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COMPLEMENTED SUBLATTICES OF wL_1 ISOMORPHIC TO CLASSICAL BANACH LATTICES

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ABSTRACT. We investigate complemented Banach subspaces of the Banach envelope of $weakL_1$. In particular, the Banach envelope of $weakL_1$ contains complemented Banach sublattices that are isometrically isomorphic to $l_p, (1 \le p < \infty)$ or c_0. Finally, we also prove that the Banach envelope of $weakL_1$ contains an isomorphic copy of $l^{p,\infty}, (1 < p < \infty)$.

1. Introduction

The space $weakL_1$, as a Lorentz space $L(1,\infty)$, was introduced in analysis because key operators of harmonic analysis do not map L_1 into L_1. Examples of such operators are the Hardy-Littlewood maximal function and the Hilbert transform. In this viewpoint, it became natural to investigate $weakL_1$, the space of measurable functions f satisfying $\mu(\{x \in \Omega : |f(x)| > y\}) \le \frac{c}{y}$.

It is known that (except for some trivial measure space), $weakL_1$ is not normable (see [C-S]). The question therefore arise as to whether any nontrivial continuous linear functionals on $weakL_1$ exists. In [C-S], the answer for this question was considered. This implies $weakL_1$ has nontrivial dual space. In [K-P], J. Kupka and T. Peck studied the structure of $weakL_1$. They showed that the space L_∞ is dense in the dual of $weakL_1$ endowed with $weak^*$-topology and showed lattice embeddings of L_1, $l_1[0,1]$, l_∞ and $c_0[0,1]$ into wL_1 where wL_1 is the Banach envelope of $weakL_1$. Later on, T. Peck and M. Talagrand ([P-T]) proved that every separable order continuous Banach lattice is lattice isometric to

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a sublattice of wL_1. Finally, H. Lotz and T. Peck ([L-P]) removed the hypothesis of order continuity in the separable case.

As a Lorentz space, we will study the space $L(1,\infty)$ which is called $weakL_1$ denoted wL_1:

$$(1.1) \qquad wL_1 = \{f \in L_0 : \mu(\{x \in \Omega : |f(x)| > y\}) < \frac{c}{y}\},$$

where $c > 0$ is independent of $y > 0$. As we mentioned in the above, wL_1 is not normable, but we can find nontrivial linear functionals on wL_1. This was first observed by M. Cwikel and Y. Sagher in [C-S].

For $0 < p < \infty$, the space $weakL_p$ taken over the measure space (Ω, Σ, μ) consists of all (equivalence classes of) measurable functions f for which the quasinorm

$$(1.2) \qquad q_p(f) = \sup_{a>0} a[\mu(\{x \in \Omega : |f(x)| > a\})]^{\frac{1}{p}}.$$

Define q to be the Minkowski functional of the convex hull of the unit ball $\{f \in wL_1 : g_1(f) \leq 1\}$ of wL_1 where $q_1(f) = \sup_{a>0} a\mu\{x \in \Omega : |f(x)| > a\}$. The fact that q is a seminorm on wL_1 is most readily seen from the alternate formulation

$$q(f) = \inf_{f=f_1+\cdots+f_n} \sum_{i=1}^{n} q_1(f_i),$$

where the infimum is taken over all finite decompositions $f = f_1 + f_2 + \cdots + f_n$ of f in wL_1. This gives the Banach envelope seminorm on wL_1. In [C-F1], if μ is nonatomic, then we can get an equivalent integral-like seminorm

$$(1.3) \qquad \|f\|_{wL_1} = \lim_{n\to\infty} \sup_{\frac{q}{p}\geq n} \frac{1}{log\frac{q}{p}} \int_{\{p\leq|f|\leq q\}} |f| d\mu.$$

Later on, in [C-F2] actually the Banach envelope seminorm on wL_1 was calculated to be exactly as above. Note that the seminorm on wL_1 defined in (1.3) is a lattice seminorm. This is not quite obvious, but using integration by parts one can readily show that the seminorm $\|\cdot\|_{wL_{\hat{1}}}$ is exactly same as (see [K-P]),

$$\lim_{n\to\infty} \sup_{\frac{q}{p}\geq n} \frac{1}{log\frac{q}{p}} \int_p^q \mu(\{x \in \mu : |f(x)| > t\})dt. \tag{1.4}$$

Even though wL_1 is complete with respect to the quasinorm q_1, it is not complete with respect to the seminorm $\|\cdot\|_{wL_{\hat{1}}}$. This is due to M. Cwikel and C. Feffreman([C-F1] and [K-P]). Let $\mathcal{N} = \{f \in wL_1 : \|f\|_{wL_{\hat{1}}} = 0\}$. Then we obtain the quotient space $wL_1/\mathcal{N}$. We define $wL_{\hat{1}}$ as the *normed envelope* (and its completion as the *Banach envelope*) of wL_1.

