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SOME ANALYSIS ON THE SUBMANIFOLDS OF MEX_n

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ABSTRACT. The purpose of this paper is to investigate a necessary and sufficient condition for submanifold of MEX_n to be einstein and to derive the generalized fundamental equations on the submanifold of MEX_n

1. Introduction

In Appendix II to his last book, "The meaning of relativity", Einstein [6] proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intend of this theory is physical, its exposition is mainly geometrical. Characterizing Einstein's four-dimensional unified field theory as a set of geometrical postulates for the space-time X_4, Hlavatý [7] gave the mathematical foundation for the first time. Since then the geometrical consequences of these postulates have been delveloped very far by a number of mathematicans and theorectical physicists. Generalization of this theory to an n-dimensional generalized Riemannian manifold X_n was considered and studied by Hlavatý, Wrede [11], and Mishra [10].

Recently, Yoo [13] introduced a new concept of ME manifold MEX_n connected to X_n an ME connection of the form (2.8), which is similar to Yano [12] and Imai's [9] semi-symmetric metric connection.

This paper contains four sections. Section 2 introduces some preliminary notations, concepts, and results, which are needed in the present paper. Section 3 derives several identities which hold on the submanifold X_m of MEX_n. In particular, we prove a necessary and sufficient

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condition for the submanifold of MEX_n to be einstein. In the last section, we derive the generalized fundamental equations on the submanifold of MEX_n, such as the generalized Gauss's formulas, generalized Weingarten's equations, and generalized Gauss-Codazzi equations.

2. Preliminaries

This section is a brief collection of definitions, notations, and basic results needed in the present paper. It is based on the results and notations of Chung et al, [2],[3],[4], and Hlavatý [7].

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys coorinate transformation $x^\nu \longrightarrow \bar{x}^\nu$, for which $Det\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0$.

The algebraic structure on X_n is endowed with a general real non-symmetric tensor $g_{\lambda\mu}$, the so-called Einstein unified field tensor. It may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.1}$$

where

$$\mathfrak{g} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = Det(h_{\lambda\mu}) \neq 0. \tag{2.2}$$

We may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu^\nu. \tag{2.3}$$

The tensor $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/oı lowering indices of holonomic components of tensors in X_n in the usual manner.

The differential geometric structure on X_n is imposed by the tensor $g_{\lambda\mu}$ by means of a real general connecting $\Gamma^\nu_{\lambda\mu}$, which satisfied the transformation rule:

$$\bar{\Gamma}^\nu_{\lambda\mu} = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial \bar{x}^\lambda}\frac{\partial x^\gamma}{\partial \bar{x}^\mu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}\right) \tag{2.4}$$

and the system of Einstein's equation

(2.5a) $$\partial_\omega g_{\lambda\mu} - \Gamma^{\alpha}_{\lambda\omega} g_{\alpha\mu} - \Gamma^{\alpha}_{\omega\mu} g_{\lambda\alpha} = 0,$$

or equivalently

(2.5b) $$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}.$$

Here $S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]}$ is the torsion tensor of $\Gamma^{\nu}_{\lambda\mu}$ and D_ω denotes the symbol of the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$.

A procedure similar to Christoffel elimination applied to the symmetric part of (2.5b) yields that if the system (2.5) admits a solution $\Gamma_{\lambda\mu}$, it must be of the form [7]

(2.6) $$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu},$$

where

(2.7) $$U^{\nu}{}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta}$$

and $\left\{ {\nu \atop \lambda\mu} \right\}$ are the Christoffel symbol defined by $h_{\lambda\mu}$.

The Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$ which takes the form

(2.8) $$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + 2\delta_{\lambda}{}^{\nu} X_{\nu} - 2g_{\lambda\mu} X^{\nu},$$

for a non-null vector X^{ν}, is called an ME connection and a generalized Riemannian manifold X_n connected by this connection is called an n-dimensional ME manifold and will denoted by MEX_n. In the representation of ME connection, the vector X^{ν} will be called an ME vector. A necessary and sufficient condition that the ME connection holds is that for a non-null vector X^{ν}, the torsion tensor $S_{\lambda\mu}{}^{\nu}$ is given by [13]

