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EXISTENCE OF SOLUTION OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS IN GENERAL BANACH SPACES

JIN GYO JEONG AND KI-YEON SHIN

ABSTRACT. The existence of a bounded generalized solution on the real line for a nonlinear functional evolution problem of the type

$$\text{(FDE)} \qquad x'(t) + A(t, x_t)x(t) \ni 0, \quad t \in R$$

in a general Banach spaces is considered. It is shown that (FDE) has a bounded generalized solution on the whole real line with well-known Crandall and Pazy's result and recent results of the functional differential equations involving the operator $A(t)$.

1. Introduction and preliminaries

Let X be a real Banach space with norm $\|\cdot\|$. The symbol $\|\cdot\|_\infty$ denotes the sup-norm of a bounded function over its domain. The symbol $\mathcal{R}$, $\mathcal{C}$ denote the sets $(-\infty, \infty)$, $\{\psi : (-\infty, 0] \to \bar{D}\ ;\ \psi$ is strongly continuous and $\|\psi\|_\infty \leq r\}$, respectly. Here $\bar{D}$ is a fixed closed subset of X and r is a positive constant. We also let $x_t(s) = x(t+s)$, $s \in (-\infty, 0]$ for $t \in \mathcal{R}$.

An operator $A : D(A) \subset X \to 2^X$ is called "ω-accretive" if

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geq (1 - \lambda\omega)\|x_1 - x_2\|$$

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for each $\lambda > 0$ such that $\omega\lambda < 1$ and every $[x_1, y_1], [x_2, y_2] \in A$. It is called "$m$-$\omega$-accretive" if it is ω- accretive and $R(I + \lambda A) = X$ for all $\lambda > 0$. Also A is said to be "accretive" if $\omega = 0$, and "strongly accretive" if $\omega < 0$. If A is m-ω-accretive, we set $|Ax| = \lim_{\lambda \to 0} \|A_\lambda x\|$, $x \in X$ where $A_\lambda = (I - J_\lambda)/\lambda$ with $J_\lambda = (I + \lambda A)^{-1}$. We also set $\hat{D} = \{x \in X : |Ax| < \infty\}$. It is well-known that $D(A) \subset \hat{D}(A) \subset \overline{D(A)}$. For other properities of these operators, the reader is referred to Barbu [1], Crandall [3], Crandall and Pazy [4] and Evans [5].

In this paper we consider functional equation of the type :

$$\text{(FDE)} \qquad x'(t) + A(t, x_t)x(t) \ni 0, \quad t \in \mathcal{R}$$

with the operator satisfying at least the following conditions.

(H.1) The domain $D(t) = D(A(t, \psi))$ is independent of $\psi \in \mathcal{C}$. Moreover, $A(t, \psi)u$ is strongly m-accretive with respect to $u \in D(t)$, i.e. $R(I + \lambda A(t, \psi)) = X$ for $\lambda > 0$, $(t, \psi) \in \mathcal{R} \times \mathcal{C}$ and

$$\|u - v + \lambda(A(t, \psi)u - A(t, \psi)v)\| \geq (1 + \lambda\alpha)\|u - v\|$$

for $u, v \in D(t)$, where α is a fixed positive constant.

(H.2) There exists a monotone increasing function $L : [0, \infty) \to [0, \infty)$ such that for every $(t, s, \psi_1, \psi_2, u) \in \mathcal{R}^2 \times \mathcal{C}^2 \times X$ we have

$$\begin{aligned} &\|A_\lambda(t, \psi_1)u - A_\lambda(s, \psi_2)u\| \\ &\qquad \leq L(\|u\|)[|t - s|(1 + \|A_\lambda(s, \psi_2)u\|) + \|\psi_1 - \psi_2\|_\infty]. \end{aligned}$$

It is very well-known that the generalized domain $\hat{D}(t)$ and the closure $\overline{D(t)}$ are fixed subset of X by the above condition [5, lemma 3.1]. For simplicity, we denote by $\hat{D} = \hat{D}(A)$ and $\overline{D} = \overline{D(t)}$.

(H.3) There exists $x_0 \in \hat{D}$ with $\|x_0\| \leq r/2$ and a constant $N > 0$ such that $|A(t, \psi)x_0| \leq N$ for every $(t, \psi) \in \mathcal{R} \times \mathcal{C}$.