To study this subject, we need some basic facts about the dual of $wL_{\hat{1}}$. We would like to convert the nonlinear limit superior expression (1.4) for $\|\cdot\|_{wL_{\hat{1}}}$ into a linear limit expression by directing the numbers $I_a^b(f) = \frac{1}{log\frac{b}{a}} \int_{\{a\leq|f|\leq b\}} |f|d\mu$ in some fashion. For this, we introduce an ultrafilter $\mathcal{U}$ so that the limit of the I_a^b along $\mathcal{U}$ determines a canonical integral-like linear functional $I_{\mathcal{U}} \in wL_{\hat{1}}^*$. We now begin with the discussion of $\mathcal{U}$. For $n = 1, 2, 3, \ldots$, let

$$F_n = \{(a, b) : 1 \leq a < b, \frac{b}{a} \geq n\}. \tag{1.5}$$

and then define $\mathcal{F} = \{F_n : n \geq 1\}$. Treating $\mathcal{F}$ as a filter of subsets of the set $S = [1,\infty) \times [1,\infty)$, we obtain from Zorn's lemma an ultrafilter $\mathcal{U}$ of subsets of S such that $\mathcal{F} \subset \mathcal{U}$.

From now on, we will fix the ultrafilter $\mathcal{F} \subset \mathcal{U}$. Define the "ersatz integral" $I_{\mathcal{U}}$ for every nonnegative function $f \in wL_{\hat{1}}$ by

$$I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} I_a^b(f) = \lim_{\mathcal{U}} \frac{1}{log\frac{b}{a}} \int_{\{a\leq f\leq b\}} f d\mu. \tag{1.6}$$

THEOREM 1.1 (J. KUPKA AND T. PECK). *Let $f, g \in wL_{\hat{1}}$ be nonnegative and let $r > 0$. Then we have*

i) $I_{\mathcal{U}}(rf) = rI_{\mathcal{U}}(f)$.

ii) $I_{\mathcal{U}}(f+g) = I_{\mathcal{U}}(f) + I_{\mathcal{U}}(g)$.

iii) If $f \leq g$, then $I_{\mathcal{U}}(f) \leq I_{\mathcal{U}}(g)$.

iv) $I_{\mathcal{U}}(f) \leq \|f\|_{wL_1}$.

From these properties, we define $I_{\mathcal{U}}(f)$ for an arbitrary function $f \in wL_1$ by $I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} \frac{1}{log\frac{b}{a}} \int_{\{a \leq |f| \leq b\}} (f^+ - f^-) d\mu$;

i) $I_{\mathcal{U}}$ is linear.

ii) $|I_{\mathcal{U}}(f)| \leq \|f\|_{wL_1}$ for all $f \in wL_1$.

iii) $I_{\mathcal{U}}$ vanishes on $\mathcal{N} = \{f \in wL_1 : \|f\|_{wL_1} = 0\}$ and hence determines a well defined, bounded linear functional on wL_1.

Similarly, N.J. Kalton in [KAL] gave a linear functional on wL_1 in the following way; for $f \in L_0$ and $x \geq 0$, we define the truncation $\tau_x f$ by

$$\begin{aligned} \tau_x f(\omega) &= f(\omega) \quad \text{if} \quad |f(\omega)| \leq x \\ &= x \quad \text{if} \quad f(\omega) > x \\ &= -x \quad \text{if} \quad f(\omega) < -x. \end{aligned}$$

Then a linear functional on wL_1 is defined by

$$\phi(f) = \lim_{\mathcal{U}} \frac{1}{logx} \mathcal{E}(\tau_x f), \tag{1.7}$$

where $\mathcal{U}$ is any ultrafilter on $(2, \infty)$ which includes each of the sets (x, ∞) for $x > 2$ and $\mathcal{E}$ is expectation.

We now have information on the dual of wL_1:

THEOREM 1.2 (J. KUPKA AND T. PECK). *Define a linear operator $T_{\mathcal{U}} : L_\infty \longrightarrow wL_1^*$ by $T_{\mathcal{U}}(m) : f \longmapsto I_{\mathcal{U}}(mf)$ for all $m \in L_\infty(\mu)$, and for all $f \in wL_1$. Then $T_{\mathcal{U}}$ constitutes an isometric, order isomorphism of $L_\infty(\mu)$ into wL_1^*.*
Moreover, the linear span of the subspace $T_{\mathcal{U}}(L_\infty(\mu))$, as $\mathcal{U}$ ranges over the collection of ultrafilters which contains $\mathcal{F}$, constitutes a norming and hence a weak dense subspace of wL_1^*.*

