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# ON THE TAYLOR-BROWDER SPECTRUM

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ABSTRACT. In this paper we extend the Zemanek s characterization of the Browder spectrum for a commuting $n$-tuple of operators in $\mathcal{L}(H)$ and show that if $\mathbf{T} = (T_1, \cdots, T_n)$ is Browder then there exists an $n$-tuple $\mathbf{K} = (K_1, \cdots, K_n)$ of compact operators and an invertible commuting $n$-tuple $(S_1, \cdots, S_n)$ for which $\mathbf{T} = \mathbf{S} + \mathbf{K}$ and $S_i K_j = K_j S_i$ for all $1 \leq i, j \leq n$.

## 1. Introduction

Suppose $H$ is a complex Hilbert space and write $\mathcal{L}(H)$ for the set of all bounded linear operators acting on $H$. Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a commuting $n$-tuple of operators in $\mathcal{L}(H)$ and let $\Lambda[e] = \{\Lambda^k[e_1, \ldots, e_n]\}_{k=0}^n$ be the exterior algebra on $n$ generators ($e_i \wedge e_j = -e_j \wedge e_i$ for all $i, j = 1, \ldots, n$). Write $\Lambda(H) = \Lambda[e] \otimes H$. Let $\Lambda(\mathbf{T}) : \Lambda(H) \longrightarrow \Lambda(H)$ be given by

$$\Lambda(\mathbf{T})(\omega \otimes x) = \sum_{i=1}^{n} (e_i \wedge \omega) \otimes T_i x.$$

The operator $\Lambda(\mathbf{T})$ can be the represented by the *Koszul complex* for $\mathbf{T}$:

$$0 \longrightarrow \Lambda^0(H) \xrightarrow{\Lambda^0(\mathbf{T})} \Lambda^1(H) \xrightarrow{\Lambda^1(\mathbf{T})} \cdots \xrightarrow{\Lambda^{n-1}(\mathbf{T})} \Lambda^n(H) \longrightarrow 0,$$

where $\Lambda^k(H)$ is the collection of $k$-forms and $\Lambda^k(\mathbf{T}) = \Lambda(\mathbf{T})|_{\Lambda^k(H)}$. Evidently, $\Lambda(\mathbf{T})^2 = 0$, so that ran $\Lambda(\mathbf{T}) \subseteq$ ker $\Lambda(\mathbf{T})$. We recall ([4],[8],[9]) that $\mathbf{T}$ is said to be (*Taylor*) *invertible* if ker $\Lambda(\mathbf{T})$ = ran$\Lambda(\mathbf{T})$ (i.e., the Koszul complex for $\mathbf{T}$ is exact at every stage.) and is said to be

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(*Taylor*) *Fredholm* if $\ker\Lambda(\mathbf{T})/\mathrm{ran}\Lambda(\mathbf{T})$ is finite dimensional (i.e., all cohomologies of the Koszul complex for $\mathbf{T}$ are finite dimensional.). We shall write $\sigma_T(\mathbf{T})$ and $\sigma_{Te}(\mathbf{T})$ for the *Taylor spectrum* and the *Taylor essential spectrum* of $\mathbf{T}$, respectively : namely,

$$\sigma_T(\mathbf{T}) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not invertible}\}$$

and

$$\sigma_{Te}(\mathbf{T}) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not Fredholm}\}.$$

We also recall ([5],[6],[8],[9]) that $\mathbf{T}$ is said to be (*Taylor*)*Browder* if $\mathbf{T}$ is Fredholm and there exists a deleted open neighborhood $N_0$ of $0 \in \mathbb{C}^n$ such that $\mathbf{T} - \lambda$ is invertible for all $\lambda \in N_0$. Then the *Taylor-Browder spectrum*, $\sigma_{Tb}(\mathbf{T})$, is defined by

$$\sigma_{Tb}(\mathbf{T}) = \sigma_{Te}(\mathbf{T}) \cup \mathrm{acc}\, \sigma_T(\mathbf{T}),$$

where $\mathrm{acc}\, \sigma_T(\mathbf{T})$ denotes the set of accumulation points of the Taylor spectrum of $\mathbf{T}$. In this paper we consider a characterization of the Taylor-Browder spectrum and a Riesz-Schauder theorem for a commuting $n$-tuple of operators.

## 2. A characterization of the Taylor-Browder spectrum

For a single operator $T$ acting on a Banach space, Zemánek ([14, Theorem 1, (2)]) characterized the Browder spectrum as the intersection of the ordinary spectra of compressions to subspaces with finite codimensions. In this section we extend the Zemánek's characterization for a commuting $n$-tuple of operators in $\mathcal{L}(H)$. If $M$ is a common invariant subspace of $H$ for each $T_i$ in $\mathcal{L}(H)$, we denote an $n$-tuple of compressions to $M$ by $\mathbf{T}_M = (T_{1\,M}, \cdots, T_{n\,M})$.