(2.9) $$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda} X_{\mu]} - 2k_{\lambda\mu} X^{\nu}$$

and the tensor field $g_{\lambda\mu}$ satisfies

$$g_{\nu(\lambda}X_{\mu)} + 2k_{\nu(\lambda}k_{\mu)}{}^{\alpha} - h_{\lambda\mu}X_{\nu} = 0. \tag{2.10}$$

We use the following abbreviation for an arbitrary vector X_λ:

$$^{(p)}X_\lambda = {}^{(p)}k_\lambda{}^\alpha X_\alpha \quad (p = 0, 1, 2. \cdots), \tag{2.11a}$$

$$^{(p)}X^\nu = (-1)^{p}{}^{(p)}k_\alpha{}^\nu X^\alpha \quad (p = 0, 1, 2, \cdots), \tag{2.11b}$$

$$^{(p)}\overset{+}{X}_\lambda = {}^{(p-1)}X_\lambda + {}^{(p)}X_\lambda \quad (p = 1, 2, 3, \cdots). \tag{2.11c}$$

Let X_m be a submanifold of $X_n (m < n)$, defined by a system of real parametric equations

$$y^\nu = y^\nu(x^1, \cdots, x^m). \tag{2.12}$$

It is assumed that the functions $y^\nu(x^i)$ are sufficiently differentiable and the rank of the matrix of derivatives $B_i{}^\nu = \frac{\partial y^\nu}{\partial x^i}$ is m. At each point of X_m, there exists the first set $\{B_i^\nu, \underset{x}{N}{}^\nu\}$ of n linearly independent non-null vectors. The m vectors $B_i{}^\nu$ are tangential to X_m and the $n - m$ vectors $\underset{x}{N}{}^\nu$ are normal to X_m and mutually orthogonal. That is,

$$h_{\alpha\beta}B_i^\alpha \underset{x}{N}{}^\beta = 0, \quad h_{\alpha\beta}\underset{x}{N}{}^\alpha \underset{y}{N}{}^\beta = 0 \text{ for } x \neq y. \tag{2.13a}$$

The process of determining the set $\{\underset{x}{N}{}^\nu\}$ is not unique unless $m = n - 1$. However, we may choose their magnitudes such that

$$h_{\alpha\beta}\underset{x}{N}{}^\alpha \underset{y}{N}{}^\beta = \varepsilon_x, \tag{2.13b}$$

where $\varepsilon_x = +1$ or -1 according as the left-hand sides of (2.13a) is positive or negative. Put

$$E_A^\nu = \begin{cases} B_i^\nu & \text{if } A = 1, \cdots, m \ (= i), \\ \underset{x}{N}{}^\nu & \text{if } A = m + 1, \cdots, n \ (= x). \end{cases} \tag{2.14}$$

Since $\{ E_A^\nu \}$ is a set of n lineary independent vectors, there exists a unique second set $\{ E_\lambda^A \}$ of n linearly independent vectors at point of X_m such that

$$E_\lambda^A E_A^\nu = \delta_\lambda^\nu, \quad E_\alpha^A E_B^\alpha = \delta_B^A. \tag{2.15}$$

Put

$$E_\lambda^A = \begin{cases} B_\lambda^i & \text{if } A = 1, \cdots, m \ (=: i), \\ \overset{x}{N}_\lambda & \text{if } A = m+1, \cdots, n \ (= x) \end{cases} \tag{2.16}$$

$$B_\lambda^\nu = B_\lambda^i B_i^\nu. \tag{2.17}$$

Then, it has been shown that the following relations hold in virtue of (2.15):

$$B_\alpha^i B_j^\alpha = \delta_j^i, \quad \overset{x}{N}_\alpha \underset{y}{N}^\alpha = \delta_y^x, \quad B_\alpha^i \underset{x}{N}^\alpha = \overset{x}{N}_\alpha B_i^\alpha = 0, \tag{2.18a}$$

$$B_\lambda^\nu = \delta_\lambda^\nu - \sum_x \overset{x}{N}_\lambda \underset{x}{N}^\nu, \tag{2.18b}$$

$$B_\lambda^\alpha \overset{x}{N}_\alpha = B_\alpha^\nu \underset{x}{N}^\alpha = 0. \tag{2.18c}$$

In the present paper, we use the following types of indices:

(1) Lowercase Greek indices $\alpha, \beta, \gamma, \cdots$, running from 1 to n and used for the holonomic components of tensors in X_n.

