν on X_n satisfies the following condition*

$$\nabla_\nu k_{\lambda\mu} = -2k_{\nu[\lambda}Y_{\mu]} + 2\ {}^{(2)}k_{\nu[\lambda}Z_{\mu]}, \tag{3.5}$$

where ∇_ν is the symbolic vector of the covariant derivative with respect to $\{\lambda{}^\nu{}_\mu\}$, Z_μ is given by (3.3), and ${}^{(2)}k_{\lambda\mu} = k_\lambda{}^\alpha k_{\alpha\mu}$.

PROOF. Substituting (2.1) and (3.2) into (2.5), and making use of $\nabla_\nu h_{\lambda\mu} = 0$, we obtain

$$\nabla_\nu k_{\lambda\mu} + 2k_{\nu[\lambda}Y_{\mu]} - 2\ {}^{(2)}k_{\nu[\lambda}Z_{\mu]} = 0 \tag{3.6}$$

by a straightforward computation. Hence the connection (3.2) is a solution of (2.5) if and only if the vector Y_ν satisfies (3.5).

LEMMA 3.4. *The curvature tensor $R^{\omega}_{\lambda\mu\nu}$ defined by the connection (2.8) is given by*

$$\begin{aligned} R^{\omega}_{\lambda\mu\nu} &= H^{\omega}_{\lambda\mu\nu} + \nabla_\mu U^{\omega}_{\lambda\nu} - \nabla_\nu U^{\omega}_{\lambda\mu} + \nabla_\mu S_{\lambda\nu}{}^\omega - \nabla_\nu S_{\lambda\mu}{}^\omega \\ &+ U^{\alpha}_{\lambda\nu}U^{\omega}_{\alpha\mu} - U^{\alpha}_{\lambda\mu}U^{\omega}_{\alpha\nu} - S_{\alpha\nu}{}^\omega U^{\alpha}_{\lambda\mu} + S_{\alpha\mu}{}^\omega U^{\alpha}_{\lambda\nu} \\ &+ S_{\lambda\nu}{}^\alpha U^{\omega}_{\alpha\mu} - S_{\lambda\mu}{}^\alpha U^{\omega}_{\alpha\nu} + S_{\lambda\nu}{}^\alpha S_{\alpha\mu}{}^\omega - S_{\lambda\mu}{}^\alpha S_{\alpha\nu}{}^\omega, \end{aligned} \tag{3.7}$$

where $H^{\omega}_{\lambda\mu\nu}$ is the Riemannian curvature tensor defined by $\{\lambda{}^\nu{}_\mu\}$.

PROOF. Substituting (2.8) into (2.7), we obtain (3.7) by a straightforward computation.

THEOREM 3.5. *Suppose that the condition (3.5) is satisfied on X_n. Then the curvature tensor $R^{\omega}_{\lambda\mu\nu}$ defined by the connection (3.2) is given by*

$$\begin{aligned} R^{\omega}_{\lambda\mu\nu} &= H^{\omega}_{\lambda\mu\nu} - 2k_{\lambda[\mu}A_{\nu]}{}^\omega + 2k_\lambda{}^\omega\nabla_{[\mu}Z_{\nu]} - 2k_{[\mu}{}^\omega B_{\nu]\lambda} \\ &- 2\ {}^{(2)}k_{\lambda[\mu}Z_{\nu]}Z^\omega + 2\ {}^{(2)}k_{[\mu}{}^\omega Z_{\nu]}Z_\lambda + 2k_{\mu\nu}(Y_\lambda Y^\omega + Z_\lambda Y^\omega), \end{aligned} \tag{3.8}$$

where

$$(3.9)\qquad (a)\quad A_\nu{}^\omega = \nabla_\nu Y^\omega - Y_\nu Y^\omega + Z^\omega Z_\nu,$$
$$(b)\quad B_{\mu\lambda} = \nabla_\mu Z_\lambda - 2Z_{(\mu}Y_{\lambda)} + 2Z_{(\mu}k_{\lambda)}{}^\alpha Z_\alpha.$$

PROOF. Substituting (3.1) and (3.4) into (3.7), and making use of (3.5), (3.9), and

$$(3.10)\qquad Y_\alpha Z^\alpha = k_{\alpha\beta}Y^\alpha Y^\beta = 0,$$

we obtain (3.8) by a straightforward computation.