Our purpose here is to show that, under an additional assumption of the constants N, α and r, the conditions (H.1)–(H.3) gurantee the existence of a bounded generalized (to be defined below) solution $x(t)$ of (FDE) such that $\|x(t)\| \leq r$.

(FDE) and equation of the type

$$x'(t) + A(t)x(t) \ni G(t, x_t)$$

have been studied by many authors for last two decades where $A(t)$ is m-accretive and $G(t, \psi)$ is locally Lipschitz continuous with respect to t, ψ. In case that $A(t)$ is strongly accretive with locally Lipschitz G, it is special case of (FDE). However, (FDE) also includes equation with multiplicative perturbation of $A(t)$. The reader is referred to that paper for various fundamental results concerning evolution equations in general Banach spaces. Important result can be found in [6,7,9].

For general result concerning linear and perturbed linear evolution equations, we cite the book of Pazy[11]. For other results concerning ordinary and functional version of (FDE), the reader is referred to [4,5,6,7,8,9,12].

Consider the equation

$$\text{(E)} \qquad x'(t) + B(t)x(t) \ni 0, \quad t \in [T_0, T],$$

where T_0, T are two fixed constants. We assume that $B(t)$, $t \in [T_0, T]$, satisfies the following conditions.

(C1) For the domain $D(B(t)) \subset X$, $\overline{D(B(t))} = \overline{D_1}$ is independent of t and for some $\omega \in \mathcal{R}$, $B(t) + \omega I$ is accretive on $[T_0, T]$: i.e. for every $\lambda \in (0, \infty)$ with $\lambda\omega < 1$ and all $u, v \in D(B(t))$, we have

$$\|u - v + \lambda(B(t)u - B(t)v)\| \geq (1 - \lambda\omega)\|u - v\|.$$

Here I denotes the identity operator in X.

(C2) $R(I + \lambda B(t)) \supset \overline{D_1}$ for $t \in [T_0, T]$ and $\lambda \in (0, \lambda_0)$, where λ_0 is a positive constant with $\lambda_0\omega < 1$.

(C3) There exists a continuous function $h : [T_0, T] \to X$ of bounded variation on $[T_0, T]$ and a monotone increasing function $L : [0, \infty) \to [0, \infty)$ such that

$$\|J_\lambda(t)u - J_\lambda(s)u\| \leq \lambda\|h(t) - h(s)\|L(\|u\|)(1 + |B(s)u|)$$

for every $\lambda \in (0, \lambda_0)$, $t, s \in [T_0, T]$, and $u \in \overline{D_1}$.

Given $u_0 \in \overline{D_1}$, $s \in [T_0, T)$, (E) has the "generalized solution" $u(t) = U(t,s)u_0$, $t \in [s,T]$ where $U(t,s)$, $T_0 \le s \le t \le T$, is the evolution operator associated the $B(t)$, $t \in [T_0, T]$, as obtained by Crandall and Pazy in [4]. We have $u(s) = u_0$.

Now, assume that $f, g : [T_0, T] \to X$ are two given continuous functions. Assume, further, that $B(t)$, $B_1(t) = B(t) - f(t)$, and $B_2(t) = B(t) - g(t)$ satisfy the conditions (C1)–(C3). Let $x(t), y(t)$ be generalized solutions of

$$x'(t) + B(t)x(t) \ni f(t), \quad t \in [T_0, T],$$
$$y'(t) + B(t)y(t) \ni g(t), \quad t \in [T_0, T],$$

respectly. We can easily see that for $T_0 \le s \le t \le T$,

$$(1) \quad \|x(t) - y(t)\| \le e^{\alpha(t-s)}\|x(s) - y(s)\| + \int_s^t e^{\alpha(t-\tau)}\|f(\tau) - g(\tau)\| d\tau.$$

Given interval J of $\mathcal{R}$, assume that the conditions (C1)–(C3) hold with $[T_0, T]$ replaced everywhere by J. Let $x(t)$, $t \in J$, be a continuous X-valued function such that for every finite interval $[T_0, T]$ of J, the restriction of $x(t)$ on $[T_0, T]$ is a generalized solution of (E). Then $x(t)$ is called a "generalized solution of (E) on J".