(2) Capital Latin indices $A, B, C, \cdots$, running from 1 to n and used for the C-nonholonomic components of tensors in X_n at point of X_m.

(3) Lowercase Latin indices $i, j, k, \cdots$, with the exception of x, y, and z, running from 1 to $m(< n)$.

(4) Lowercase Latin italtic indices x, y, and z, running from $m+1$ to n.

The summation convention is operative with respect to each set of the above indices within their range, with the exception of x, y, and z. We note that the vector B^i_λ and $\overset{x}{N}_\lambda$ are also tangential and normal to X_n, respectively.

The set $\{E^\nu_A\}$ and $\{E^A_\lambda\}$ will be referred to as a C-nonholonomic frame of reference in X_n at points of X_m. This frame of reference gives rise to C-nonholonomic components of a tensor in X_n. If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor in X_n, then at points of X_m, its C-nonholonomic components $T^{A\cdots}_{B\cdots}$ are defined by

$$T^{A\cdots}_{B\cdots} = T^{\alpha\cdots}_{\beta\cdots} E^A_\alpha \cdots E^\beta_B \cdots \tag{2.19}$$

In virtue of (2.14), an easy inspection show that

$$T^{\nu\cdots}_{\lambda\cdots} = T^{A\cdots}_{B\cdots} E^\nu_A \cdots E^B_\lambda \cdots . \tag{2.20}$$

In particular, the quantities

$$T^{i\cdots}_{j\cdots} = T^{\alpha\cdots}_{\beta\cdots} B^i_\alpha \cdots B^\beta_j \cdots \tag{2.21}$$

are components of a tensor in X_m and are called the components of the induced tensor of $T^{\nu\cdots}_{\lambda\cdots}$ on $X_m$of X_n. As a consequence of (2.20), we have

$$X_\lambda = X_i B^i_\lambda + \sum_x X_x \overset{x}{N}_\lambda, \tag{2.22a}$$

$$X^\nu = X^i B^\nu_i + \sum_x X^x \underset{x}{N}^\nu, \tag{2.22b}$$

where

$$X_i = X_\alpha B^\alpha_i, \quad X_x = X_\alpha \underset{x}{N}^\alpha, \quad X_x = \varepsilon_x X^x, \tag{2.22c}$$

$$X^i = X^\alpha B^i_\alpha, \quad X^x = X^\alpha \overset{x}{N}_\alpha. \tag{2.22d}$$

The induced tensor g_{ij} of $g_{\lambda\mu}$ is given by

$$g_{ij} = g_{\alpha\beta} B_i^\alpha B_j^\beta, \tag{2.23a}$$

where its symmetric part h_{ij} and skew-symmetric part k_{ij} are

$$h_{ij} = h_{\alpha\beta} B_i^\alpha B_j^\beta, \quad k_{ij} = k_{\alpha\beta} B_i^\alpha B_j^\beta. \tag{2.23b}$$

It has been shown that the induced tensors h_{ij} of $h_{\lambda\mu}$ and h^{ik} of $h^{\lambda\nu}$ satisfy

$$h_{ij} h^{ik} = \delta_j^k. \tag{2.24}$$

Therefore, they may be used for raising and/or lowering indices of the induced tensors on X_m in the usual manner.

If $\Gamma_{\lambda\mu}^\nu$ is a connection on X_n, the connection Γ_{ij}^k defined by

$$\Gamma_{ij}^k = B_\gamma^k \left(B_{ij}^\gamma + \Gamma_{\alpha\beta}^\gamma B_i^\alpha B_j^\beta \right), \tag{2.25a}$$

where

$$B_{ij}^\gamma = \frac{\partial B_i^\gamma}{\partial x^j} = \frac{\partial^2 y^\gamma}{\partial x^i \partial x^j} \tag{2.25b}$$

is called the induced connection of $\Gamma_{\lambda\mu}^\nu$ on X_m of X_n. It should be remarked that the torsion tensor $S_{ij}{}^k$ of the induced connection Γ_{ij}^k is the induced tensor of the torsion tensor $S_{\lambda\mu}^\nu$ of the connection $\Gamma_{\lambda\mu}^\nu$. That is

$$S_{ij}{}^k = S_{\alpha\beta}{}^\gamma B_i^\alpha B_j^\beta B_\gamma^k. \tag{2.26}$$