The following observation was, basically, noticed in [4] and [12]:

LEMMA 1. *Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a commuting $n$-tuple of operators in $\mathcal{L}(H)$. If $P$ is a continuous projection commuting with each $T_i$, then*

$$\sigma_T(\mathbf{T}) = \sigma_T(\mathbf{T}_{PH}) \;\cup\; \sigma_T(\mathbf{T}_{(I-P)H}). \tag{1.1}$$

PROOF. Since the subspace $PH$ reduces each $T_i$ and $\Lambda(H)$ is just $\Lambda[e] \otimes (PH \oplus (I-P)H)$, $\Lambda(\mathbf{T})$ admits the following operator matrix representation:

$$\Lambda(\mathbf{T}) = \begin{pmatrix} \Lambda(\mathbf{T})_{\Lambda(PH)} & 0 \\ 0 & \Lambda(\mathbf{T})_{\Lambda((I-P)H)} \end{pmatrix}$$

with respect to the decomposition $\Lambda(H) = \Lambda(PH) \oplus \Lambda((I-P)H)$. Thus we see that the Koszul complex for $\mathbf{T}$ is exact if and only if the Koszul complex for $\mathbf{T}_{PH}$ and $\mathbf{T}_{(I-P)H}$ are both exact, which implies (1.1). □

Let $\mathcal{P}(H)$ denote the set of all projections in $H$ with a finite codimension. For any $P \in \mathcal{P}(H)$ the compression of $T$ is a bounded linear operator on the closed subspace $P(H)$ defined by $T_{PH}x = PTx$ for each $x \in PH$. The following is a generalization of Zemánek's result for a commuting $n$-tuple of operators in $\mathcal{L}(H)$:

THEOREM 2. *If $\mathbf{T} = (T_1, \cdots, T_n)$ is a commuting $n$-tuple of operators in $\mathcal{L}(H)$, then*

$$\sigma_{Tb}(\mathbf{T}) = \bigcap_{P \in \mathcal{P}(H)} \{\sigma_T(\mathbf{T}_{PH}) | \, PT_i = T_iP\}. \tag{2.1}$$

PROOF. Suppose that $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ is in $\sigma_{Tb}(\mathbf{T})$. If $\lambda$ does not in the right-hand side of (2.1), then there exists a projection $P \in \mathcal{P}(H)$ such that $\lambda \notin \sigma_T(\mathbf{T}_{PH})$ and $PT_i = T_iP$. Thus we have a spectral formula (1.1):

$$\sigma_T(\mathbf{T}) = \sigma_T(\mathbf{T}_{PH}) \cup \sigma_T(\mathbf{T}_{(I-P)H}). \tag{2.2}$$

Thus $\lambda \in \sigma_T(\mathbf{T}_{(I-P)H})$, and hence $\lambda \in \text{iso}\sigma_T(\mathbf{T})$ because $\lambda \notin \sigma_T(\mathbf{T}_{PH})$ and $(I-P)H$ is a finite subspace of $H$. Thus an argument of Fialkow([7, Lemma 2.1]) gives that $\lambda \notin \sigma_{Te}(\mathbf{T})$ because the spectral subspace of $\lambda$ is contained in $(I-P)H$. This leads a contradiction. Conversely, suppose that $\lambda \notin \sigma_{Tb}(\mathbf{T})$. Since $\lambda \in \text{iso}\sigma_T(\mathbf{T})$, we can find a projection $Q$ with respect to $\lambda$ ([12, Theorem 4.8]) such that $QT_i = T_iQ$, the spectral subspace of $\lambda$ is $QH$, and $\lambda \notin \sigma_T(\mathbf{T}_{(I-Q)H})$. Putting $P = I - Q$ gives that $P \in \mathcal{P}(H)$ and $\lambda \notin \sigma_T(\mathbf{T}_{PH})$. Thus the proof is completed. □

## 3. A Riesz-Schauder theorem

For a single operator $T \in \mathcal{L}(H)$, the *Riesz-Schauder theorem* ([8],[13]) says that

$$T \text{ is Browder} \Longleftrightarrow T = S + K,$$

where $S$ is invertible, $K$ is compact and $SK = KS$. Therefore we meet a natural question:

QUESTION. *Can be the Riesz-Schauder theorem extended for commuting $n$-tuples of operators?*

In this section we discuss the above question. R.E. Harte ([8, Problem 11.10.5, p.557]) suggested a half of the above question. The following theorem gives the "only if" part of the Riesz-Schauder theorem and also answers a question of R.E.Harte; the answer is yes.