4. A solution of (2.5) and (2.6)

In this section, we shall display a particular solution of (2.5) and (2.6) on X_n $(n \geq 3)$. Assume that $((h_{\lambda\mu}))$ is defined by an $n \times n$ diagonal matrix

$$(4.1)\qquad ((h_{\lambda\mu})) = \begin{pmatrix} +1 & 0 & \dots & 0 & 0 \\ 0 & +1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & +1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix}.$$

Then all the Christoffel symbols $\{{}_\lambda{}^\nu{}_\mu\}$ vanish. Define two n-vectors on X_n by

$$(4.2)\qquad (a)\quad Y_\lambda : (0, \dots, 0, 1, -1), \qquad (b)\quad W_\lambda : (e^t, 0, \dots, 0),$$

where $t = x^{n-1} - x^n$. Then they satisfy the conditions

$$(4.3)\quad (a)\ Y_\lambda Y^\lambda = Y_\lambda W^\lambda = 0, \quad (b)\ \nabla_\lambda Y_\mu = 0, \quad (c)\ \nabla_\lambda W_\mu = Y_\lambda W_\mu,$$

Now, define a basic tensor on X_n by

$$(4.4)\qquad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where $h_{\lambda\mu}$ is defined by (4.1) and $k_{\lambda\mu}$ is defined by

$$k_{\lambda\mu} = 2Y_{[\lambda}W_{\mu]}. \tag{4.5}$$

Then we obtain, in virtue of (4.3),

$$k_{\lambda\alpha}Y^{\alpha} = 0 \tag{4.6}$$

$$\nabla_{\omega}k_{\lambda\mu} = 2Y_{\omega}Y_{[\lambda}W_{\mu]} = -2k_{\omega[\lambda}Y_{\mu]}. \tag{4.7}$$

With the vector Y_{ν} defined by (4.2)(a), define a connection on X_n by (3.2). Since the condition (3.5) is satisfied on X_n in virtue of (4.6) and (4.7), the connection (3.2) is a solution of the system (2.5), and this solution reduces in our case to

$$\Gamma^{\nu}_{\lambda\mu} = k_{\lambda\mu}Y^{\nu} = 2Y_{[\lambda}W_{\mu]}Y^{\nu}. \tag{4.8}$$

Furthermore, in virtue of (4.6), its torsion tensor $S_{\lambda\mu}{}^{\nu}$ satisfies

$$S_{\lambda} = S_{\lambda\alpha}{}^{\alpha} = k_{\lambda\alpha}Y^{\alpha} = 0. \tag{4.9}$$

Hence the system (2.6)(a) is satisfied automatically. Next, substituting (4.6) into (3.8), and making use of (4.3)(b) and $H^{\omega}_{\lambda\mu\nu} = 0$, the curvature tensor $R^{\omega}_{\lambda\mu\nu}$ defined by the connection (4.8) is given by

$$R^{\omega}_{\lambda\mu\nu} = 2k_{\lambda[\mu}Y_{\nu]}Y^{\omega} + 2k_{\mu\nu}Y_{\lambda}Y^{\omega}. \tag{4.10}$$

Contracting ω and ν in (4.10), and making use of (4.3)(a) and (4.6), its contracted curvature $R_{\lambda\mu}$ is given by

$$R_{\lambda\mu} = 0, \tag{4.11}$$

which implies that

$$(a)\quad R_{[\lambda\mu]} = 0, \quad (b)\quad R_{(\lambda\mu)} = 0. \tag{4.12}$$

Hence the system (2.6)(b) is satisfied by

$$P_{\lambda} = \partial_{\lambda}P, \tag{4.13}$$

and the system (2.6)(c) is satisfied automatically.

Consequently, the Einstein's unified field equations on X_n are satisfied by a basic tensor (4.4), a connection (4.8), and an Einstein vector (4.13).

References

1. Chung, K. T. and Cho, J. H., *On n-dimensional SE-manifold and its conformal change*, Nuovo Cimento **100B (4)** (1987), 537-550.
2. Einstein, A., *The meaning of relativity*, Princeton University Press, Princeton, New Jersey, 1950.
3. Eisenhart, L. P., *Generalized Riemannian spaces and general relativity*, Proceedings of the National Academy of Sciences, U.S.A. **39** (1953), 546-551.
4. Geroch, R. P., *Space-time structure from a global relativity and cosmology*, Academic Press, New York, 1971.
5. Hlavatý, V., *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd., 1957.
6. Lee, J. W., *Field Equations on SE(k)-manifold X_n*, International journalof Theoretical Physics **30** (1991), 1343-1353.
7. ———, *Geometric studies of n-dimensional E(k)-manifold*, Tensor, N. S. **51** (1992), 78-82.
8. Mishra, R. S., *Recurrence relations in Einstein's unified field theory*, Tensor, N. S. **12** (1962), 90-96.
9. Wrede, R. C., *n-dimensional considerations of the basic principles A and B of the unified theory of relativity*, Tensor **8** (1958), 95-122.

Jong Woo Lee
Department of Mathematics
Yonsei University
Wonju Kangwondo 222-701, Korea

Kyung Tae Chung
Department of Mathematics
Yonsei University
Seoul 120-749, Korea