Furthermore, the induced connection $\left\{ {k \atop ij} \right\}$ of $\left\{ {\nu \atop \lambda\mu} \right\}$ is the Christoffel symbol defined by h_{ij}. That is

$$\left\{ {k \atop ij} \right\} = \frac{1}{2} h^{kp} \left(\partial_i h_{jp} + \partial_j h_{ip} - \partial_p h_{ij} \right). \tag{2.27}$$

In our subsequent considerations, we frequently use the following C-nonholonomic components:

$$S_{ij}{}^x = S_{\alpha\beta}{}^\gamma B_i^\alpha B_j^\beta \overset{x}{N}_\gamma, \tag{2.28a}$$

$$U^x{}_{ij} = U^\gamma{}_{\alpha\beta} B_i^\alpha B_j^\beta \overset{x}{N}_\gamma. \tag{2.28b}$$

3. The submanifold of MEX_n

In this section we shall prove that the induced connection is the ME connection and that several identities which hold on the submanifold of MEX_n. In particular, we find a necessary and sufficient comdition for the submanifold of MEX_n is to be einstein.

Let $\overset{x}{\Omega}_{ij}$ be the generalized coefficients of the second fundamental form of X_m and $\overset{\circ}{D}_j$ be the symmetric vector of the generalized covariant derivative with respect to $x's$. Then

$$(3.1) \qquad \overset{\circ}{D}_j\left(B_i^\alpha\right) = B_{ij}^\alpha + \Gamma_{\beta\gamma}^\alpha B_i^\beta B_j^\gamma - \Gamma_{ij}^k B_k^\alpha .$$

The vector $\overset{\circ}{D}_j B_i^\alpha$ in X_n is normal to X_m and may be given by [3]

$$(3.2) \qquad \overset{\circ}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Omega}_{ij} \underset{x}{N}^\alpha ,$$

where

$$(3.3) \qquad \overset{x}{\Omega}_{ij} = -\left(\overset{\circ}{D}_j B_i^\alpha\right) \overset{x}{N}_\alpha .$$

Furthermore, the tensor $\overset{x}{\Omega}_{ij}$ is the induced tensor on X_m of the tensor $D_\beta \overset{x}{N}_\alpha$ in X_n. That is,

$$(3.4) \qquad \overset{x}{\Omega}_{ij} = \left(D_\beta \overset{x}{N}_\alpha\right) B_i^\alpha B_j^\beta .$$

Let $\overset{x}{\Lambda}_{ij}$ be the generalized coefficients of the second fundamental form with respect to the Christoffel symbol $\left\{ {\nu \atop \lambda\mu} \right\}$. That is

$$(3.5) \qquad \overset{x}{\Lambda}_{ij} = \left(\nabla_\beta \overset{x}{N}_\alpha\right) B_i^\alpha B_j^\beta .$$

Here ∇_β denotes the symmetric vector of the covariant derivative with respect to $\left\{ {\nu \atop \lambda\mu} \right\}$.

THEOREM 3.1. *The coefficients* $\overset{x}{\Omega}_{ij}$ *of the submanifold* X_m *of* MEX_n *are given by*

$$\overset{x}{\Omega}_{ij} = \overset{x}{\Lambda}_{ij} - 2\delta^x_{(i}X_{j)} + 2g_{ij}X^x. \tag{3.6}$$

PROOF. In virtue of (2.6), (2.28), (3.4), and (3.6), we have

$$\overset{x}{\Omega}_{ij} = \overset{x}{\Lambda}_{ij} - S_{ij}{}^x - U^x{}_{ij}. \tag{3.7}$$

Also, on an X_m of MEX_n, making use of (2.6), (2.8), (2.9), (2.22), and (2.28), we have

$$S_{ij}{}^x = -2k_{ij}X^x, \tag{3.8}$$

$$U^x{}_{ij} = 2\delta^x_{(i}X_{j)} - 2h_{ij}X^x. \tag{3.9}$$

Our assertion (3.6) immediately follows by substituting (3.8) and (3.9) into (3.7).