We now are in a position to prove the results. For the proof of those, we need several lemmas. Note that if for $f, g \in wL_1$, $|f| \wedge |g| = 0$ with $\|f\|_{wL_1} = \|g\|_{wL_1} = 1$, then by the Hahn-Banach theorem, we can find $\phi, \psi \in wL_1^*$ with $\|\phi\| = \|\psi\| = 1$, $\phi(f) = \psi(g) = 1$ and $\phi(g) = \psi(f) = 0$. Then for arbitrary $h \in wL_1$, we have

$$\begin{aligned}|\phi(h) + \psi(h)| &\leq |\phi(h)| + |\psi(h)| \\ &\leq 2\|h\|_{wL_1}.\end{aligned}$$

This estimate is not good enough for our purpose.

This theorem gives some faverable information for $wL_1(\mathcal{U})^*$ and the very last part of theorem says

$$\overline{T_{\mathcal{U}}(B_{L_\infty})}^{wL_1(\mathcal{U})^*} = B_{wL_1(\mathcal{U})^*}. \tag{1.8}$$

From this theorem, for any $m \in L_\infty(\mu)$, we have

$$\widehat{m} = T_{\mathcal{U}}(m) \in wL_1(\mathcal{U})^*. \tag{1.9}$$

Clearly, every linear functional $\varphi \in wL_1(\mathcal{U})^*$ is a linear functional on wL_1 (see more detail in [K-P, 2.20]).

We now give a lemma about linear functionals on wL_1 which is actually due to J. Kupka and T. Peck (see [K-P,2.20]).

LEMMA 1.3. *For a fixed ultrafilter $\mathcal{U}$ in (1.5), let $f \in wL_1$ be a nonnegative function with $\|f\|_{\mathcal{U}} = 1$. Then for any $g \in wL_1$, disjointly supported from f, there exists a $\phi \in wL_1^*$ such that $\|\phi\| = 1$, $\phi(f) = 1$, and $\phi(g) = 0$.*

We can now generalize this lemma for arbitrary pairwise disjointly supported elements in wL_1.

COROLLARY 1.4. *Let $(f_n)_{n=1}^\infty$ be a sequence of nonnegative elements in wL_1 with $\|f_n\|_{wL_1} = 1$, for all $n = 1, 2, 3, \cdots$. and such that the f_n have pairwise disjoint supports. Then for each n, there exists a linear functional ϕ_n on wL_1 such that $\phi_n(f) = 1$, $\|\phi_n\| = 1$ and $\phi_n(f_m) = 0$ if $n \neq m$.*

PROOF. We can show this by an induction with Lemma 1.3 for each f_n. For given $f_1 \in wL_1$, by Lemma 1.3, we can choose ϕ_1 with $\phi_1(f_1) = \|\phi\| = 1$ and $\phi_1(f_j) = 0$, for all $j = 2, 3, \ldots$. If we selected $\phi_1, \phi_2, \ldots, \phi_n$ satisfying all the conclusions of corollary, then φ_{n+1} can be selected by applying Lemma 1.3 again. This proves the corollary.

REMARK 1.5. Define $wL_1(\mathcal{U}) = \{f \in wL_1 : \|f\|_{\mathcal{U}} < \infty\}$ where $\mathcal{U}$ is the ultrafilter defined in (1.1). Then we have $\|f\|_{\mathcal{U}} \leq \|f\|_{wL_1}$ where $\|f\|_{\mathcal{U}} = I_{\mathcal{U}}(|f|) = \lim_{\mathcal{U}} \frac{1}{log\frac{b}{a}} \int_{\{a \leq |f| \leq b\}} |f| d\mu$. Hence we have $wL_1 \subset wL_1(\mathcal{U})$. Moreover $\|f\|_{\mathcal{U}} = I_{\mathcal{U}}(f)$ has the following properties:

i) $\|\cdot\|_{\mathcal{U}}$ is a lattice seminorm on wL_1,

ii) $\|f+g\|_{\mathcal{U}} = \|f\|_{\mathcal{U}} + \|g\|_{\mathcal{U}}$ whenever f and g are nonnegative,

iii) $\|f\|_{wL_1} = \sup\{\|f\|_{\mathcal{U}} : \mathcal{U}$ is an ultrafilter, $\mathcal{F} \subset \mathcal{U}\}$ for all $f \in wL_1$.