2. Main results

THEOREM 1. *Assume that the conditions* (H.1)–(H.3) *are satisfied with* $\alpha > \max\{2L(r), 2N/r\}$. *Then there exists a bounded generalized solution* $x(t)$ *of* (FDE) *such that* $\|x(t)\| \le r$ *on* $\mathcal{R}$. *This solution is globally Lipschitz continuous and lies in* $\hat{D}$.

PROOF. Let $K > 0$ be a constant such that $N\alpha + L(r) < (\alpha - 2L(r))K$. Such a constant exists by our assumption $\alpha > 2L(r)$. Let

$$\mathcal{S} = \{f : \mathcal{R} \to \overline{D}; \|f\| \le r \text{ and } \|f(t) - f(s)\| \le K|t - s| \text{ for all } t, s \in \mathcal{R}\}.$$

Then $\mathcal{S}$ is a closed and bounded subset of the Banach space of all bounded continuous functions on $\mathcal{R}$. Thus, $\mathcal{S}$ is a complete metric space with the sup-norm. Let $f \in \mathcal{S}$ be given and $B(t) \equiv A(t, f_t)$. We consider the equation

$$(2) \qquad x'(t) + A(t, f_t)x(t) \ni 0.$$

Clearly, the operator $B(t) - \alpha I$ is accretive for every $t \in \mathcal{R}$ by (H.1). It is easy to see that $R(I + \lambda B(t)) = X$ for all $(t, \lambda) \in \mathcal{R} \times (0, \infty)$. Since

$$\|f_t - f_s\|_\infty = \sup_{\theta \le 0} \|f(t+\theta) - f(s+\theta)\| \le K|t-s|,$$

we actually obtain by (H.2) that

$$\begin{aligned}\|B_\lambda(t)u - B_\lambda(s)u\| &\le L(\|u\|)[|t-s|(1 + \|B_\lambda(s)u\|) + \|f_t - f_s\|_\infty] \\ &\le L(\|u\|)|t-s|(1+K)|B(s)u|.\end{aligned}$$

It has shown that $B(t)$, $t \in [-n, n]$, $n = 1, 2, \cdots$, satisfies the conditions (C1)–(C3) so that there exists a unique generalized solution $x_n(t)$, $t \in [-n, n]$, of (2) with $x_n(-n) = x_0$ by Crandall and Pazy [4, theorem 2.1].

In what follows, we let $B_k(t) = k(I - J_k(t))$, $J_k(t) = (I + (1/k)B(t))^{-1}$. We consider the equation

$$(3) \qquad x'(t) + B_k(t)x(t) = 0, \quad t \in [-n, n], \quad x(-n) = x_0.$$

For each $k = 1, 2, \cdots$, since $B_k(t) - \beta_k I$, where $\beta_k = (k\alpha)/(k + \alpha)$, is accretive by [4, lemma 1.2] and $R(I + \lambda B_k(t)) = X$ for all $\lambda > 0$, $B_k(t)$ is also satisfies (C1)-(C2). Moreover, $B_k(t)$ is single-valued and $D(B_k(t)) = X$ is independent of $t \in [-n, n]$, $B_k(t)$ satisfies (C3) by [4, lemma 3.2]. We let $x_n^k(t)$, $t \in [-n, n]$, be a unique generalized solution of (3) for each $k = 1, 2, \cdots$.

We note that $x_n^k(t)$ converges uniformly to $x_n(t)$ on $[-n, n]$ by Crandall and Pazy [4, comments after the proof of lemma 4.2]. We also note that $\|x_n(t) - x_0\| \le N/\alpha$ for each $t \in [-n, n]$. To show this, we first consider the equation

$$(4) \qquad y'(t) + B_k(t)y(t) = B_k(t)x_0, \quad t \in [-n, n], \quad y(-n) = x_0.$$