THEOREM 3. *If $\mathbf{T}$ is Browder , then there exists an $n$-tuple $\mathbf{K} = (K_1, \cdots, K_n)$ of compact operators such that $\mathbf{T} = \mathbf{S} + \mathbf{K}$ where $\mathbf{S} = (S_1, \cdots, S_n)$ is an invertible commuting $n$-tuple and $S_i K_j = K_j S_i$ for all $1 \le i, j \le n$.*

PROOF. If $\mathbf{T}$ is Browder but not invertible then $\mathbf{T}$ is Fredholm and $0 \in \text{iso}\sigma_T(\mathbf{T})$. The arguments of Taylor ([12, Theorem 4.9; Corollary 4.10]) give that there exists a projection $P \in \mathcal{L}(H)$ satisfying that $P$ commutes with each $T_i$, $\mathbf{T}_{PH}$ is a commuting $n$-tuple of quasinilpotent operators and $0 \notin \sigma_T(\mathbf{T}_{(I-P)H})$. In particular, $P$ is of finite rank and hence compact. Now consider the following commuting $n$-tuple:

$$\mathbf{T} + \mathbf{P} = (T_1 + P, \cdots, T_n + P),$$

where $\mathbf{P} = (P, \cdots, P)$. Then by Lemma 1,

$$\sigma_T(\mathbf{T} + \mathbf{P}) = \sigma_T((\mathbf{T} + \mathbf{P})_{PH}) \cup \sigma_T((\mathbf{T} + \mathbf{P})_{(I-P)H}).$$

Clearly, we have $0 \notin \sigma_T((\mathbf{T}+\mathbf{P})_{(I-P)H})$. Since a commuting quasinilpotent perturbation of an invertible operator is also invertible, it immediately follows that $0 \notin \sigma_T((\mathbf{T} + \mathbf{P})_{PH})$. Thus $0 \notin \sigma_T(\mathbf{T} + \mathbf{P})$, which says that $\mathbf{T} + \mathbf{P}$ is invertible. Hence the proof is completed with $\mathbf{S} = \mathbf{T} + \mathbf{P}$ and $\mathbf{K} = -\mathbf{P}$. $\square$

REMARK. A joint Weyl spectrum, $\omega(\mathbf{T})$, for a commuting $n$-tuple $\mathbf{T}$ is defined by ([1],[2], [3])

(3.1)

$$\omega(\mathbf{T}) = \bigcap\{\sigma_T(\mathbf{T}+\mathbf{K}) : \mathbf{K} \text{is an } n-\text{tuple of compact operators and}$$
$$\mathbf{T}+\mathbf{K} = (T_1+K_1, \cdots, T_n+K_n) \text{is a commuting } n-\text{tuple}\}.$$

In [9], we suggested a question : does it follow that $\omega(\mathbf{T}) \subseteq \sigma_{Tb}(\mathbf{T})$? Theorem 2 answers the question: the answer is yes.

We were unable to decide whether or not the converse of Theorem 2 is true. However we can show that the converse is true for an interesting class containing normal $n$-tuples(cf. [10]). To see this, we recall that a point $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ is called a *joint eigenvalue* of $\mathbf{T} = (T_1, \cdots, T_n)$ if there exists a nonzero vector $x$ in $H$ for which

$$(T_i - \lambda_i)\, x = 0 \quad \text{for each } i = 1, \cdots, n.$$

We write $\sigma_p(\mathbf{T})$ for the set of all joint eigenvalues of $\mathbf{T}$ and $\pi_0(\mathbf{T})$ for the set of all joint eigenvalues of $\mathbf{T}$ of finite multiplicity. We now consider the following property that $\mathbf{T} = (T_1, \cdots, T_n)$ may satisfy:

($\alpha$) $\pi_0(\mathbf{T}) = \overline{\pi_0(\mathbf{T}^*)}$ and the corresponding joint eigenspaces of $\lambda \in \pi_0(\mathbf{T})$ and $\overline{\lambda} \in \pi_0(\mathbf{T}^*)$ are all equal.

Here $\mathbf{T}^* = (T_1^*, \cdots, T_n^*)$ and $\overline{K}$ denotes the set of complex conjugates of elements in $K$.

COROLLARY 4. *If* $\mathbf{T} = (T_1, \cdots, T_n)$ *is a doubly commuting* $n$*-tuple of dominant operators and satisfies the property* ($\alpha$). *Then* $\mathbf{T}$ *is Browder if and only if there exists an* $n$*-tuple* $\mathbf{K} = (K_1, \cdots, K_n)$ *of compact operators such that* $\mathbf{T} = \mathbf{S} + \mathbf{K}$ *where* $\mathbf{S} = (S_1, \cdots, S_n)$ *is an invertible commuting* $n$*-tuple of operators and* $S_iK_j = K_jS_i$ *for all* $1 \le i, j \le n$.

PROOF. In the view of Theorem 3, it suffices to show "if" part. If there exists an $n$-tuple $\mathbf{K}$ of compact operators such that $\mathbf{T} = \mathbf{S} + \mathbf{K}$, where $\mathbf{S}$ is an invertible commuting $n$-tuple of operators, then it is evident that $\mathbf{T}$ is Fredholm. In turn, by [10, Theorem 2.2], 0 is an isolated point of $\sigma_T(T)$, which implies that $\mathbf{T}$ is Browder. □

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