THEOREM 3.2. *On an* X_m *of* MEX_n*, the induced connection* Γ^k_{ij} *is of the form*

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + 2\delta^k_i X_j - 2g_{ij}X^k. \tag{3.10}$$

PROOF. Substituting (2.8) into (2.25), we obtain

$$\begin{aligned}\Gamma^k_{ij} &= B^k_\gamma \left(B^\gamma_{ij} + \left\{ {\gamma \atop \alpha\beta} \right\} B^\alpha_i B^\beta_j \right) \\ &= 2(\delta^\gamma_\alpha X_\beta - g_{\alpha\beta}X^\gamma) B^\alpha_i B^\beta_j B^k_\gamma.\end{aligned}$$

Making use of (2.17), (2.21), (2.22), and (2.25), we have (3.10).

PROPOSITION 3.3. *In MEX_n, the system of equation (2.5b) may be given by*

$$D_\omega g_{\lambda\mu} = -4k_{\omega\mu}\overset{+}{X}_\lambda, \tag{3.11}$$

which can be split into

$$D_\omega h_{\lambda\mu} = -4k_{\omega(\mu}\overset{+}{X}_{\lambda)}, \tag{3.12a}$$

$$D_\omega k_{\lambda\mu} = -4k_{\omega[\mu}\overset{+}{X}_{\lambda]}. \tag{3.12b}$$

Furthermore, in MEX_n, we also have

$$D_\omega h^{\lambda\nu} = -4h^{\lambda\alpha}h^{\nu\beta}k_{\omega(\beta}\overset{+}{X}_{\alpha)}. \tag{3.13}$$

PROOF. Substituting (3.8) into (2.5b) and making use of (2.1) and (2.11), we get (3.11).

THEOREM 3.4. *The induced connection Γ_{ij}^k on X_m, given by (3.10), of $\Gamma_{\lambda\mu}^\nu$ on MEX_n is an ME connection.*

PROOF. In virtue of (2.5b), (2.9), (2.22), and (2.23a), it follows from (2.19).

$$\begin{aligned} D_k g_{ij} &= (D_\omega g_{\lambda\mu})\, B_k^\omega B_i^\lambda B_j^\mu \\ &= 2\left((\delta_\omega^\alpha X_\mu - \delta_\mu^\alpha X_\omega) - 2k_{\omega\mu}X^\alpha\right) g_{\lambda\alpha} B_k^\omega B_i^\lambda B_j^\mu \\ &= 2\,(X_\mu g_{\lambda\omega} - X_\omega g_{\lambda\mu})\, B_k^\omega B_i^\lambda B_j^\mu \\ &= 4\left(\delta_{[k}^p X_{j]} - k_{kj}X^p\right) g_{ip} \\ &= 2S_{kj}{}^p g_{ip}. \end{aligned}$$

THEOREM 3.5. *On an X_m of MEX_n, a necessary and sufficient condition for the induced connection Γ_{ij}^k to be einstein is*

$$\sum_x \left(k_{x[i}\overset{x}{\Omega}_{j]k} - 2X^x k_{xi}k_{jk} \right) = 0. \tag{3.14}$$

PROOF. In virtue of (2.13), (2.18), and (3.2), we have

$$D_k g_{ij} = (D_\omega g_{\lambda\mu})B_k^\omega B_i^\lambda B_j^\mu - 2\sum_x k_{x[j}\overset{x}{\Omega}_{i]k}. \tag{3.15}$$

If Γ_{ij}^k is einstein, then the relations (2.5b), (2.11), (2.17), (2.18b), (2.21), and (3.15) gives the following relation

$$\sum_x \left(k_{x[i}\overset{x}{\Omega}_{j]k} - S_{jk}{}^x k_{ix} \right) = 0. \tag{3.16}$$

Substituting (3.9a) into (3.16), we have (3.14). The reverse calculatings give the proof of the sufficiency.

4. The generalized fundamental equations for submanifold of MEX_n

This section is devoted to the derivation of the generalized fundamental equations for submanifold of MEX_n, such as the generalized Gauss formulas, Weingarten equations, and Gauss-Codazzi equations.