Clearly, $y(t) \equiv x_0$, $t \in [-n, n]$, is a strongly continuously differentiable solution of (4). Since $B_k(t)$ satisfies the conditions (C1) -(C3) on $[-n, n]$

for $n, k = 1, 2, \cdots$, we have a inequality which is very similar to (1). It means that, for $t \in [-n, n]$ and $k = 1, 2, \cdots$,

$$\begin{aligned}
\|x_n^k(t) - x_0\| &= \|x_n^k(t) - y(t)\| \\
&\le e^{-\beta_k(t+n)}\|x_n^k(-n) - y(-n)\| + \int_{-n}^{t} e^{-\beta_k(t-s)}\|B_k(s)x_0\|ds \\
&\le \int_{-n}^{t} e^{-\beta_k(t-s)} \frac{k}{k+\beta_k}|B(s)x_0|ds \\
&\le \frac{Nk}{k+\beta_k}\int_{-\infty}^{t} e^{-\beta_k(t-s)}ds \\
&= \frac{Nk}{\beta_k(k+\beta_k)} = \frac{N(k^3+2k^2\alpha+k\alpha^2)}{\alpha(k^3+2k^2\alpha)} = \frac{N\gamma_k}{\alpha},
\end{aligned}$$

where $\gamma_k = (k^3 + 2k^2\alpha + k\alpha^2)/(k^3 + 2k^2\alpha)$. We note that $\gamma_k > 1$ for $k = 1, 2, \cdots$ and $\gamma_k \to 1$ as $k \to \infty$. Here we have used $\beta_k = (k\alpha)/(k+\alpha)$ and condition (H.3). Thus, by letting $k \to \infty$, we may conclude that $\|x_n(t) - x_0\| \le N/\alpha$ for $t \in [-n, n]$.

Now, we show that $|B(t)x_n(t)| \le K$ for $t \in [-n, n]$ and $x_n(t)$ is Lipschitz continuous on $[-n, n]$. We fix $t \in (-n, n)$ and let $h \neq 0$ be such that $t + h \in (-n, n)$. We also let $\phi_n^k(t) = x_n^k(t+h) - x_n^k(t)$. Since $x_n^k(t)$ is strongly continuously differentiable, so does $\phi_n^k(t)$. By [1, proposition 9.4], we have

$$\limsup_{h\to 0+}(\|\phi_n^k(t)\|^2 - \|\phi_n^k(t-h)\|^2)/h \le 2 < (\phi_n^k)'(t), j >,$$

where j is any element of $F(\phi_n^k(t))$. Here $F : X \to 2^{X^*}$ is the duality mapping and $< u, u^* >$ denotes by the value of $u^* \in X^*$ at $u \in X$. We see that $F(\phi_n^k(t))$ is non-empty set by Hahn-Banach theorem.

From the above inequality, we have

$$\begin{aligned}
(d^-/dt)\|\phi_n^k(t)\|^2 &\le 2 < (x_n^k)'(t+h) - (x_n^k)'(t), j > \\
&= -2 < B_k(t+h)x_n^k(t+h) - B_k(t)x_n^k(t), j > \\
&= -2 < B_k(t+h)x_n^k(t+h) - B_k(t+h)x_n^k(t), j > \\
&\quad - 2 < B_k(t+h)x_n^k(t) - B_k(t)x_n^k(t), j > .
\end{aligned}$$

But, since $< B_k(t)u - B_k(t)v, j^* > \geq \beta_k \|u-v\|^2$ for $j^* \in F(u-v)$, the above inequality becomes

$$\begin{aligned}&(d^-/dt)\|\phi_n^k(t)\|^2\\&\leq -2\beta_k\|\phi_n^k(t)\|^2 + 2\|B_k(t+h)x_n^k(t) - B_k(t)x_n^k(t)\|\|\phi_n^k(t)\|\\&\leq -2\beta_k\|\phi_n^k(t)\|^2\\&\qquad + 2L(\|x_n^k(t)\|)[|h|(1+\|B_k(t)x_n^k(t)\|) + \|f_t - f_s\|_\infty]\|\phi_n^k(t)\|.\end{aligned}$$