Again, we convert $\|\cdot\|_{\mathcal{U}}$ into a norm by forming the ideal

$$\mathcal{N}_{\mathcal{U}} = \{f \in wL_1 : \|f\|_{\mathcal{U}} = 0\}. \tag{1.10}$$

and then the quotient vector lattice $wL_1(\mathcal{U}) = wL_1/\mathcal{N}(\mathcal{U})$ on which $\|\cdot\|_{\mathcal{U}}$ acts as a lattice norm. We need one technical lemma, namely;

LEMMA 1.6. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in wL_1 such that the f_n have pairwise disjoint supports with $\|f_n\|_{wL_1} = 1$, for all $n = 1, 2, 3, \ldots$ and let $(\phi_n)_{n=1}^{\infty}$ be a sequence of linear functionals on wL_1 selected as in Corollary 1.4. Then for any $f \in wL_1$, we have $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_1}$.*

PROOF. For an arbitrary function $f \in wL_1$, the number $\phi_n(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)\}$ where $(E_{n,k})_{k=1}^{\infty}$ is a decreasing sequence of subsets of $E_n = supp(f_n)$, and f_n is bounded on $E_{n,k}^c$ for all k (see Corollary 1.4). Fix $n \neq m$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $(E_{m,k})_{k=1}^{\infty}$ the corresponding sequence for f_m.

Let $r = sgnI_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)$, $s = sgnI_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)$. Put $m = r\chi_{E_{n,k}} + s\chi_{E_{m,k}}$ so that $\|m\|_{\infty} = 1$. By (1.10) we have that for $m \in L_{\infty}$, $\widehat{m} \in wL_1$. Then we have

$$\begin{aligned} \widehat{m}(f) &= |I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)| + |I_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)| \\ &= I_{\mathcal{U}}(m \cdot f) \\ &\leq \|m\|_{\infty}\|f\|_{\mathcal{U}} \quad \text{since} \quad \|m\|_{\infty} = 1 \\ &= \|f\|_{\mathcal{U}} \\ &\leq \|f\|_{wL_1}. \end{aligned} \tag{1.11}$$

By the additive rule for nets, we have that in the limit

$$|\phi_n(f)| + |\phi_m(f)| \le \|f\|_{\mathcal{U}} \le \|f\|_{wL_1} . \tag{1.12}$$

To show $\sum_{n=1}^{\infty} |\phi_n(f)| \le \|f\|_{wL_1}$, it suffices to show that for any $N \in \mathbf{N}$, $\sum_{n=1}^{N} |\phi_n(f)| \le \|f\|_{wL_1}$. For $n = 1, 2, 3, \ldots$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $E_n = supp(f_n)$. Let $r_n = sgnI_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)$. Put $m = \sum_{n=1}^{N} r_r \chi_{E_{n,k}}$. Then we have $\|m\|_{\infty} = 1$. By the same argument as the above, we have

$$\begin{aligned} \widehat{m}(f) &= \sum_{m=1}^{N} |I_{\mathcal{U}}(\chi_{m,k} \cdot f)| \\ &= I_{\mathcal{U}}(mf) \\ &\le \|m\|_{\infty} \|f\|_{\mathcal{U}} \\ &\le \|f\|_{wL_1} . \end{aligned} \tag{1.13}$$

By the additive rule for nets, we have

$$\sum_{n=1}^{N} |\phi_n(f)| \le \|f\|_{\mathcal{U}} \le \|f\|_{wL_1} . \tag{1.14}$$

We can therefore have that $\sum_{n=1}^{\infty} |\phi_n(f)| \le \|f\|_{wL_1}$. This proves the lemma.

2. Isomorphic copies of l_p, $(1 \le p < \infty)$ in wL_1.

We will investigate a complemented sublattice of wL_1 that is isometrically isomorphic to l_p for $(1 \le p < \infty)$. The space l_p is a classical Banach lattice which is separable and order continuous. Also the usual basis $(e_i)_{i=1}^{\infty}$ is an unconditional basis for l_p. To prove this, we need a theorem which was done by H. Lotz and T. Peck

THEOREM 2.1 (H. LOTZ AND T. PECK). *Let E be a separable Banach lattice. Then E is lattice isometric to a closed sublattice of the Banach envelope of* $weakL_1$.

PROPOSITION 2.2. *For $1 \leq p < \infty$, there is a lattice isometry T from l_p i wL_1, that is, l_p can be embedded isometrically into wL_1.*

PROOF. For $1 \leq p < \infty$, l_p is a separable Banach lattice with order continuous norm. Hence by Theorem 2.1, there exists a lattice isometry T from l_p into wL_1. Then T is the desired lattice isometric embedding linear map. This proves the proposition.

REMARK 2.3. In [K-P], J. Kupka and T. Peck proved that there exists an isometric, order isomorphism T from $l_1[0,1]$ into wL_1. Moreover, the range of T is a complemented subspace of wL_1.