THEOREM 4.1. *(The generalized Gauss formulas for X_m of MEX_n) On an X_m of MEX_n, the following relation holds:*

$$\overset{\circ}{D}_j B_i^\alpha = \sum_x \left(-\overset{x}{\Lambda}_{ij} + 2\delta_{(i}^x X_{j)} - 2g_{ij}X^x \right) \underset{x}{N}^\alpha. \tag{4.1}$$

PROOF. Substituting (3.5) into (3.2), we have (4.1).

In order to derive the generalized Weingarten equations, we need the following preparations. Let

$$M_{x}{}^{\alpha}_{j} = \overset{\circ}{D}_j \underset{x}{N}^{\alpha}. \tag{4.2}$$

LEMMA 4.2. *The vector* $\underset{x}{M}{}^{\alpha}_{j}$ *may be decomposed as*

$$\underset{x}{M}{}^{\alpha}_{j} = \underset{x}{M}{}^{i}_{j} B^{\alpha}_{i} + \sum_{y} \underset{x}{M}{}^{y}_{j} \underset{y}{N}^{\alpha}. \tag{4.3}$$

Furthermore, $\underset{x}{M}{}^{i}_{j}$ *is also the induced tensor of* $D_\gamma \underset{x}{N}^{\alpha}$ *and* $\underset{x}{M}{}^{y}_{j}$ *is the induced vector of* $\left(D_\gamma \underset{x}{N}^{\alpha}\right) \overset{y}{N}_{\alpha}$. *That is,*

$$\underset{x}{M}{}^{i}_{j} = \underset{x}{M}{}^{\alpha}_{j} B^{i}_{\alpha} = \left(D_\gamma \underset{x}{N}^{\alpha}\right) B^{i}_{\alpha} B^{\gamma}_{j}, \tag{4.4a}$$

$$\underset{x}{M}{}^{y}_{j} = \underset{x}{M}{}^{\alpha}_{y} \overset{y}{N}_{\alpha} = \left(D_\gamma \underset{x}{N}^{\alpha}\right) \overset{y}{N}_{\alpha} B^{\gamma}_{j}. \tag{4.4b}$$

PROOF. The first assertion (4.3) follows from (2.22) and the relations (4.4) obtain from (2.21).

LEMMA 4.3. *On an* X_m *of* MEX_n, *the induced tensor* $\underset{x}{M}{}^{i}_{j}$ *of* $\underset{x}{M}{}^{\alpha}_{j}$ *is given by*

$$\underset{x}{M}{}^{i}_{j} = -4h^{im} k_{\beta(\sigma} \overset{+}{X}_{\delta)} \underset{x}{N}^{\sigma} B^{\delta}_{m} B^{\beta}_{j} + \varepsilon_x h^{im} \overset{x}{\Omega}_{mj}. \tag{4.5}$$

PROOF. Equation (4.4a) gives

$$\begin{aligned} \underset{x}{M}{}^{i}_{j} &= \left(D_\beta (h^{\alpha\gamma} \overset{x}{N}_{\gamma})\right) B^{i}_{\alpha} B^{\beta}_{j} \\ &= D_\beta (h^{\alpha\gamma}) \overset{x}{N}_{\gamma} B^{i}_{\alpha} B^{\beta}_{j} + h^{\alpha\gamma} \left(D_\beta \overset{x}{N}_{\gamma}\right) B^{i}_{\alpha} B^{\beta}_{j}. \end{aligned} \tag{4.6}$$

Substituting (3.13) into (4.6) and making use of (2.21) and (3.4), we have (4.5).

LEMMA 4.4. *On an X_m of MEX_n, the induced vector $\underset{x}{M}{}_j^y$ of $\underset{x}{M}{}_j^\alpha$ is given by*

$$\underset{x}{M}{}_j^y = 2k_j{}^y \overset{+}{X}_x. \tag{4.7}$$

PROOF. Generalized covariant differentiation of both sides of (2.12b) with respect to x^j gives

$$D_\gamma(h_{\alpha\beta})\underset{x}{N}{}^\alpha \underset{x}{N}{}^\beta B_j^\gamma + 2h_{\alpha\beta}\left(D_\gamma \underset{x}{N}{}^\alpha\right)\underset{x}{N}{}^\beta B_j^\gamma = 0. \tag{4.8}$$

Substituting (3.12a) and (4.4b) into (4.8) and making use of (2.22) and (2.23), we have (4.7).