We note that, since $2N/\alpha < r$ and $\gamma_k(>1) \to 1$ as $k \to \infty$, there exists an index $\hat{k}_0$ such that $2N\gamma_k/\alpha < r$ for all $k \geq \hat{k}_0$. Also, we have an index $\tilde{k}_0$ such that

$$N\alpha + L(r) < (\beta_k - 2L(r))K < (\alpha - 2L(r))K \tag{5}$$

for all $k \geq \tilde{k}_0$ by the fact $\beta_k < \alpha$ for all $k = 1, 2, \cdots$ and $\beta_k \to \alpha$ as $k \to \infty$. Let $k_0 = \max\{\hat{k}_0, \tilde{k}_0\}$. From now on, we only consider such values of $k \geq k_0$. For all $k \geq k_0$ we get

$$L(\|x_n^k\|) \leq L(\|x_0\| + N\gamma_k/\alpha) \leq L(r/2 + r/2) = L(r).$$

Then, we have

$$\begin{aligned}(d^-/dt)\|\phi_n^k(t)\|^2 &\leq -2\beta_k\|\phi_n^k(t)\|^2\\&\quad + 2[L(r)|h|(1+\|(x_n^k)'(t)\|) + L(r)|h|(1+K)]\|\phi_n^k(t)\|.\end{aligned}$$

We now apply [1, proposition 9.5] to have

$$\begin{aligned}\|\phi_n^k(t)\| &\leq \|\phi_n^k(-n)\|e^{-\beta_k(t+n)}\\&\quad + L(r)|h|\int_{-n}^t e^{-\beta_k(t-s)}(\|(x_n^k)'(s)\| + 1 + K)ds.\end{aligned}$$

Dividing by $|h|$ and then letting $h \to 0$, we get

$$\begin{aligned}
\|(x_n^k)'(t)\| &\le \|(x_n^k)'(-n)\|e^{-\beta_k(t+n)} \\
&\quad + L(r)[(1+K)\sup_{s\in[-n,t]}\|(x_n^k)'(s)\|]\int_{-n}^{t} e^{-\beta_k(t-s)}ds \\
&\le \frac{k}{k+\alpha}|B(-n)x_0|e^{-\beta_k(t+n)} + L(r)(1+K)\int_{-\infty}^{t} e^{-\beta_k(t-s)}ds \\
&\quad + L(r)(\sup_{t\in[-n,n]}\|(x_n^k)'(t)\|)\int_{-\infty}^{t} e^{-\beta_k(t-s)}ds \\
&\le Nk/(k+\alpha) + L(r)(1+K)/\beta_k \\
&\quad + (L(r)/\beta_k)(\sup_{t\in[-n,n]}\|(x_n^k)'(t)\|).
\end{aligned}$$

Thus we have

$$(1 - L(r)/\beta_k)\sup_{t\in[-n,n]}\|(x_n^k)'(t)\| \le Nk/(k+\alpha) + L(r)(1+K)/\beta_k.$$

Since $L(r) < 2L(r) < \beta_k < \alpha$, $1 - L(r)/\beta_k > 0$. It follows that

$$\begin{aligned}
\sup_{t\in[-n,n]}\|(x_n^k)'(t)\| &\le \frac{Nk/(k+\alpha) + L(r)(1+K)/\beta_k}{(1 - L(r)/\beta_k)} \\
&= \frac{N\beta_k^2/\alpha + L(r)(1+K)}{\beta_k - L(r)} \le K
\end{aligned}$$

by (5). In other words, we obtain that

$$\sup_{t\in[-n,n]}\|B_k(t)x_n^k(t)\| = \sup_{t\in[-n,n]}\|(x_n^k)'(t)\| \le K.$$

Since $x_n^k(t) \to x_n(t)$ as $n \to \infty$ uniformly on $[-n, n]$ by [4, comments after the proof of lemma 4.2] and

$$(1+\alpha/j)\|B_j(t)x_n^k(t)\| \le (1+\alpha/k)\|B_k(t)x_n^k(t)\|$$

for $k_0 \le j \le k$ by [4, lemma 1.2], we have $\|B_j(t)x_n(t)\| \le K$ and $|B(t)x_n(t)| \le K$, $t \in [-n, n]$. Consequently, for $n = 1, 2, \cdots$, we have shown that the equation (2) has a unique generalized solution $x_n(t)$, $t \in [-n, n]$ such that $x_n(-n) = x_0$, $\|x_n(t)\| \leq \|x_0\| + N/\alpha \leq r$, $x_n(t) \in \hat{D}$, and $x_n(t)$ is Lipschitz continuous on $[-n, n]$ with Lipschitz constant K.