Thank to Proposition 2.2, we can embed l_p into wL_1 under a lattice isometry. So one can ask the natural question: What can we say about the range of T for l_p, $1 \leq p < \infty$ in Proposition 2.2 ? We can now answer this as follows:

THEOREM 2.4. *For $1 \leq p < \infty$, let $T : l_p \longrightarrow wL_1$ be a lattice isometry given by $T(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} a_i f_i$ where $(e_i)_{i=1}^{\infty}$ is the usual basis of l_p. Then the range of T is a complemented sublattice of wL_1.*

PROOF. Let $T : l_p \longrightarrow wL_1$ be a lattice isometry, and let $(e_i)_{i=1}^{\infty}$ be the usual basis for l_p. Define $Te_i = f_i$, $i = 1, 2, 3, \ldots$. Then since T preserves lattice structure, f_i are nonnegative pairwise disjoint and $\|f_i\|_{wL_1} = 1$, for all $i = 1, 2, 3, \ldots$. Now, by Corollary 1.4, we can find linear functionals ϕ_n on wL_1 such that $\phi_n(f_n) = 1$, $\phi_n(f_m) = 0$ if $n \neq m$ and $\|\phi_n\| = 1$, for all $n = 1, 2, 3, \ldots$. For an arbitrary function $f \in wL_1$, the number $\phi_n(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)\}$ where $(E_{n,k})_{k=1}^{\infty}$ is a decreasing sequence of subsets of $E_n = supp(f_n)$ and f_n is bounded on $E_{n,k}^c$ for all k and for all $n = 1, 2, 3, \ldots$.
Now, we can define a contractive projective P from wL_1 onto Tl_p, $P : wL_1 \longrightarrow Tl_p$ by

$$P(f) = \sum_{n=1}^{\infty} \phi_n(f) f_n. \tag{2.1}$$

First, we need to show that P is well defined. Since $(f_n)_{n=1}^{\infty}$ is a copy of the usual l_p basis in $wL_{\hat{1}}$, we have for $1 \leq p < \infty$,

$$(2.2) \qquad \begin{aligned} \|\sum_{n=1}^{\infty} \phi_n(f) f_n\|_{wL_{\hat{1}}} &= (\sum_{n=1}^{\infty} |\phi_n(f)|^p)^{\frac{1}{p}} \\ &\leq \sum_{n=1}^{\infty} |\phi_n(f)| \\ &\leq \|f\|_{\mathcal{U}} \qquad \text{(by Lemma 1.6)} \\ &\leq \|f\|_{wL_{\hat{1}}}. \end{aligned}$$

Hence, $P(f) = \sum_{n=1}^{\infty} \phi_n(f) f_n \in Tl_p$, for all $f \in wL_{\hat{1}}$.

Next we need to show that $\|P\| = 1$. By (2.2), we have $\|P(f)\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}$. Hence $\|P\| \leq 1$. On the other hand, since $f_m \in Tl_p \subset wL_{\hat{1}}$, we have

$$(2.3) \qquad \begin{aligned} P(f_m) &= \sum_{n=1}^{\infty} \phi_n(f_m) f_n \\ &= \phi_m(f_m) f_m \\ &= f_m. \end{aligned}$$

Therefore we have $\|P(f_m)\|_{wL_{\hat{1}}} = \|f_m\|_{wL_{\hat{1}}} = 1$. This shows $\|P\| = 1$.

Finally, we need to show that $P^2 = P$. For $f \in wL_{\hat{1}}$,

$$(2.4) \qquad \begin{aligned} P^2(f) &= P(\sum_{n=1}^{\infty} \phi_n(f) f_n) \\ &= \sum_{j=1}^{\infty} \phi_j(\sum_{n=1}^{\infty} \phi_n(f) f_n) f_j \quad \text{by} \quad \phi_n(f_j) = \delta_{n,j} \\ &= \sum_{j=1}^{\infty} \phi_j(f) f_j \\ &= P(f). \end{aligned}$$

Therefore P is a norm one projection from $wL_{\hat{1}}$ onto Tl_p. This proves the theorem.

Now, we can have the main result.

COROLLARY 2.5. *For $1 \leq p < \infty$, the Banach envelope of $wL_{\dot{1}}$ contains a complemented subspace that is isometrically isomorphic to l_p.*

PROOF. Immediate from Theorem 2.4.