Now, we are ready to prove the generalized Weingarten equations for on an X_m of MEX_n.

THEOREM 4.5. *(The first representation of the generalized Weingarten equations on an X_m of MEX_n.)*

$$\begin{aligned} \overset{\circ}{D}_j \underset{x}{N}{}^\alpha &= -2\left(h^{\alpha\delta}k_{\beta\sigma}\overset{+}{X}_\delta \underset{x}{N}{}^\sigma + k_\beta{}^\alpha \overset{+}{X}_x\right)B_j^\beta \\ &\quad + \sum_y 2k_j{}^y \overset{+}{X}_x \underset{y}{N}{}^\alpha + \varepsilon_x h^{im}\overset{x}{\Omega}_{mj}B_i^\alpha. \end{aligned} \tag{4.8}$$

PROOF. Substituting (4.5) and (4.7) into (4.3) and making use of (2.22), we have (4.8).

THEOREM 4.6. *(The second representation of the generalized Weingarten equations on an X_m of MEX_n.)*

$$\begin{aligned} \overset{\circ}{D}_j \overset{x}{N}_\alpha &= \overset{x}{\Omega}_{ij}B_\alpha^i + 2\left(k^\beta{}_\gamma \overset{+}{X}_\alpha \overset{x}{N}_\beta + \varepsilon_x k_{\alpha\gamma}\overset{+}{X}_x\right)B_j^\gamma \\ &\quad + 2\left(k^\sigma{}_\alpha \overset{+}{X}_j \overset{x}{N}_\sigma + \sum_y \varepsilon_x h_{\alpha\beta}k_j{}^y \overset{+}{X}_x \underset{y}{N}{}^\beta\right). \end{aligned} \tag{4.9}$$

PROOF. Substituting (3.12a) and (4.8) into

$$\overset{\circ}{D}_j \overset{x}{N}_\alpha = \overset{\circ}{D}_j \left(h_{\alpha\beta} \underset{x}{N}^\beta \right) = h_{\alpha\beta} \overset{\circ}{D}_j \underset{x}{N}^\beta + (D_\gamma h_{\alpha\beta}) \underset{x}{N}^\beta B_j^\gamma$$

and making use of (2.3) and (2.22), we have (4.9).

In order to derive the generalized Gauss-Codazzi equations, we need the following curvature $R_{\omega\mu\lambda}{}^\nu$ of MEX_n and $R_{ijk}{}^m$ of X_m of MEX_n.

$$R_{\omega\mu\lambda}{}^\nu = 2 \left(\partial_{[\mu} \Gamma^\nu_{|\lambda|\omega]} + \Gamma^\alpha_{\lambda[\omega} \Gamma^\nu_{|\alpha|\mu]} \right), \tag{4.10}$$

$$R_{ijk}{}^h = 2 \left(\partial_{[j} \Gamma^h_{|k|i]} + \Gamma^p_{k[i} \Gamma^h_{|p|j]} \right) \tag{4.11}$$

THEOREM 4.7. *(The generalized Gauss-Codazzi equations for an X_m of MEX_n) On an X_m of MEX_n, the curvature tensor defined by (4.10) and (4.11) are involved in the following identities:*

$$\begin{aligned} R_{ijk}{}^h = \; & R_{\beta\gamma\varepsilon}{}^\alpha B^h_\alpha B^\beta_i B^\gamma_j B^\varepsilon_k + 2\varepsilon_x \overset{x}{\Omega}_{i[k} \overset{x}{\Omega}_{|m|j]} h^{hm} \\ & - 4 \sum_x \overset{x}{\Omega}_{i[k} B^\beta_{j]} B^h_\alpha \left(h^{\alpha\delta} k_{\beta\sigma} \overset{+}{X}_\delta N^\sigma + k_\beta{}^\alpha \overset{+}{X}_x \right), \end{aligned} \tag{4.12}$$

$$\begin{aligned} 2\overset{\circ}{D}_{[k} \overset{x}{\Omega}_{j]i} = & R_{\beta\gamma\varepsilon}{}^\alpha \overset{x}{N}_\alpha B^\beta_k B^\gamma_j B^\varepsilon_i + 4\overset{x}{\Omega}_{i[k} \left(X_{j]} + \sum_y k_{j]}{}^y \overset{+}{X}_y \right) \\ & - 4\overset{x}{\Omega}_{i[k} B^\beta_{j]} \left(k_\beta{}^\delta \overset{+}{X}_\delta + \sum_x k_\beta{}^c \overset{+}{X}_x \overset{x}{N}_\alpha \right). \end{aligned} \tag{4.13}$$