Now, let the numbers a, b be such that $-n \leq -m \leq a < b \leq m \leq n$. Then

$$\begin{aligned}\|x_n(t) - x_m(t)\| &\leq e^{-\alpha(t+m)}\|x_n(-m) - x_0\| \\ &\leq (N/\alpha)e^{-\alpha(t+m)}\end{aligned}$$

for all $t \in [a, b]$. Hence $\{x_n(t)\}$ is a Cauchy sequence in the sup-norm on $[a, b]$. Let $x(t)$ be the uniform limit of $x_n(t)$ and $y(t)$ be a unique generalized solution of (2) on $[a, b]$ such that $x(a) = y(a)$. By (1),

$$\|x_n(t) - y(t)\| \leq e^{-\alpha(t-a)}\|x_n(a) - y(a)\|, \qquad t \in [a, b].$$

This shows $x(t) \equiv y(t)$ on $[a, b]$. Since $[a, b]$ is arbitrary, we conclude that there is a generalized solution $x(t)$ of (2) on $\mathcal{R}$ such that $x \in \mathcal{S}$. To show the uniqueness of such a solution, we let $x_1(t)$, $x_2(t)$ be two generalized solutions of (2). By (1),

$$\begin{aligned}\|x_1(t) - x_2(t)\| &\leq e^{-\alpha(t-s)}\|x_1(s) - x_2(s)\| \\ &\leq e^{-\alpha(t-s)}(\|x_1(s)\| + \|x_2(s)\|) \leq 2re^{-\alpha(t-s)}\end{aligned}$$

for every $t, s \in \mathcal{R}$ with $s \leq t$. Letting $s \to -\infty$, we obtain $x_1(t) \equiv x_2(t)$ on $\mathcal{R}$.

We now define an operator $T : \mathcal{S} \to \mathcal{S}$ such as Tf is a unique bounded solution of (2) on $\mathcal{R}$ for given $f \in \mathcal{S}$. We have shown that $Tf \in \mathcal{S}$. We show T is a strict contraction on $\mathcal{S}$. For sufficiently large k and fixed n, we let $[a, b] \subset [-n, n]$ and consider the equations

$$\begin{aligned}&x'(t) + A_k(t, f_t)x(t) = 0, \quad t \in [-n, n], \quad x(-n) = x_0, \\ &y'(t) + A_k(t, g_t)y(t) = 0, \quad t \in [-n, n], \quad y(-n) = x_0,\end{aligned}$$

where $f, g \in \mathcal{S}$ are two given functions and $A_k(t, f_t) = k(I - J_k(t, f_t))$ with $J_k(t, f_t) = (I + A(t, f_t)/k)^{-1}$. Let $x_n^k(t)$, $y_n^k(t)$ be the unique strongly continuously differentiable solutions of the above equations, respectively. Since we may handle the second equation as

$$y'(t) + A_k(t, f_t)y(t) = A_k(t, f_t)y(t) - A_k(t, g_t)y(t),$$

we obtain that

$$\begin{aligned}\|x_n^k(t) - y_n^k(t)\| &\le \int_{-\infty}^{t} e^{-\beta_k(t-s)}\|A_k(s, f_s)y_n^k(t) - A_k(s, g_s)y_n^k(s)\|ds \\ &\le \int_{-\infty}^{t} e^{-\beta_k(t-s)} L(\|y_n^k(s)\|)\|f_s - g_s\|_\infty ds \\ &\le \int_{-\infty}^{t} e^{-\beta_k(t-s)} L(r)\|f - g\|_\infty ds \\ &= (L(r)/\beta_k)\|f - g\|_\infty.\end{aligned}$$

Therefore, letting $k \to \infty$, we have

$$\|(Tf)(t) - (Tg)(t)\| \le (L(r)/\alpha)\|f - g\|_\infty < \|f - g\|_\infty.$$

It shows that the unique fixed point $x(t)$, $t \in \mathcal{R}$ is a generalized solution of (FDE) by Banach's contraction principle.

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Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: kyshin@hyowon.cc.pusan.ac.kr