3. An isomorphic copy of c_0 in $wL_{\dot{1}}$.

The Banach space $c_0 = \{(a_n)_{n=1}^{\infty} : \lim_{n\to\infty} a_n = 0\}$ is a separable space with basis $(e_n)_{n=1}^{\infty}$ where $e_i = (0,0,0,\dots,1_i,0,\dots)$. The space c_0 is not reflexive, but is $\sigma-$complete, and a $\sigma-$order continuous Banach lattice, which means c_0 is order continuous. As a universal Banach space for separable Banach spaces $C([0,1])$ contains a complemented subspace that is isomorphic to c_0. The goal of this section is to prove the Banach envelope wL_1 contains a complemented subspace that is isomorphic to c_0. In [K-P], J. Kupka and T. Peck proved that $c_0[0\ 1]$ can be embedded lattice isometrically into $wL_{\dot{1}}$. Since we need only a countable index set, we give a modification of [K-P, Theorem 4.4].

PROPOSITION 3.1. *There is an isometric, order isomorphic linear embedding of the space c_0 into $wL_{\dot{1}}$.*

PROOF. Let $\mathbf{N} = \cup_{i=1}^{\infty} S_i$ where the S_i are infinite and pairwise disjoint subsets of $\mathbf{N}$. Let $h \in wL_{\dot{1}}$, $h \geq 0$, and $\|h\|_{wL_{\dot{1}}} = 1$. Then by definition of $\|\cdot\|_{wL_{\dot{1}}}$, there exist pairwise disjoint closed intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ such that $\frac{b_n}{a_n} \to \infty$, $a_n \to \infty$ and such that

$$I_{a_n}^{b_n}(h) = \frac{1}{log\frac{b_n}{a_n}} \int_{\{a_n \leq h \leq b_n\}} hd\mu \geq \|h\|_{wL_{\dot{1}}} - \frac{1}{n} \quad \text{for all } n. \tag{3.1}$$

Define for $i \in \mathbf{N}$, $B_i = \cup_{n \in S_i}[a_n, b_n]$. Now, define $T : c_0 \longrightarrow wL_{\dot{1}}$ by

$$T((a_n)) = \text{pointwise} - \sum_{n=1}^{\infty} a_n h \chi_{B_n}(h), \quad \text{for} \quad (a_n) \in c_0. \tag{3.2}$$

where $\chi_{B_n}(h)$ is the composition of χ_{B_n} and h. Then we need to check T is well defined. Since $a_n \to 0$ as $n \to \infty$, it follows that given $\epsilon > 0$, there exists $\mathbf{M} > 0$ such that if $n \geq \mathbf{M}$ then $|a_n| < \epsilon$. Then we have

$$(3.3) \qquad \|\sum_{\mathbf{M}+1}^{\infty} a_n h\chi_{B_n}(h)\|_{wL_1} \leq \max_{n \geq \mathbf{M}+1} |a_n| < \epsilon.$$

Hence T is well defined.

Clearly, T is linear and preserves the lattice operations. Finally, we need to check that it is an isometry:

$$(3.4) \qquad \begin{aligned} \|T(a_n)\|_{wL_1} &= \|\sum_{n=1}^{\infty} a_n h\chi_{B_n}(h)\|_{wL_1} \\ &= \max_n |a_n| \qquad (\text{by } (3.1)) \\ &= \|(a_n)\|. \end{aligned}$$

Therefore T is a linear and lattice isometry. This proves the proposition.

For the proof of main result, we need one technical lemma.

LEMMA 3.2. *In the proof of Proposition 3.1, there is a $h \in wL_1$ with $\|h\|_{wL_1} = 1$ as a strictly increasing nonnegative function with no atoms.*

PROOF. Let $h \in wL_1$, $\|h\|_{wL_1} = 1$ and $h \geq 0$. If the function h satisfies $\mu(h = r) > 0$ for some r, then we embed the measure algebra of Lebesgue measure λ on $[0,1]$ into the measure algebra of normalized $\mu-$measure on the measurable set $\{h = r\}$. We replace h on this set by the image of the function $\psi(t) = t + r$. Since there are at most countably many points $r \geq 0$ for which $\mu\{h = r\} > 0$, the performance of such replacement for each of these points will change h by at most a bounded measurable function. This means that h can be everywhere strictly positive. This proves the lemma.

THEOREM 3.3. *Let $T : c_0 \longrightarrow wL_1$ be a lattice isometry given by*

$$(3.5) \quad T(a_n) = pointwise - \sum_{n=1}^{\infty} a_n h \chi_{B_n}(h), \qquad \text{as in Proposition 3.1.}$$

Then the range of T is a complemented sublattice of wL_1.