PROOF. In virtue of (3.1), (3.2), (4.10), and (4.11), we have

(4.14)
$$\begin{aligned} 2\overset{\circ}{D}_{[k} \overset{\circ}{D}_{j]} B^\alpha_i &= 2 \left(\partial_{[k} \overset{\circ}{D}_{j]} B^\alpha_i - \Gamma^m_{[jk]} (\overset{\circ}{D}_m B^\alpha_i) - \Gamma^m_{i[k} (\overset{\circ}{D}_{j]} B^\alpha_m + \Gamma^\alpha_{\beta\gamma} (\overset{\circ}{D}_{[j} B^\beta_{|i|}) B^\gamma_{k]} \right) \\ &= -R_{\varepsilon\gamma\beta}{}^\alpha B^\beta_k B^\gamma_j B^\varepsilon_i + R_{kji}{}^m B^\alpha_m + 4 \sum_x \overset{x}{\Omega}_{i[j} X_{k]} \underset{x}{N}^\alpha. \end{aligned}$$

On the other hand, the relations (3.2) and (4.9) give

$$
\begin{aligned}
2\overset{\circ}{D}_{[k}\overset{\circ}{D}_{j]}B_i^\alpha &= -2\sum_x \overset{\circ}{D}_{[k}\overset{x}{\Omega}_{j]i}\underset{x}{N}^\alpha \\
&= 2\sum_x \left(\overset{\circ}{D}_{[j}\overset{x}{\Omega}_{k]i}\right)\underset{x}{N}^\alpha + 2\sum_x\sum_y \overset{x}{\Omega}_{i[k}k_{j]}{}^y\overset{+}{X}_x\underset{y}{N}^\alpha \\
&\quad - 4\sum_x \overset{x}{\Omega}_{i[k}k_{j]}{}^\beta\left(h^{\alpha\delta}k_{\beta\sigma}\overset{+}{X}_\delta\underset{x}{N}^\sigma + k_\beta{}^\alpha\overset{+}{X}_x\right) \\
&\quad + 2\varepsilon_x\overset{x}{\Omega}_{i[k}\overset{x}{\Omega}_{|m|j]}h^{im}B_i^\alpha.
\end{aligned} \tag{4.15}
$$

Hence comparing (4.14) and (4.15), we have

(4.16)

$$
\begin{aligned}
R_{kji}{}^m B_m^\alpha =& R_{\varepsilon\gamma\beta}{}^\alpha B_k^\beta B_j^\gamma B_i^\varepsilon + 2\sum_x\left(\overset{\circ}{D}_{[j}\overset{x}{\Omega}_{k]i} - 2\overset{x}{\Omega}_{i[j}X_{k]}\right)\underset{x}{N}^\alpha \\
&+4\sum_x\sum_y \overset{x}{\Omega}_{i[k}k_{j]}{}^y\overset{+}{X}_x\underset{y}{N}^\alpha \\
&- 4\sum_x \overset{x}{\Omega}_{i[k}B_{j]}^\beta\left(h^{\alpha\delta}k_{\beta\sigma}\overset{+}{X}_\delta\underset{x}{N}^\sigma + k_\beta{}^\alpha\overset{+}{X}_x\right) \\
&+2\varepsilon_x\overset{x}{\Omega}_{i[k}\overset{x}{\Omega}_{|m|j]}h^{im}B_i^\alpha.
\end{aligned}
$$

Multiplying B_α^h on both sides of (4.16) and making use of (2.17), we have the identity (4.12). Similary, multiplying $\overset{x}{N}_\alpha$ on both sides of (4.16), we have (4.13).

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