PROOF. By Lemma 3.2, we can assume that h is everywhere strictly positive with $\|h\|_{wL_1} = 1$. Considering the proof of Proposition 3.1, we defined $T : c_0 \longrightarrow wL_1$ by $T((a_n)) = \sum_{n=1}^{\infty} a_n h \chi_{B_n}$. Now let $h_n = h\chi_{B_n}(h)$, where $B_n = \cup_{i \in S_n}[a_i, b_i]$. Then we have $(h_n)_{n=1}^{\infty}$ pairwise disjointly supported nonnegative functions in wL_1 with $\|h_n\|_{wL_1} = 1$, for all $n = 1, 2, 3, \ldots$. By the Corollary 1.4 we can find linear functionals (ϕ_n) on wL_1 such that $\phi_n(h_m) = \delta_{n,m}$ and $\|\phi_n\| = 1$. Now for arbitrary $f \in wL_1$, the number $\phi(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{D_{n,k}} \cdot f)\}$, where $(D_{n,k})_{k=1}^{\infty}$ is a decreasing sequence of subsets of $D_n = supp(h_n)$ and h_n is bounded on $D_{n,k}^c$, for all k. For fixed n, let $r = sgn I_{\mathcal{U}}(\chi_{D_{n,k}} \cdot f)$. Put $m = r\chi_{D_{n,k}}$. Then we have $\|m\|_{\infty} = 1$. Moreover,

$$(3.6) \qquad \begin{aligned} |I_{\mathcal{U}}(\chi_{D_{n,k}} \cdot f)| = \widehat{m}(f) &= I_{\mathcal{U}}(mf) \\ &\leq \|m\|_{\infty}\|f\|_{\mathcal{U}} \\ &= \|f\|_{\mathcal{U}} \\ &\leq \|f\|_{wL_1}. \end{aligned}$$

Now define a projection $P : wL_1 \longrightarrow Tc_0$ by

$$(3.7) \qquad P(f) = \sum_{n=1}^{\infty} \phi_n(f) h_n.$$

First, we need to show that this is well defined. It suffices to show that for all $f \in wL_1$, the sequence $(|\phi_n(f)|)_{n=1}^{\infty}$ converges to zero (so that $(|\phi_n(f)|) \in c_0$). Now

$$
\begin{aligned}
\|\sum_{n=1}^{\infty} \phi_n(f)h_n\|_{wL_1} &\leq \sum_{n=1}^{\infty} |\phi_n(f)|\|h_n\|_{wL_1} \\
&= \sum_{n=1}^{\infty} |\phi_n(f)| \quad (\text{since} \quad \|h_n\|_{wL_1} = 1, \quad n = 1, 2, \cdots) \\
&\leq \|f\|_{\mathcal{U}} \qquad (\text{by Lemma 1.6}) \\
&\leq \|f\|_{wL_1},
\end{aligned} \tag{3.8}
$$

hence we have $\phi_n(f) \to 0$ as $n \to \infty$, for all $f \in wL_1$. Therefore P is well defined. Moreover we have $\|P(f)\|_{wL_1} \leq \|f\|_{wL_1}$. This implies $\|P\| \leq 1$. On the other hand, if we take $h_m \in T(c_0) \subset wL_1$, then $P(h_m) = \sum_{n=1}^{\infty} \phi_n(h_m)h_n = \phi_m(h_m)h_m = h_m$. Then we have $\|P(h_m)\|_{wL_1} = 1$. Hence P is a norm one onto map. The proof of $P^2 = P$ is just routine. Therefore P is the desired norm one projection from wL_1 onto $T(c_0)$. This proves the theorem.

Now we can have the main result.

COROLLARY 3.4. *The Banach envelope of wL_1 contains a complemented subspace that is isometrically isomorphic to c_0.*

PROOF. Immediate from Theorem 3.3.

4. A lattice isomorphic copy of $l^{p,\infty}$ in wL_1

For $1 < p < \infty$, the $weakL_p$ space is the space of all $\Sigma-$measurable functions f such that $\{\omega \in \Omega : |f(\omega)| > 0\}$ is $\sigma-$finite and

$$\|f\| = \sup_{B} \frac{1}{\mu(B)^{1-\frac{1}{p}}} \int_B |f| d\mu < \infty. \tag{4.1}$$

where the supremum is taken over all measurable sets B with $0 < \mu(B) < \infty$. It is well known that the expression

$$|||f||| = \sup_{t>0} t(\mu\{|f| > t\})^{\frac{1}{p}} \tag{4.2}$$

is equivalent to the norm $\|\cdot\|$ defined in (4.1).

In this section, we want to show the existence of a copy of $l^{p,\infty}$ in wL_1.

THEOREM 4.1 (H.P. LOTZ AND T. PECK). *Let (Ω, Σ, μ) be a separable measure space and let $1 < p < \infty$. Then weakL_p is lattice isometric to a closed sublattice of wL_1.*

First, we want to embed $l^{p,\infty}$ into wL_1. If we use Theorem 4.1, it is an easy task because the $l^{p,\infty}$ space is a separable Banach space. Now let T be a lattice isometry given by Theorem 4.1. Does the range of T a complemented sublattice in wL_1 ? We do not know whether the range of T is complemented. But if we use another lattice isomorphism, we can have that the range of the lattice embedding map is a complement sublattice of wL_1. Hence we will use another embedding of $l^{p,\infty}$ into wL_1.

Now, referring to [LEU], fix p $(1 < p < \infty)$ and let $q = \frac{p}{p-1}$. For $n \geq 3$, let

$$(4.3) \qquad g_n = \sup_{j \geq n} (j!)^{\frac{1}{p}} \chi_{A_j^n},$$

$$\text{where} \quad A_j^n = [2^{1-n} + \sum_{m=j+1}^{\infty} \frac{1}{m!},\ 2^{1-n} + \sum_{m=j}^{\infty} \frac{1}{m!}).$$

Then (g_n) is a pairwise disjoint sequence of measurable functions on $[0,1]$.

THEOREM 4.2 (DENNY H. LEUNG). *The map $T : l^{p,\infty} \longrightarrow$ weak $L_p[0,1]$ defined by $T((a_n)_{n=3}^{\infty}) =$ pointwise $-\sum_{n=3}^{\infty} a_n g_n$ is a lattice isomorphism.*

It is not hard to see that T is not a lattice isometry (see [LEU]).

PROPOSITION 4.3. *For $1 < p < \infty$, the $l^{p,\infty}$ space is lattice isomorphic to a closed sublattice of wL_1.*

PROOF. Immediate from Theorem 4.1 and Theorem 4.2.

Let $U : l^{p,\infty} \longrightarrow wL_1$ be a lattice isomorphism with $\|U\| = C < \infty$. Then U preserves the lattice structure of $l^{p,\infty}$ in wL_1. Let S : weak$L_p[0,1] \longrightarrow wL_1$ be a lattice isometry given by Theorem 4.1. Then we can define $S(g_n) = f_n$ foach $n \geq 3$. Since (g_n) are pairwise disjoint, (f_n) are also pairwise disjoint. Now we can have the main result.

THEOREM 4.4. *For $1 < p < \infty$, the Banach envelope of wL_1 contains a complemented sublattice which is isomorphic to $l^{p,\infty}$.*

PROOF. Define $U : l^{p,\infty} \longrightarrow wL_1$ by

$$U((a_n)_{n=3}^{\infty}) = \sum_{n=3}^{\infty} a_n f_n, \quad \text{for each} (a_n)_{n=3} \in l^{p,\infty}, \tag{4.4}$$

where $f_n = Sg_n$ for each $n \geq 3$ and g_n is defined in Theorem 4.2. Without loss of generality, we can assume $\|f_n\|_{wL_1} = 1$, by normalizing. It suffices to show that the range of U is complemented in wL_1. Since (f_n) are pairwise disjoint and $\|f_n\|_{wL_1} = 1$ foe each $n \geq 3$, we can find linear functionals $\phi_n \in wL_1^*$ with $\phi_n(f_m) = \delta_{n,m}$, and $\|\phi_n\| = 1$, for each $n = 3, 4, \ldots$ .by Corollary 1.4.

Define $P : wL_1 \longrightarrow Ul^{p,\infty}$ by

$$P(f) = \sum_{n=3}^{\infty} \phi_n(f) f_n. \tag{4.5}$$

We need to show that P is well defined.

$$\begin{aligned} \|P(f)\|_{wL_1} &= \|\sum_{n=3}^{\infty} \phi_n(f) f_n\|_{wL_1} \\ &= \|\sum_{n=3}^{\infty} \phi_n(f) g_n\|, \quad \text{where} Sg_n = f_n \qquad (4.6) \\ &\leq C\|\|(\phi_n(f))\|\| \quad \text{where } C \text{ is an isomorphism constant} \\ &\leq D \sup_B \frac{1}{\mu(B)^{1-\frac{1}{p}}} \sum_{n \in B} |\phi_n(f)| \\ &\leq D \sum_{n=3} |\phi_n(f)| \\ &\leq D\|f\|_{wL_1} \quad \text{(by Lemma 1.6)}, \end{aligned}$$

where the equality follows from the fact that $(a_n)_{n=3} \mapsto \sum_{n=3}^{\infty} a_n g_n$ is an isophic lattice embedding, the first inequality comes from the equivalence of the quasinorm $|||\cdot|||$ defined in (4.3) and the norm $\|\cdot\|$ defined in (4.2), and the seconed inequality follows from the fact that $\mu(B) \geq 1$ and μ in counting measure. Therefore $\|P\| \leq D$ and P is well defined.

The proofs of linearity and $P^2 = P$ are just routine. Therefore $Ul^{p,\infty}$ is a complemented sublattice of wL_1. This proves the theorem.

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